Cointeracting bialgebras

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Let G a (proto)-algebraic monoid. The algebra $\mathbb{C}[G]$ of polynomial functions on G inherits a coproduct $\Delta : \mathbb{C}[G] \longrightarrow \mathbb{C}[G] \otimes \mathbb{C}[G] \approx \mathbb{C}[G \times G]$ such that:

$$\forall f \in \mathbb{C}[G], \forall x, y \in G, \qquad \Delta(f)(x, y) = f(xy).$$

This makes $\mathbb{C}[G]$ a bialgebra. It is a Hopf algebra if, and only if, G is a group. Moreover, G is isomorphic to the monoid $\textbf{char}(\mathbb{C}[G])$ of characters of $\mathbb{C}[G]$.

Characters of a bialgebra B

A character of a bialgebra B is an algebra morphism $\lambda : B \longrightarrow \mathbb{C}$. The set of characters $\textbf{char}(B)$ is given an associative convolution product:

$$\lambda * \mu = m_{\mathbb{C}} \circ (\lambda \otimes \mu) \circ \Delta.$$

Let G and G' be two (proto)-algebraic monoids, such that G' acts polynomialy on G by monoid endomorphisms (on the right). Then:

Interacting bialgebras

1. $A = (\mathbb{C}[G], m_A, \Delta)$ is a bialgebra.
2. $B = (\mathbb{C}[G'], m_B, \delta)$ is a bialgebra.
3. B coacts on A by a coaction $\rho : A \longrightarrow A \otimes B$.
4. A is a bialgebra in the category of B-comodules.

In other words, for any $f, g \in A$:

$$
\begin{aligned}
(\mathrm{Id} \otimes \rho) \circ \rho &= (\Delta \otimes \mathrm{Id}) \circ \rho,\\
\rho(fg) &= \rho(f)\rho(g),\\
\rho(1_A) &= 1_A \otimes 1_B,\\
(\varepsilon_A \otimes \mathrm{Id}_B) \circ \rho(f) &= \varepsilon_A(f) 1_B,\\
(\Delta \otimes \mathrm{Id}) \circ \rho(f) &= m_{1,3,24} \circ (\rho \otimes \rho) \circ \Delta(f).
\end{aligned}
$$

where

$$
m_{1,3,24} : \left\{ \begin{array}{rcl} A^{\otimes 4} & \longrightarrow & A^{\otimes 3} \\ a_1 \otimes a_2 \otimes a_3 \otimes a_4 & \longrightarrow & a_1 \otimes a_3 \otimes a_2 a_4. \end{array} \right.
$$

The algebra $\mathbb{C}[X]$

The group $(\mathbb{C}^*, \times)$ acts on $(\mathbb{C}, +)$ by group automorphisms.

- $A = (\mathbb{C}[X], m, \Delta)$ with $\Delta(X) = X \otimes 1 + 1 \otimes X$, is a Hopf algebra.
- $B = (\mathbb{C}[X, X^{-1}], m, \delta)$ with $\delta(X) = X \otimes X$, is a Hopf algebra.
- $\rho(X) = X \otimes X$ defines a coaction of B on A, and A is a bialgebra in the category of B-comodules.

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- $\rho(X) = X \otimes X$ defines a coaction of B on A, and A is a bialgebra in the category of B-comodules.

From now, we shall consider only examples where $A = B$ as algebras: we obtain objects (A, m, Δ, δ), with one product and two coproducts. The coaction ρ and the coproduct δ are equal. These objects will be called double bialgebras.

The algebra $\mathbb{C}[X]$

$(A, m) = (B, m) = (\mathbb{C}[X], m)$, where m is the usual product of polynomials, and the coproducts are given by:

$$\Delta(X) = X \otimes 1 + 1 \otimes X,$$
$$\delta(X) = X \otimes X.$$

Then $(\mathbb{C}[X], m, \Delta, \delta)$ is a double bialgebra.

