Noncommutative $U(1)$ gauge theory in the semiclassical limit

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Introduction

In the standard approach to noncommutative $U(1)$ gauge theories

- $(\mathbb{M}, \mathcal{A})$, $\mathbb{M} = \mathbb{C} \otimes \mathcal{A}$, is a 1-dim module over the NC algebra $(\mathcal{A}, \star)$
- $\nabla : Der\mathcal{A} \times \mathbb{M} \to \mathbb{M}$ covariant derivative
- $([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})m = F(X, Y)m$ curvature
- $A(X) = \nabla_X(\mathbb{I})$ gauge potential
- $U(\mathbb{M})$ Unitary automorphisms of $\mathbb{M}$ gauge transformations.

Via gauge transformations

$$\nabla_X^g : \mathbb{M} \to \mathbb{M} \qquad \nabla_X^g = g^{-1} \circ \nabla_X \circ g$$
$$F(X, Y)^g : \mathbb{M} \to \mathbb{M} \qquad F(X, Y)^g = g^{-1} \circ F(X, Y) \circ g$$

The Moyal plane $(\mathcal{A},\star)=\mathbb{R}^2_\Theta$

Constant noncommutativity

$$F_{\mu\nu}=F(\partial_\mu,\partial_\nu)=\partial_\mu A_\nu-\partial_\nu A_\mu-i[A_\mu,A_\nu]_\star \qquad A_\mu=i\partial_\mu(\mathbb{I})$$

$$A_\mu^{g_f}=g_f\star A_\mu\star g_f^\dagger-i\partial_\mu g_f\star g_f^\dagger$$

and $\quad F_{\mu\nu}^{g_f}=g_f\star F_{\mu\nu}\star g_f^\dagger$

where

$$g_f:=\exp_\star(if)=\sum_n^\infty\frac{(i)^n}{n!}f\star\cdots\star f$$

$$\longrightarrow \text{Infinitesimally}\begin{cases}\delta A_\mu &= \partial_\mu f+i[f,A_\mu]_\star\xrightarrow{\Theta\to 0}\partial_\mu f\\ \delta F_{\mu\nu} &= i[f,F_{\mu\nu}]_\star\end{cases}$$

Constant noncommutativity is an exceptional case

In general

- $\star$-Derivations do not reduce to standard derivations when $\Theta \to 0$
- standard derivatives are not derivations of the Kontsevich $\star$-product

$$\begin{aligned} f \star g &= f \cdot g + \frac{i}{2}\theta^{ab}(x)\partial_a f \partial_b g + \dots \\ \partial_c(f \star g) &= (\partial_c f) \cdot g + f \cdot (\partial_c g) + \frac{i}{2}\partial_c \theta^{ab}(x)\partial_a f \partial_b g + \dots \end{aligned}$$

QUESTION: Can we suitably modify the definition of gauge fields and gauge transformations in order to make them compatible with spacetime noncommutativity and reproduce the correct commutative limit?

PROPOSAL: In the semiclassical approximation→ Poisson Electrodynamics

A first look at Poisson Electrodynamics

$$(M, \Theta) \begin{cases} M \text{ space-time manifold} \\ \Theta \text{ Poisson bivector field (noncommutative parameter)} \end{cases}$$

Standard $U(1)$ gauge theory has $\delta_f^0 A = df$ and $[\delta_f^0, \delta_h^0] = 0$
The presence of a non-trivial Θ should modify the algebra as follows

$$[\delta_f, \delta_h]A = \delta_{\{f,h\}_\Theta} A$$

A solution was obtained

- using a bootstrap procedure, based on completing an L_∞ algebra [Blumenhagen et al '18, Kupriyanov '19, Hohm-Zwiebach, '17]
- introducing a deformation of gauge transformations, with a field dependent parameter [Kupriyanov-Vitale '20]

$$\delta_f A_a = \gamma_a^k(x, A)\partial_k f + \{A_a, f\}_\Theta$$

with $\gamma_a^k(x, A) = \sum_n \gamma_a^{k(n)}(x, A) = \delta_a^k - \frac{1}{2}(\partial_a \Theta^{kb})A_b + O(\Theta^2)$ which can be related with a symplectic embedding of (M, Θ) in (T^*M, Λ) [Kupriyanov-Szabo '22]

$$\delta_f A_a == \{f, p_a - A_a\}_{p=A} : \qquad \Lambda = \Theta + \gamma$$

Standard Electrodynamics

For standard $U(1)$ gauge theory on $(\mathbb{R}^4, \Theta = 0)$ Faraday's law can be obtained as follows:

A symplectic embedding is $(T^*\mathbb{R}^4, \omega_0)$, $\quad \omega_0 = dp_\mu \wedge dx^\mu$ canonical 2-form

Faraday tensor:

