Quasilinear singular SPDEs and paracontrolled calculus

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Joint work with Ismaël Bailleul

Higher structures emerging from renormalisation

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Introduction

Singular SPDEs

Semilinear SPDEs with **irregular noise**

$$\begin{aligned} (\partial_t - \Delta)\, u &= f(u)\xi && \text{on } [0,T] \times \mathbb{T}^3 \quad \text{(gPAM)} \\ \left(\partial_t - \partial_x^2\right) u &= g(u)\zeta + h(u)(\partial_x u)^2 && \text{on } [0,T] \times \mathbb{T} \quad \text{(gKPZ)} \end{aligned}$$

with ξ/ζ noises of parabolic Hölder regularity $\alpha - 2$.

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$\to$ Singular PDEs : **multiplication of distributions.** Given $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$,

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Schauder estimates : u is expected to be α-Hölder

Singular if $\alpha + (\alpha - 2) \leq 0$

Quasilinear singular SPDEs

Quasilinear associated SPDEs

$$\partial_t u - d(u)\Delta u = f(u)\xi \quad \text{on } [0,T] \times \mathbb{T}^3 \quad \text{(QgPAM)}$$

$$\partial_t u - d(u)\partial_x^2 u = g(u)\zeta + h(u)(\partial_x u)^2 \quad \text{on } [0,T] \times \mathbb{T} \quad \text{(QgKPZ)}$$

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Δu of Hölder-regularity $(\alpha - 2)$ $\Longrightarrow$ $d(u)\Delta u$ ill-defined.

- **Otto/Weber** and **Otto/Sauer/Smith/Weber** : rough paths flavoured variant of regularity structures
- **Furlan/Gubinelli** : paracomposition operators
- **Bailleul/Debussche/Hofmanovà** : first order paracontrolled expansion
- **Gerencser/Hairer** : regularity structures

Solving singular SPDEs : analysis

Rough path philosophy : One can multiply ξ with a distribution u that “looks like” Z if one can multiply two distributions Z and ξ.
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Relation between the two approaches : Martin/Perkowski 2018 and Bailleul/Hoshino $2018/2019$.

Solving singular SPDEs : probability

Renormalisation : define the product of two random distributions, for example $Z\xi$ with $Z := (\partial_t - \Delta)^{-1}\xi$. The product is not almost surely well-defined so

$$(Z\xi)(\omega) \neq Z(\omega)\xi(\omega).$$

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$$(Z\xi)(\omega) \neq Z(\omega)\xi(\omega).$$

We consider a regularisation of the noise $\xi_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} \xi$ and add counter-terms to the ill-defined quantity. For example, we have

$$(Z\xi)(\omega) := \lim_{\varepsilon \to 0} \Big(Z_\varepsilon(\omega)\xi_\varepsilon(\omega) - \mathbb{E}[Z_\varepsilon \xi_\varepsilon] \Big).$$

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The renormalisation procedure might seem unnatural since one changes the equation under investigation to solve it. **What is renormalisation ?** A way to deal with:

- divergence of random systems described by singular PDEs on a macroscopic level.
- presence of infinity in Quantum Field Theory with stochastic quantization.

High order paracontrolled calculus

Paraproduct with Fourier analysis

For any distribution $f \in \mathcal{D}'(\mathbb{T}^d)$, we have the Paley-Littlewood decomposition

$$f = \lim_{N \to \infty} S_N f = \sum_{n \geq 0} \Delta_n f$$

where δ_n are projectors on the annulus of frequencies of order 2^n.

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Given two distributions f and g, we have

$$fg = \lim_{N\to\infty} (S_N f \cdot S_N g) = \sum_{n,m\geq 0} \Delta_n f \cdot \Delta_m g = P_f g + P_g f + \Pi(f,g)$$

where $P_f g = \sum_{n<m-1} \Delta_n f \cdot \Delta_m g$ is always well-defined.

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with $Q_t := -t\partial_t P_t$.