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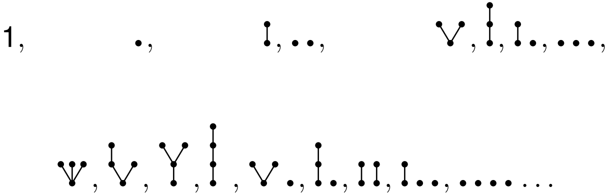
The algebra $\mathbb{C}[X]$

$(A, m) = (B, m) = (\mathbb{C}[X], m)$, where m is the usual product of polynomials, and the coproducts are given by:

$$\Delta(X^n) = \sum_{k=0}^{n} \binom{n}{k} X^k \otimes X^{n-k},$$
$$\delta(X^n) = X^n \otimes X^n.$$

Then $(\mathbb{C}[X], m, \Delta, \delta)$ is a double bialgebra.

The Connes-Kreimer Hopf algebra of trees is based on rooted forests:



As algebras, $A = B = \mathcal{H}_{CK}$ with the disjoint union product.

The first coproduct Δ is given by admissible cuts (Connes-Kreimer coproduct).

Example

$$\Delta\left(\begin{smallmatrix}\bullet\;\bullet\\\bigvee\\\bullet\end{smallmatrix}\right)=\begin{smallmatrix}\bullet\;\bullet\\\bigvee\\\bullet\end{smallmatrix}\otimes 1+1\otimes\begin{smallmatrix}\bullet\;\bullet\\\bigvee\\\bullet\end{smallmatrix}+2\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}\otimes\bullet+\bullet\otimes\bullet\bullet,$$

$$\Delta\left(\begin{smallmatrix}\bullet\\|\\\bullet\\|\\\bullet\end{smallmatrix}\right)=\begin{smallmatrix}\bullet\\|\\\bullet\\|\\\bullet\end{smallmatrix}\otimes 1+1\otimes\begin{smallmatrix}\bullet\\|\\\bullet\\|\\\bullet\end{smallmatrix}+\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}\otimes\bullet+\bullet\otimes\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}.$$

Counit:

$$\varepsilon(F)=\begin{cases}1 \text{ if } F=1,\\ 0 \text{ otherwise.}\end{cases}$$

The second coproduct δ is given by contraction-extraction.

Example

$$\delta(\tdtroisun) = \tdtroisun \otimes \tdun\tdun\tdun + 2\tddeux \otimes \tdun\tddeux + \tdun \otimes \tdtroisun,$$
$$\delta(\tdtroisdeux) = \tdtroisdeux \otimes \tdun\tdun\tdun + 2\tddeux \otimes \tdun\tddeux + \tdun \otimes \tdtroisdeux.$$

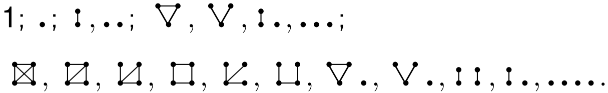
Its counit is:

$$\varepsilon'(F) = \begin{cases} 1 \text{ if } F = \tdun \ldots \tdun, \\ 0 \text{ otherwise.} \end{cases}$$

(Calaque, Ebrahimi-Fard, Manchon, 2008). Then $(\mathcal{H}_{CK}, m, \Delta, \delta)$ is a double bialgebra.

This construction can be extended to finite posets or to finite topologies.

$\mathcal{H}_{\mathcal{G}}$ has for basis the set of graphs:



The product is the disjoint union. The unit is the empty graph 1.

(Schmitt, 1994). The first coproduct Δ is defined by

$$\Delta(G) = \sum_{V(G)=I\sqcup J} G_{|I} \otimes G_{|J}.$$

Examples

$$\Delta(\bullet) = \bullet \otimes 1 + 1 \otimes \bullet,$$

$$\Delta(\mathrm{I}) = \mathrm{I} \otimes 1 + 1 \otimes \mathrm{I} + 2\bullet \otimes \bullet,$$

$$\Delta(\nabla) = \nabla \otimes 1 + 1 \otimes \nabla + 3\mathrm{I} \otimes \bullet + 3\bullet \otimes \mathrm{I},$$

$$\Delta(\vee) = \vee \otimes 1 + 1 \otimes \vee + 2\mathrm{I} \otimes \bullet + \bullet\bullet \otimes \bullet + 2\bullet \otimes \mathrm{I} + \bullet \otimes \bullet\bullet.$$

(Schmitt, 1994– Manchon, 2011). The second coproduct δ is defined by

$$\delta(G)=\sum_{\sim}(G/\sim)\otimes(G\mid\sim),$$

where:

- $\sim$ runs in the set of equivalences on $V(G)$ which classes are connected.
- $G|\sim$ is the union of the equivalence classes of $\sim$.
- $G/\sim$ is obtained by the contraction of the equivalence classes of $\sim$.

