Double bialgebra of noncrossing partitions

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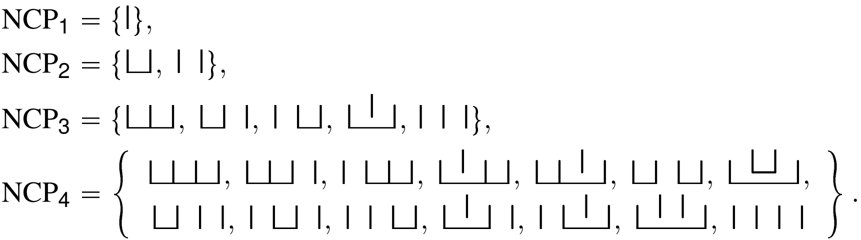
February 19th 2025–Wien

We denote by $\mathrm{NCP}(n)$ the set of noncrossing partitions on $[n]$. They are represented by diagrams as these:



Let G a (proto)-algebraic monoid. The algebra $\mathbb{C}[G]$ of polynomial functions on G inherits a coproduct

$$\Delta : \mathbb{C}[G] \longrightarrow \mathbb{C}[G] \otimes \mathbb{C}[G] \approx \mathbb{C}[G \times G]$$

defined by

$$\forall f \in \mathbb{C}[G], \forall x, y \in G, \qquad \Delta(f)(x, y) = f(xy).$$

This makes $\mathbb{C}[G]$ a bialgebra. The monoid of characters of $\mathbb{C}[G]$ is isomorphic to G.

Characters of a bialgebra B

A character of a bialgebra B is an algebra morphism $\lambda : B \longrightarrow \mathbb{C}$. The set of characters $\mathbf{char}(B)$ is given an associative convolution product:

$$\lambda * \mu = m_{\mathbb{C}} \circ (\lambda \otimes \mu) \circ \Delta.$$

If G is a (proto)-algebraic group, then the map $g \longrightarrow g^{-1}$ induces a map

$$S : \left\{ \begin{array}{rcl} \mathbb{C}[G] & \longrightarrow & \mathbb{C}[G] \\ f & \longmapsto & S(f) : \left\{ \begin{array}{rcl} G & \longrightarrow & G \\ x & \longmapsto & f(x^{-1}). \end{array} \right. \end{array} \right.$$

This map is called the antipode of $\mathbb{C}[G]$. A bialgebra with an antipode is called a Hopf algebra.
Dually, if B is a Hopf algebra, then the monoid $\mathbf{char}(B)$ is a group.

Question

What if G is a ring?

We now assume that G has two products $+$ and $\times$, such that

- $+$ and $\times$ are associative and unitary, with respective units 0_G and 1_G.
- $\times$ is distributive on the right over $+$: for any $x, y, z \in G$,

$$(x + y) \times z = (x \times z) + (y \times z).$$

We do not assume that $+$ and $\times$ are commutative, nor the left distributivity.

We consider the algebra $B = \mathbb{C}[G]$. Under some conditions, one can dualize the products $+$ and $\times$ as two coproducts

$$\Delta : B \longrightarrow B \otimes B, \qquad \delta : B \longrightarrow B \otimes B,$$

with the following conditions:

- Δ and δ are algebra morphisms.
- Δ and δ are coassociative:

$$(\Delta \otimes \mathrm{Id}_B) \circ \Delta = (\mathrm{Id}_B \otimes \Delta) \circ \Delta, \quad (\delta \otimes \mathrm{Id}_B) \circ \delta = (\mathrm{Id}_B \otimes \delta) \circ \delta.$$

- Δ and δ have counits ε_Δ and ϵ_δ, which are linear forms on B.

$$(\varepsilon_\Delta \otimes \mathrm{Id}_B) \circ \Delta = (\mathrm{Id}_B \otimes \varepsilon_\Delta) \circ \Delta = \mathrm{Id}_B,$$
$$(\epsilon_\delta \otimes \mathrm{Id}_B) \circ \delta = (\mathrm{Id}_B \otimes \epsilon_\delta) \circ \delta = \mathrm{Id}_B.$$

- Δ and δ cointeract:

$$\begin{aligned}(\Delta \otimes \mathrm{Id}_B) \circ \delta &= m_{1,3,24} \circ (\delta \otimes \delta) \circ \Delta,\\ (\varepsilon_\Delta \otimes \mathrm{Id}_B) \circ \delta(f) &= \varepsilon_\Delta(f)1.\end{aligned}$$

We shall say that (B, m, Δ, δ) is a double bialgebra.