- $A : \mathbb{R}^4 \to T^*\mathbb{R}^4$ is a 1-form (a section of $T^*\mathbb{R}^4$)
- $A^*(\lambda) =: A$, $\quad \lambda$ tautological 1-form
- $A^*(\omega_0) =: F = dA^*(\lambda) = dA$ Faraday tensor
  $\Longrightarrow dF = 0$

gauge transformations: Take $f \in C^\infty(\mathbb{R}^4)$; the vector field $X_f = \Lambda_0(df) = \partial_\mu f \frac{\partial}{\partial p_\mu}$ generates a 1-parameter group of symplectomorphisms

$$\psi_t^f : \begin{cases} y_t^\mu & = & x^\mu \\ \pi_\mu^t & = & p_\mu + t\partial_\mu f \end{cases} \longrightarrow (\psi_1^f)^*(\lambda) = \lambda + df$$

$$\begin{cases} (\psi_1^f \circ A)^*(\lambda) & = & A + df \\ (\psi_1^f \circ A)^*(\omega_0) & = & F \end{cases} \longrightarrow U(1)\,\text{gauge transformations}$$

Generalization for $\Theta \neq 0$

The previous point of view can be generalized

$$
\begin{array}{ccc}
(M,\Theta) & \dashrightarrow & (T^*M,\Lambda) \\
 & & \big\downarrow \pi \\
 & & (M,\Theta)
\end{array}
$$

Λ a non-degenerate Poisson tensor s.t. $\{\pi^* f, \pi^* h\}_\Lambda = \pi^*(\{f,h\}_\Theta)$

- gauge fields are sections $A : M \to T^*M$
- the Faraday tensor is $F = A^*(\omega)$ with $\omega = \Lambda^{-1}$

Given $f \in C^\infty(M)$ $\left.\begin{array}{lll} Y_f & := \Theta(df,\cdot) & \in \mathcal{X}(M) \\ X_f & := \Lambda(d\pi^* f,\cdot) & \in \mathcal{X}(T^*M) \end{array}\right\}$ π-related vector fields

$$
\begin{array}{ccc}
T^*M & \xrightarrow{\psi_t^f} & T^*M, \\
\big\downarrow \pi & & \big\downarrow \pi \\
M & \xrightarrow{\bar\psi_t^f} & M
\end{array}
\quad \Rightarrow \quad
\begin{array}{l}
\text{transformation on sections} \\
(A_t^f) := \hat\psi_t^f(A) = \psi_t^f \circ A \circ \bar\psi_t^f
\end{array}
$$

The infinitesimal generator of the one-parameter group $\hat{\psi}_t^f(A)$ is

$$(\delta_f A)_x = \frac{d}{dt}(A_t^f(x)|_{t=0} = X^f_{A(x)} - A_*(Y_x^f)$$
$$= \left(\gamma_\nu^\mu \partial_\mu f - \Theta^{\mu\rho}\partial_\mu f \partial_\rho A_\nu\right)\frac{\partial}{\partial p_\nu} \in \mathcal{X}^V(T^*M)$$

same gauge transformation as in the L_∞-algebra approach

- $F_t^f := (A_t^f)^*(\omega) = (\bar{\psi}_t^f)^*(F) \Rightarrow \delta_f F = (\mathcal{L}_{Y_f} F)_{\mu\nu}\frac{\partial}{\partial v_{\mu\nu}}$

REMARK **1**: Also $\delta_f A$ can be interpreted as a Lie derivative

$$\delta_f A = (\mathcal{L}_{X_f} A)_\mu \frac{\partial}{\partial p_\mu} \rightarrow \text{ the gauge algebra closes w.r.t. PB's}$$

REMARK **2** For $\Theta = const$ $\gamma_\nu^\nu \rightarrow \delta_\nu^\mu$ (back to known Moyal case)

Symplectic Realizations

A symplectic realization of (M, Θ) is a pair (S, ω) s.t.

- $S \xrightarrow{\pi} M$ a surjective submersion
- ω a symplectic 2-form s.t. $\{\pi^* f, \pi^* h\}_\Lambda = \pi^* \{f, h\}_\Theta$

  Consider:

  - $U \subset T^*M, \quad \omega_0 = d\lambda$ canonical symplectic structure
  - $V^\Theta \in \mathcal{X}(T^*M)$ a Poisson spray

    $\forall \xi \in T^*M$, $\pi_*(V^\Theta_\xi) = \Theta(\xi, \cdot)$

    $(m_\nu)_*(V^\Theta) = \nu V^\Theta$ (homogeneous of degree 1)

Theorem:

The pair (U, ω)
$$\begin{cases} U \xrightarrow{\pi} & M \\ \omega = & \int_0^1 (\phi_t^\Theta)^* \omega_0 dt \end{cases}$$

is a symplectic realization of (M, Θ).
Here ϕ_t^Θ is the flow of V^Θ and $\Lambda = \omega^{-1}$ satisfies $\pi_* \Lambda = \Theta$