Examples

$$\delta(\bullet) = \bullet \otimes \bullet,$$

$$\delta(\mathtt{I}) = \bullet \otimes \mathtt{I} + \mathtt{I} \otimes \bullet\bullet,$$

$$\delta(\nabla) = \bullet \otimes \nabla + 3\,\mathtt{I} \otimes \bullet\mathtt{I} + \nabla \otimes \bullet\bullet\bullet,$$

$$\delta(\vee) = \bullet \otimes \vee + 2\,\mathtt{I} \otimes \bullet\mathtt{I} + \vee \otimes \bullet\bullet\bullet.$$

Its counit is given by:

$$\varepsilon'(G) = \begin{cases} 1 \text{ if } G \text{ has no edge,} \\ 0 \text{ otherwise.} \end{cases}$$

Then $(\mathcal{H}_{\mathcal{G}}, m, \Delta, \delta)$ is a double bialgebra.

Questions

- Theoretical consequences?
- Examples and applications?

We consider a double bialgebra (A, m, Δ, δ).

Proposition

Let B be a bialgebra and let $E_{A \to B}$ be the set of bialgebra morphisms from A to B. The monoid of characters M_A of (A, m, δ) acts on $E_{A \to B}$:

$$\phi \leftarrow \lambda = (\phi \otimes \lambda) \circ \delta.$$

If (A, Δ) is a connected coalgebra:

Theorem

1. There exists a unique $\phi_1 : A \longrightarrow \mathbb{C}[X]$, compatible with both bialgebraic structure.
2. The following maps are bijections, inverse one from the other:

$$\left\{ \begin{array}{rcl} M_A & \longrightarrow & E_{A\rightarrow\mathbb{C}[X]} \\ \lambda & \longrightarrow & \phi_1 \leftarrow \lambda, \end{array} \right. \qquad \left\{ \begin{array}{rcl} E_{A\rightarrow\mathbb{C}[X]} & \longrightarrow & M_A \\ \phi & \longrightarrow & \varepsilon' \circ \phi \\ & & = \phi(\cdot)(1). \end{array} \right.$$

Let us apply this result on the double bialgebra of forests.
As $\bullet$ is primitive, $\phi_1(\bullet)$ is primitive, so $\phi_1(\bullet)=\lambda X$. As $\phi_1(\bullet)(1)=\varepsilon'(\bullet)=1$, $\phi_1(\bullet)=X$.

$$\Delta(\tddeux)=\tddeux\otimes 1+1\otimes\tddeux+\bullet\otimes\bullet,$$

$$\Delta(\phi_1(\tddeux))=\phi_1(\tddeux)\otimes 1+1\otimes\phi_1(\tddeux)+X\otimes X,$$

so $\phi_1(\tddeux)=\dfrac{X^2}{2}+\lambda X$. As $\phi_1(\tddeux)(1)=\varepsilon'(\tddeux)=0$, we obtain $\lambda=-\dfrac{1}{2}$.

$$\phi_1(\tddeux)=\frac{X(X-1)}{2}.$$

Let us apply this result on the double bialgebra of forests.
As $\bullet$ is primitive, $\phi_1(\bullet)$ is primitive, so $\phi_1(\bullet) = \lambda X$. As $\phi_1(\bullet)(1) = \varepsilon'(\bullet) = 1$, $\phi_1(\bullet) = X$.