A simple example

We consider $G = (\mathbb{C}, +, \times)$. Then $\mathbb{C}[G] = \mathbb{C}[X]$, with its usual product and the coproducts given by

$$\Delta(X^n) = \sum_{k=0}^{n} \binom{n}{k} X^k \otimes X^{n-k}, \qquad \delta(X^n) = X^n \otimes X^n.$$

The counits are given by

$$\varepsilon_\Delta(P) = P(0), \qquad \epsilon_\delta(P) = P(1).$$

Then $(\mathbb{C}[X], m, \Delta, \delta)$ is a double bialgebra.

A simple example

We consider $G = (\mathbb{C}, +, \times)$. Then $\mathbb{C}[G] = \mathbb{C}[X]$, with its usual product and the multiplicative coproducts given by

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \qquad \delta(X) = X \otimes X.$$

The counits are given by

$$\varepsilon_\Delta(P) = P(0), \qquad \epsilon_\delta(P) = P(1).$$

Then $(\mathbb{C}[X], m, \Delta, \delta)$ is a double bialgebra.

Graphs

$\mathcal{H}_G$ has for basis the set of all finite graphs. The product is the disjoint union. The first coproduct separate the vertices of the graph into two separate parts, whereas the second one contracts some connected parts.

Examples

$$\Delta(\nabla) = \nabla \otimes 1 + 1 \otimes \nabla + 3\,\mathrm{I} \otimes . + 3 . \otimes \mathrm{I},$$

$$\Delta(\vee) = \vee \otimes 1 + 1 \otimes \vee + 2\,\mathrm{I} \otimes . + .. \otimes . + 2 . \otimes \mathrm{I} + . \otimes ..,$$

$$\delta(\nabla) = . \otimes \nabla + 3\,\mathrm{I} \otimes .\mathrm{I} + \nabla \otimes ...,$$

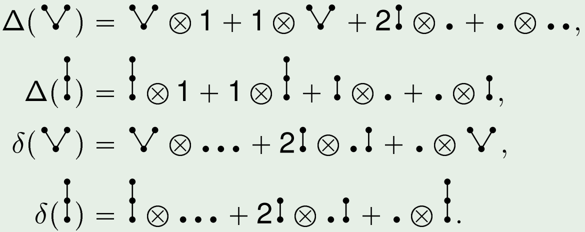
$$\delta(\vee) = . \otimes \vee + 2\,\mathrm{I} \otimes .\mathrm{I} + \vee \otimes ....$$

Rooted trees

$\mathcal{H}_T$ has for basis the set of rooted forests. The product is the disjoint union. The first coproduct separates the tree into a lower and a upper part (Butcher–Connes–Kreimer coproduct), the second one contracts subforests (Calaque–Ebrahimi-Fard–Manchon coproduct).

These structures are used in quantum field theory (renormalisation), in the theory of operads (pre-Lie algebras), and numerical analysis (Butcher's group of Runge-Kutta methods), among others.

Examples



The algebra $\mathcal{H}_{\mathrm{NCP}}$ is the free commutative algebra generated by NCP. Its (commutative) product is denoted by $\cdot$.
A basis of $\mathcal{H}_{\mathrm{NCP}}$ is given by commutative monomials of elements of NCP. Note that

$$\mid \cdot \mid \neq \mid\ \mid, \qquad \sqcup\ \mid \neq \mid\ \sqcup, \qquad \sqcup\ \mid \neq \sqcup \cdot \mid, \qquad \mid\ \sqcup \neq \mid \cdot \sqcup,$$

$$\sqcup \cdot \mid = \mid \cdot \sqcup.$$

The first coproduct is given by separation of blocks into a lower part and an upper part:

$$\begin{aligned}
\Delta(\sqcup\!\sqcup) &= \sqcup\!\sqcup \otimes 1 + 1 \otimes \sqcup\!\sqcup,\\
\Delta(\sqcup\ |) &= \sqcup\ | \otimes 1 + 1 \otimes \sqcup\ | + \sqcup \otimes | + | \otimes \sqcup,\\
\Delta(|\ \sqcup) &= |\ \sqcup \otimes 1 + 1 \otimes |\ \sqcup + \sqcup \otimes | + | \otimes \sqcup,\\
\Delta(\overset{|}{\sqcup}) &= \overset{|}{\sqcup} \otimes 1 + 1 \otimes \overset{|}{\sqcup} + \sqcup \otimes |,\\
\Delta(|\ |\ |) &= |\ |\ | \otimes 1 + 1 \otimes |\ |\ | + | \otimes \left({\color{red} | \cdot | + 2|\ |}\right) + 3|\ | \otimes |.
\end{aligned}$$

For any noncrossing partition P, $\varepsilon_\Delta(P) = 0$.

The second coproduct is given by fusions of blocks:

$$\delta(\text{└┴┘}) = \text{└┴┘} \otimes \text{└┴┘},$$

$$\delta(\text{└┘ |}) = \text{└┘ |} \otimes \text{└┘} \cdot \text{|} + \text{└┴┘} \otimes \text{└┘ |},$$

$$\delta(\text{| └┘}) = \text{| └┘} \otimes \text{└┘} \cdot \text{|} + \text{└┴┘} \otimes \text{| └┘},$$

$$\delta(\text{└╵┘}) = \text{└╵┘} \otimes \text{└┘} \cdot \text{|} + \text{└┴┘} \otimes \text{└╵┘},$$

$$\delta(\text{| | |}) = \text{| | |} \otimes \text{|} \cdot \text{|} \cdot \text{|} + (\text{└┘ |} + \text{| └┘} + \text{└╵┘}) \otimes \text{|} \cdot \text{| |} + \text{└┴┘} \otimes \text{| | |}.$$

For any noncrossing partition P,

$$\epsilon_\delta(P) = \begin{cases} 1 \text{ if } P \text{ has one block,} \\ 0 \text{ otherwise.} \end{cases}$$

Theorem

With these two coproduct, $\mathcal{H}_{\mathrm{NCP}}$ is a double bialgebra.

We consider a double bialgebra (B, m, Δ, δ).

Proposition

Let (B', m', Δ') be a bialgebra and let $E_{B\to B'}$ be the set of bialgebra morphisms from (B, m, Δ) to (B', m', Δ').
The monoid of characters M_B of (B, m, δ) acts on $E_{B\to B'}$ by

$$\phi \leftsquigarrow \lambda = (\phi \otimes \lambda) \circ \delta.$$

When $B' = \mathbb{C}[X]$, under a condition of connectivity on the coproduct Δ:

Theorem

1. There exists a unique $\phi_1 : B \longrightarrow \mathbb{C}[X]$, compatible with both bialgebraic structures.
2. The following maps are bijections, inverse one from the other:

$$\left\{ \begin{array}{rcl} M_B & \longrightarrow & E_{B\to\mathbb{C}[X]} \\ \lambda & \longmapsto & \phi_1 \leftarrow\!\!\!\!\sim\!\!\sim \lambda, \end{array} \right. \qquad \left\{ \begin{array}{rcl} E_{B\to\mathbb{C}[X]} & \longrightarrow & M_B \\ \phi & \longmapsto & \phi(\cdot)(1). \end{array} \right.$$

The bialgebra morphisms from (B, m, Δ) to $(\mathbb{C}[X], m, \Delta)$ will be called polynomial invariants, and the morphism ϕ_1 will be called the fundamental polynomial invariant.