In a local chart, in Darboux coordinates $\omega = dy^\mu \wedge dp_\mu$, $y^\mu(x,p) = \int_0^1 x^\mu \circ \phi_t^\Theta dt$ the Jacobian $\epsilon = (\frac{\partial y^\mu}{\partial x^\nu})$ is formally invertible

$$\begin{aligned} \omega &= \epsilon_\mu^\nu(x,p) dx^\mu \wedge dp_\nu + \frac{1}{2}\epsilon_\mu^\alpha \Theta^{\mu\nu} \epsilon_\nu^\beta dp_\alpha \wedge dp_\beta \\ \Lambda &= \Theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial x^\nu} + \gamma_\nu^\mu \frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial p_\nu} \end{aligned}$$

$\gamma = \epsilon^{-1}$

the gauge potentials are identified with (local) sections of the symplectic realization $A : M \to S$

the Faraday tensor F is the pullback $A^*(\omega)$; in local coordinates

$$F = \frac{1}{2}\Big(\epsilon_\sigma^\mu(x,A)\partial_\rho A_\mu - \epsilon_\rho^\mu(x,A)\partial_\sigma A_\mu + \epsilon_\alpha^\mu \Theta^{\alpha\beta} \epsilon_\beta^\nu \partial_\rho A_\mu \partial_\sigma A_\nu \Big) dx^\rho \wedge dx^\sigma$$

Action of Lie algebroids

The infinitesimal gauge transformations are related to Lie algebroid actions
T^*M besides hosting a symplectic realization of M can be given the structure of a Lie algebroid $\mathcal{A} = (T^*M, \pi, a, M, [-,-])$ with

- $a : T^*M \to TM$, $a(\alpha) := \Theta(\alpha, -)$ the anchor map
- $[-,-]_\Theta : \Gamma(T^*M) \times \Gamma(T^*M) \to \Gamma(T^*M)$ given by
  $[\alpha, \beta] = \mathcal{L}_{a(\alpha)}\beta - \mathcal{L}_{a(\beta)}\alpha - d(\Theta(\alpha, \beta))$

The Lie algebroid acts on the sections of the symplectic realization

$$\begin{array}{ccc} & & \mathcal{A} \\ & & \big\downarrow \pi \\ S & \xrightarrow{\ \phi\ } & M \end{array}$$

An action of $\mathcal{A}$ over ϕ is a map $\uparrow : \Gamma(\mathcal{A}) \to \mathcal{X}(S)$

- $(X+Y)^\uparrow = X^\uparrow + Y^\uparrow$
- . $[X, Y]^\uparrow = [X^\uparrow, Y^\uparrow]$
- $(fX)^\uparrow = \phi^*(f) X^\uparrow$
- $(X)^\uparrow$ is ϕ-related to $a(X)$

Back to Poisson Electrodynamics

We can now reinterpret the previous results

$S \to M$ symplectic realization (full and complete, namely $(M'\Theta)$ is integrable)

mathcalA is a Lie algebroid

(M, Θ) is integrable if it exists $\Sigma(M)$ (smooth symplectic groupoid and $\mathcal{A} = A\Sigma$

$S \xrightarrow{\pi} M$ carries an action of $A\Sigma$ given by

$$\uparrow (df) = X_{\pi^* f} \in \mathcal{X}(S)$$

This action can be integrated. The exponential map of the algebroid associates with closed sections a 1−parameter group $\exp tdf$ of bisections of the symplectic groupoid. These are Lagrangian bi-sections

[Kupriyanov-Sharapov-Szabo '23, Di Cosmo-Ibort-Marmo-Vitale '23]

- $F = A^*(\omega)$ is interpreted as a curvature. It measures the deviation from $A : M \to S$ being a Lagrangian submanifold.

Conclusions and Perspectives

- Poisson Electrodynamics is an attempt of conciling noncommutative spacetime and gauge theories
- The gauge potentials have been interpreted as sections of a symplectic realization S of semiclassical spacetime M
- Lie groupoids and Lie algebroids provide the geometric framework where to understand gauge transformations of the model, previous derived via L_∞ algebras. The gauge transformations are associated with an action of the Lie algebroid T^*M which integrated to an action of the corresponding symplectic groupoid $\Sigma(M)$
- The pullback of ω on S via a section A, defines a 2-form F on M which measures the deviation of A from being a Lagrangian submainifold.

To do

- Understand the relation of the groupoids approach with L_∞ algebras (e.g. F)
- Introduce matter fields and construct dynamical gauge theories
- Generalize to non-Abelian gauge theories
- Go to the full NC regime

Happy Birthday Harald!

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