$$\Delta(\overset{\bullet}{\underset{\bullet}{|}}) = \overset{\bullet}{\underset{\bullet}{|}} \otimes 1 + 1 \otimes \overset{\bullet}{\underset{\bullet}{|}} + \bullet \otimes \bullet,$$

$$\Delta(\phi_1(\overset{\bullet}{\underset{\bullet}{|}})) = \phi_1(\overset{\bullet}{\underset{\bullet}{|}}) \otimes 1 + 1 \otimes \phi_1(\overset{\bullet}{\underset{\bullet}{|}}) + X \otimes X,$$

so $\phi_1(\overset{\bullet}{\underset{\bullet}{|}}) = \dfrac{X^2}{2} + \lambda X$. As $\phi_1(\overset{\bullet}{\underset{\bullet}{|}})(1) = \varepsilon'(\overset{\bullet}{\underset{\bullet}{|}}) = 0$, we obtain $\lambda = -\dfrac{1}{2}$.

$$\phi_1(\overset{\bullet}{\underset{\bullet}{|}}) = \frac{X(X-1)}{2}.$$

$$\Delta(\ttroisdeux) = \ttroisdeux \otimes 1 + 1 \otimes \ttroisdeux + \tdeux \otimes \tun + \tun \otimes \tdeux,$$

$$\Delta(\phi_1(\ttroisdeux)) = \phi_1(\ttroisdeux) \otimes 1 + 1 \otimes \phi_1(\ttroisdeux) + \frac{1}{2}(X^2 \otimes X + X \otimes X^2) - X \otimes X,$$

so $\phi_1(\ttroisdeux) = \frac{X^3}{6} - \frac{X^2}{2} + \lambda X$. As $\phi_1(\ttroisdeux)(1) = \varepsilon'(\ttroisdeux) = 0$, $\lambda = \frac{1}{3}$.

$$\phi_1(\ttroisdeux) = \frac{X(X-1)(X-2)}{6}.$$

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$$\phi_1(\ttroisun) = \frac{X(X-1)(2X-1)}{6}.$$

Proposition

For any $a \in A$, with $\varepsilon(a) = 0$:

$$\phi_1(a) = \sum_{n=1}^{\infty} \varepsilon'^{\otimes n} \circ \tilde{\Delta}^{(n-1)}(a) \frac{X(X-1)\ldots(X-n+1)}{n!}.$$

Here, $\tilde{\Delta}$ is the reduced coproduct:

$$\tilde{\Delta}(a) = \Delta(a) - a \otimes 1 - 1 \otimes a,$$

and $\tilde{\Delta}^{(n-1)}$ is defined by

$$\tilde{\Delta}^{(n-1)} = \begin{cases} \mathrm{Id}_A \text{ if } n = 1, \\ (\tilde{\Delta}^{(n-2)} \otimes \mathrm{Id}_A) \circ \tilde{\Delta} \text{ otherwise.} \end{cases}$$

Let F be a forest with k vertices, indexed by $[k]$. We associate to F a polytope of dimension k:

$$pol(F) = \{(x_1, \ldots, x_k) \in [0,1]^k \mid i \leqslant_h j \Longrightarrow x_i \leqslant x_j\}.$$

For any $n \in \mathbb{N}$:

- $ehr_F(n)$ is the number of integral points of $(n-1).pol(F)$.
- $ehr_F^{str}(n)$ is the number of integral points in the interior of $(n+1).pol(F)$.

This defines two polynomials $ehr_F(X)$ and $ehr_F^{str}(X)$, the Ehrhart and the stric Ehrhart polynomials attached to F.

Example 1

For $F =$ :

$$pol(F) = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leqslant x \leqslant y \leqslant z \leqslant 1\},$$

$$\begin{aligned} ehr_F(n) &= \sharp\{(x, y, z) \in \mathbb{N}^3 \mid 0 \leqslant x \leqslant y \leqslant z \leqslant n-1\} \\ &= \frac{n(n+1)(n+2)}{6}, \\ ehr_F^{str}(n) &= \sharp\{(x, y, z) \in \mathbb{N}^3 \mid 0 < x < y < z < n+1\} \\ &= \frac{n(n-1)(n-2)}{6}. \end{aligned}$$