Proposition

For any $x \in B$, with $\varepsilon_\Delta(x) = 0$,

$$\phi_1(x) = \sum_{n=1}^{\infty} \epsilon_\delta^{\otimes n} \circ \tilde{\Delta}^{(n-1)}(x) \frac{X(X-1)\ldots(X-n+1)}{n!}.$$

Here, $\tilde{\Delta}$ is the reduced coproduct:

$$\tilde{\Delta}(x) = \Delta(x) - x \otimes 1 - 1 \otimes x,$$

and $\tilde{\Delta}^{(n-1)}$ is defined by

$$\tilde{\Delta}^{(n-1)} = \begin{cases} \mathrm{Id}_B \text{ if } n = 1, \\ (\tilde{\Delta}^{(n-2)} \otimes \mathrm{Id}_B) \circ \tilde{\Delta} \text{ otherwise.} \end{cases}$$

The connectivity condition means that the reduced coproduct is locally nilpotent.

Example: $\mathcal{H}_G$

For any graph G, for any $n \in \mathbb{N}$, $\phi_1(G)(n)$ is the number of maps $c : V(G) \longrightarrow [n]$ such that if $\{x, y\}$ is an edge of G, then $c(x) \neq c(y)$.

In other terms, $\phi_1(G)$ is the chromatic polynomial.

Example: $\mathcal{H}_T$

For any rooted tree T, for any $n \in \mathbb{N}$, $\phi_1(T)(n)$ is the number of maps $c : V(T) \longrightarrow [n]$ such that if (x, y) is an edge of T, then $c(x) < c(y)$.

In other terms, $\phi_1(T)$ is the strict Ehrhart polynomial.

Usually, it is quite difficult to compute the antipode of a given Hopf algebra. Under the connectivity condition, one can use Takeuchi's formula: if $\varepsilon_\Delta(x) = 0$,

$$S(x) = \sum_{k=1}^{\infty} (-1)^k m^{(k-1)} \circ \tilde{\Delta}^{(k-1)}(x).$$

Inconvenient: a lot of cancellations. For example, in $\mathbb{C}[X]$,

$$\begin{aligned} S(X^3) &= -X^3 + (3XX^2 + 3X^2X) - 6(XXX) = -X^3, \\ S(X^4) &= -X^4 + (4XX^3 + 6X^2X^2 + 4X^3X) \\ &\quad - (12XXX^2 + 12XX^2X + 12X^2XX) + 24XXXX = X^4. \end{aligned}$$

Antipode

Let (B, m, Δ, δ) be a double bialgebra. We assume that the counit ε_δ has an inverse α in the monoid of characters of (B, m, Δ). Then (B, m, Δ) is a Hopf algebra, of antipode

$$S = (\alpha \otimes \mathrm{Id}) \circ \delta.$$

Link with morphisms to $\mathbb{C}[X]$

Let $\phi_1 : B \longrightarrow \mathbb{C}[X]$ be a double bialgebra morphism. Then ε_δ has an inverse α in the monoid of characters of (B, m, Δ), given by

$$\alpha(a) = \phi_1(a)(-1).$$

Therefore, (B, m, Δ) is a Hopf algebra, and its antipode is given by

$$S(x) = \underbrace{(\phi_1 \otimes \mathrm{Id}) \circ \delta(x)}_{\in \mathbb{C}[X] \otimes B}\Big|_{X=-1}.$$

If P is a noncrossing partition, we can write the iterated coproducts of P under the form

$$\Delta^{(n-1)}(P) = \sum P_{|f^{-1}(1)} \otimes \ldots \otimes P_{|f^{-1}(n)},$$

where the sum is over all maps $f : P \longrightarrow [n]$ such that if B' is a block of P nested in the block B of P, then

$$f(P) \leqslant f(P').$$

If P is a noncrossing partition, we can write the iterated reduced coproducts of P under the form

$$\tilde{\Delta}^{(n-1)}(P) = \sum P_{|f^{-1}(1)} \otimes \ldots \otimes P_{|f^{-1}(n)},$$

where the sum is over all surjective maps $f : P \longrightarrow [n]$ such that if B' is a block of P nested in the block B of P, then

$$f(P) \leqslant f(P').$$

Moreover, $\epsilon_\delta(P_{f^{-1}(i)}) = 1$ if $P_{|f^{-1}(i)}$ is a monomial of noncrossing partitions reduced to a single block, and 0 otherwise.