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Example 2

For $F =$  :

$$
\begin{aligned}
pol(F) &= \{(x,y,z) \in \mathbb{R}^3 \mid 0 \leqslant x \leqslant y, z \leqslant 1\},\\
ehr_F(n) &= \sharp\{(x,y,z) \in \mathbb{N}^3 \mid 0 \leqslant x \leqslant y, z \leqslant n-1\}\\
&= 1^2 + \ldots + n^2\\
&= \frac{n(n+1)(2n+1)}{6},\\
ehr_F^{str}(n) &= \sharp\{(x,y,z) \in \mathbb{N}^3 \mid 0 < x < y \neq z < n+1\}\\
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\end{aligned}
$$

Theorem

$ehr^{str} : \mathcal{H}_{CK} \longrightarrow \mathbb{C}[X]$ is the morphism ϕ_1.

Another morphism from $\mathcal{H}_{CK}$ compatible with m, Δ and δ is defined by:

$$\phi(F) = (-1)^{|F|} ehr_F(-X).$$

Hence:

Duality principle

For any forest F,

$$ehr^{str}_F(F) = (-1)^{|F|} ehr_F(-X).$$

All this can be extended to finite posets and to finite topologies.

Proposition

For any $a \in A$, with $\varepsilon(a) = 0$:

$$\phi_1(a) = \sum_{n=1}^{\infty} \varepsilon'^{\otimes n} \circ \tilde{\Delta}^{(n-1)}(a) \frac{X(X-1)\ldots(X-n+1)}{n!}.$$

Let G be a graph.

- A n-coloration of G is a map from $V(G)$ to $\{1, \ldots, n\}$.
- A n-coloration is valid if any two neighbors in G have different colors.
- A n-coloration c is packed if c is surjective.

Let G be a graph. The chromatic polynomial of G is defined by:

$$\forall n \in \mathbb{N}, \qquad chr_G(n) = \sharp\{\text{valid } n\text{-colorations of } G\}.$$

Theorem

$chr : \mathcal{H}_{\mathcal{G}} \longrightarrow \mathbb{C}[X]$ is the morphism ϕ_1.

Antipode

Let (A, m, Δ, δ) be a double bialgebra. We assume that the counit ε' has an inverse α in the monoid of characters of (A, m, Δ). Then (A, m, Δ) is a Hopf algebra, of antipode

$$S = (\alpha \otimes \mathrm{Id}) \circ \delta.$$

As a consequence, if (A, m, Δ) is connected, then (A, m) is commutative.

Link with morphisms to $\mathbb{C}[X]$

Let $\phi_1 : A \longrightarrow \mathbb{C}[X]$ be a double bialgebra morphism. Then ε' has an inverse α in the monoid of characters of (A, m, Δ), given by:

$$\alpha(a) = \phi_1(a)(-1).$$

Moreover, (A, m, Δ) is a Hopf algebra, and its antipode is given by:

$$S(a) = \underbrace{(\phi_1 \otimes \mathrm{Id}) \circ \delta(a)}_{\in \mathbb{C}[X] \otimes A}\Big|_{X=-1}.$$

Antipode of $\mathcal{H}_{CK}$

For any rooted forest F:

$$\alpha(F) = (-1)^{|F|}, \qquad S(F) = \sum_{c \text{ cut of } F} (-1)^{|c|+\lg(F)} W^c(F).$$

We recover the formula proved by Connes and Kreimer.

Antipode of $\mathcal{H}_\mathcal{G}$

$(\mathcal{H}_\mathcal{G}, m, \Delta)$ is a Hopf algebra. Its antipode is given by:

$$S(G) = \sum_{\sim} P_{chr}(G/\sim)(-1)^{|G/\sim|}(G\mid\sim)$$
$$= \sum_{\sim}(-1)^{|cl(\sim)|}\sharp\{\text{acyclic orientations of } G/\sim\}(G\mid\sim).$$

This formula was proved by Benedetti, Bergeron and Machacek in 2019 with combinatorial methods and a Möbius inversion.