If P is a noncrossing partition, we can write the iterated reduced coproducts of P under the form

$$\tilde{\Delta}^{(n-1)}(P) = \sum P_{|f^{-1}(1)} \otimes \ldots \otimes P_{|f^{-1}(n)},$$

where the sum is over all surjective maps $f : P \longrightarrow [n]$ such that if B' is a block of P nested in the block B of P, then

$$f(P) \leqslant f(P').$$

Moreover, $\epsilon_\delta(P_{f^{-1}(i)}) = 1$ if $P_{|f^{-1}(i)}$ is a monomial of noncrossing partitions reduced to a single block, and 0 otherwise.

This gives:

Theorem

For any noncrossing partition P, for any $n \in \mathbb{N}$, $\phi_1(P)(n)$ is the number of maps $f : P \longrightarrow [n]$ such that:

- If B' is nested in B, then $f(B) < f(B')$.
- If $f(B) = f(B')$ and $\max(B) < \min(B')$, then there exists $B'' \in \pi$ such that
$$]\max(B), \min(B')[\cap B'' \neq \emptyset \quad \text{and} \quad f(B'') < f(B) = f(B').$$

Such a map f will be called a valid coloration of P.

The combinatorics of noncrossing partition allows to inductively compute ϕ_1. We denote by $\mathrm{Base}(P)$ the set of blocks of P which are minimal for the nesting.

Proposition

Let π be a noncrossing partition. Then

$$\phi_1(P)(X+1) = \phi_1(P) + \sum_{B \in \mathrm{Base}(P)} \phi_1(P_1^B)\phi_1(P_2^B)\phi_1(P_3^B),$$

with

$$P_1^B = P_{|[1,\min(B)[}, \quad P_2^B = P_{|[\min(B),\max(B)]\setminus B}, \quad P_3^B = P_{|]\max(B),\infty[}$$

Idea of the proof

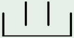
If $f : P \longrightarrow [n]$ is a valid coloration of P, then, either $f^{-1}(1)$ is empty or is obtained on a unique element of $\mathrm{Base}(P)$.

Examples

$\phi_{\mathrm{NCP}}($  $) = \frac{X(X-1)}{2},$

$\phi_{\mathrm{NCP}}($  $) = X(X-1)\left(X-\frac{3}{2}\right),$

$\phi_{\mathrm{NCP}}($  $) = \frac{X(X-1)^2}{2},$

$\phi_{\mathrm{NCP}}($  $) = \frac{X(X-1)(X-2)}{3},$

$\phi_{\mathrm{NCP}}($  $) = X(X-1)(X-2)\left(X-\frac{4}{3}\right).$

For any $x \in \mathcal{H}_{\mathrm{NCP}}$, we put $\mu(x) = \phi_1(x)(-1)$. This defines a character of $\mathcal{H}_{\mathrm{NCP}}$. The antipode of $\mathcal{H}_{\mathrm{NCP}}$ is given by

$$S = (\mu \otimes \mathrm{Id}) \circ \delta.$$

Proposition

For any noncrossing partition P, such that $\mathrm{Base}(\pi) = \pi$,

$$\mu(P) = \mu\left(P_{|\mathrm{Base}(\pi)}\right) \mu\left(P_{|.\pi \setminus \mathrm{Base}(\pi)}\right).$$

It is needed now to compute for noncrossing partitions reduced to their bases. We denote by $\mathrm{Cat}_n = \dfrac{1}{n+1}\dbinom{2n}{n}$ is the n-th Catalan number.

n	0	1	2	3	4	5	6	7	8	9	10
Cat_n	1	1	2	5	14	42	132	429	1430	4862	16796

Note that the number of noncrossing partitions of $[n]$ is Cat_n.

Proposition

Let P be a noncrossing partition with k blocks such that $\mathrm{Base}(P) = P$. Then

$$\mu(P) = (-1)^k \mathrm{Cat}_k.$$

Examples of antipode

$$\begin{aligned}\delta(\mid\;\mid\;\mid) &= \mid\;\mid\;\mid \otimes \mid \cdot \mid \cdot \mid + (\sqcup\;\mid + \mid\;\sqcup + \sqcup\!\!\!\!\mid\;\,) \otimes \mid \cdot \mid\;\mid \\ &\quad + \sqcup\!\sqcup \otimes \mid\;\mid\;\mid,\end{aligned}$$

$$S(\mid\;\mid\;\mid) = -5 \mid \cdot \mid \cdot \mid + 5 \mid \cdot \mid\;\mid - \mid\;\mid\;\mid.$$

We are now interested in the fundamental polynomial invariant of a noncrossing partition P such that $P = \mathrm{Base}(P)$. In this case, $\phi_1(P)$ only depends on the number k of blocks of P. This defines a sequence $(\psi_k)_{k\geqslant 0}$ of polynomials.

Examples

$$\psi_1 = X,$$
$$\psi_2 = X^2 - X,$$
$$\psi_3 = X^3 - \frac{5X^2}{2} + \frac{3X}{2},$$
$$\psi_4 = X^4 - \frac{13X^3}{3} + 6X^2 - \frac{8X}{3},$$
$$\psi_5 = X^5 - \frac{77X^4}{12} + \frac{89X^3}{6} - \frac{175X^2}{12} + \frac{31X}{6},$$
$$\psi_6 = X^6 - \frac{87X^5}{10} + \frac{175X^4}{6} - \frac{281X^3}{6} + \frac{215X^2}{6} - \frac{157X}{15}.$$

Proposition

Let $M = (M_{k,l})_{k,l \geqslant 1}$ be the infinite matrix defined by

$$M_{k,l} = \begin{cases} \binom{l}{k-l} \text{ if } k \geqslant l, \\ 0 \text{ if } k < l. \end{cases}$$

Then

$$\begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \vdots \end{pmatrix} = e^{X \ln(M)} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}.$$

A remark on M

M is the Riordan array of $(1+t, t(1+t))$.

Definition

If $P, Q \in \mathbb{C}[[t]]$, with P of order 0 and Q of order 1, the Riordan array of P is Q is given by

$$M_{k,l} = [t^k](PQ^l).$$

Riordan arrays form a group of infinite triangular matrices.

We develop the polynomial ψ_k on the basis of Hilbert polynomials:

$$\psi_k(X) = \sum_{i=0}^{\infty} a_{i,n} \frac{X(X-1)\ldots(X-i+1)}{i!}.$$

$i \backslash n$	1	2	3	4	5	6	7	8	9	10
1	1	0	0	0	0	0	0	0	0	0
2	0	2	1	0	0	0	0	0	0	0
3	0	0	6	10	8	4	1	0	0	0
4	0	0	0	24	86	172	254	302	298	244
5	0	0	0	0	120	756	2734	7484	17164	34612
6	0	0	0	0	0	720	7092	40148	172168	621348
7	0	0	0	0	0	0	5040	71856	585108	3589360
8	0	0	0	0	0	0	0	40320	787824	8720136
9	0	0	0	0	0	0	0	0	362880	9329760
10	0	0	0	0	0	0	0	0	0	3628800

These coefficients are not so well known. For example, no entry of the OEIS contains the term $172168 = a_{6,9}$.

if $n, k_1, \ldots, k_p \geqslant 1$, we put

$$\zeta_n(k_1, \ldots, k_p) = \sum_{1 \leqslant n_1 < \ldots < n_p \leqslant n} \frac{1}{n_1^{k_1} \ldots n_p^{k_p}}.$$

Proposition

For any $n \geqslant 2$,

$$\forall n \geqslant 1, \quad a_{n,n} = n!,$$

$$\forall n \geqslant 2, \quad a_{n-1,n} = n! \left(\frac{n+1}{2} - \zeta_n(1) \right),$$

$$\forall n \geqslant 3, \quad a_{n-2,n} = n! \left(\zeta_n(1,1) - \frac{n}{2} \zeta_n(1) + \frac{(3n-2)(n^2+n+6)}{24n} \right).$$

References

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Thank you for your attention!