There exists another Hopf algebra morphism $\phi_0 : \mathcal{H}_\mathcal{G} \longrightarrow \mathbb{C}[X]$, defined by

$$\phi_0(G) = X^{|G|}.$$

If λ is the character defined by $\lambda(G) = 1$ for any graph G, then

$$\phi_0 = \phi_1 \leftarrow \lambda.$$

λ is invertible in M_A, and we denote its inverse by λ_{chr}.

$$\phi_1 = \phi_0 \leftarrow \lambda_{chr}.$$

Theorem

For any graph G:

$$chr_G(X) = \sum_{\sim} \lambda_{chr}(G|\sim) X^{cl(\sim)},$$

with $\lambda_{chr} = \lambda_0^{-1}$.

G										
$\lambda_{chr}(G)$	1	-1	2	1	-6	-4	-2	-3	-1	-1

Contraction-extraction. For any graph G, for any edge e of G:

$$chr_G(X) = chr_{G\backslash e}(X) - chr_{G/e}(X),$$

$$\lambda_{chr}(G) = \begin{cases} -\lambda_{chr}(G/e) \text{ if } e \text{ is a bridge,} \\ \lambda_{chr}(G\backslash e) - \lambda_{chr}(G/e) \text{ otherwise.} \end{cases}$$

Corollary

For any graph G, $\lambda_{chr}(G)$ is nonzero, of sign $(-1)^{|G|-cc(G)}$.
Putting $chr_G(X) = a_0 + \ldots + a_n X^n$:

- $a_k \neq 0 \iff cc(G) \leqslant k \leqslant |G|$.
- a_k is of sign $(-1)^{n-k}$ (Rota).
- $-a_{n-1}$ is the number of edges of G.

One can can replace the double bialgebra $\mathbb{C}[X]$ by the double bialgebra of quasisymmetric functions **QSym.**

Examples

$$\begin{aligned}(a_1) \uplus (a_2) &= (a_1 a_2) + (a_2 a_1) + (a_1 + a_2),\\ (a_1) \uplus (a_2 a_3) &= (a_1 a_2 a_3) + (a_2 a_1 a_3) + (a_2 a_3 a_1)\\ &+ ((a_1 + a_2) a_3) + (a_2(a_1 + a_3)),\end{aligned}$$

$$\begin{aligned}\Delta(a_1) &= (a_1) \otimes 1 + 1 \otimes (a_1),\\ \Delta(a_1 a_2) &= (a_1 a_2) \otimes 1 + (a_1) \otimes (a_2) + 1 \otimes (a_1 a_2),\\ \Delta(a_1 a_2 a_3) &= (a_1 a_2 a_3) \otimes 1 + (a_1 a_2) \otimes (a_1)\\ &+ (a_1) \otimes (a_2 a_3) + 1 \otimes (a_1 a_2 a_3).\end{aligned}$$

The second coproduct si given by extraction and contraction of subwords.

Examples

$$\begin{aligned}
\delta(a_1) &= (a_1) \otimes (a_1),\\
\delta(a_1 a_2) &= (a_1 a_2) \otimes (a_1) \uplus (a_2) + (a_1 + a_2) \otimes (a_1 a_2),\\
\delta(a_1 a_2 a_3) &= (a_1 a_2 a_3) \otimes (a_1) \uplus (a_2) \uplus (a_3)\\
&+ ((a_1 + a_2) a_3) \otimes (a_1 a_2) \uplus (a_3)\\
&+ (a_1 (a_2 + a_3)) \otimes (a_1) \uplus (a_2 a_3)\\
&+ (a_1 + a_2 + a_3) \otimes (a_1 a_2 a_3).
\end{aligned}$$

Under conditions of graduations, we obtain a unique homogeneous morphism from A to **QSym**, compatible with both bialgebraic structure.

For graphs, we obtain the chromatic symmetric function. For forests, we obtain an Ehrhart quasisymmetric function.

We obtain noncommutative versions of these results, replacing graphs by indexed graphs, trees by indexed trees, . . ., and, quasisymmetric functions by packed words.
This gives noncommutative versions of chromatic polynomials and of Ehrhart polynomials, with a generalization of the duality principle.

In these noncommutative versions, we lose the compatibility between the two coproducts. To explicit the obtained compatibility, one has to work in the category of species.

Thank you for your attention!