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with $Q_t := -t\partial_t P_t$. In the case $M = \mathbb{T}^d$ and $L = -\Delta$, we have

$$\widehat{P_t}(\lambda) = e^{-t|\lambda|^2} \quad \text{and} \quad \widehat{Q_t}(\lambda) = t|\lambda|^2 e^{-t|\lambda|^2}$$

hence $\widehat{P_t}$ is approximately localised in a ball $|\lambda| \lesssim t^{-\frac{1}{2}}$ and $\widehat{Q_t}$ in an annulus $\lambda \simeq t^{-\frac{1}{2}}$.

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Paraproduct with heat analysis

Given two distributions f and g, we have

$$\begin{aligned} fg &= \lim_{t \to 0} P_t \left(P_t f \cdot P_t g \right) \\ &= \int_0^1 \left\{ Q_t \left(P_t f \cdot P_t g \right) + P_t \left(Q_t f \cdot P_t g \right) + P_t \left(P_t f \cdot Q_t g \right) \right\} \frac{\mathrm{d}t}{t} \\ &= \mathsf{P}_f g + \mathsf{P}_g f + \mathsf{\Pi}(f, g). \end{aligned}$$

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with

$$\begin{aligned} \mathsf{P}_f g: &\quad \int_0^1 Q_t\Big(P_t f \cdot Q_t g\Big)\frac{\mathrm{d}t}{t}, \\ \Pi(f,g): &\quad \int_0^1 P_t\Big(Q_t f \cdot Q_t g\Big)\frac{\mathrm{d}t}{t}. \end{aligned}$$

(Para)product and Hölder spaces

The product of two distributions is well-defined as soon as the sum of their regularity is large enough.

Theorem

Let $\alpha < 0 < \beta$ *such that* $\alpha + \beta > 0$. *Then the multiplication* $(f, g) \mapsto fg$ *extends from smooth function into a continuous bilinear operators from* $\mathcal{C}^\alpha \times \mathcal{C}^\beta$ *to* $\mathcal{C}^\alpha$.

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More precisely, we have

$$fg = \mathsf{P}_f g + \Pi(f, g) + \mathsf{P}_g f = (\alpha + \beta) + (\alpha + \beta) + (\alpha)$$

and the condition $\alpha + \beta > 0$ is necessary **only for the resonant term** $\Pi(f, g)$.

First order paracontrolled expansion

Proposition (Bony paralinearisation)

Let $f : \mathbb{R} \to \mathbb{R}$ *be a* C_b^2 *function and* $u \in \mathcal{C}^\alpha$ *with* $0 < \alpha < 1$*. Then*

$$f(u) = \mathsf{P}_{f'(u)} u + f(u)_2^\sharp$$

for some remainder $f(u)_2^\sharp$ *of Hölder regularity* 2α*.*

Higher order paracontrolled expansion

Proposition

Let $f : \mathbb{R} \to \mathbb{R}$ be a C_b^4 function and $u \in \mathcal{C}^\alpha$ with $0 < \alpha < 1$. Then

$$f(u) = \mathsf{P}_{f'(u)} u + \frac{1}{2!} \left\{ \mathsf{P}_{f^{(2)}(u)} u^2 - 2\mathsf{P}_{f^{(2)}(u)u} u \right\} + \frac{1}{3!} \left\{ \mathsf{P}_{f^{(3)}(u)} u^3 - 3\mathsf{P}_{f^{(3)}(u)u} u^2 + 3\mathsf{P}_{f^{(3)}(u)u^2} u \right\} + f(u)^\sharp$$

for some remainder $f(u)^\sharp \in \mathcal{C}^{4\alpha}$.

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We introduce the corrector from [GIP]

$$\mathsf{C}(u_1, Z, \xi) := \mathsf{\Pi}(\mathsf{P}_{u_1} Z, \xi) - u_1 \mathsf{\Pi}(Z, \xi)$$

so one has $\mathsf{\Pi}(u, \xi) = u_1 \mathsf{\Pi}(Z, \xi) + \mathsf{C}(u_1, Z, \xi) + \mathsf{\Pi}(u^\sharp, \xi)$.

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Proposition

Let $\alpha \in (0, 1)$ and $\beta, \gamma \in \mathbb{R}$ and assume that $0 < \alpha + \beta + \gamma < 1$ and $\beta + \gamma < 0$. Then the corrector C *extends continuously from $\mathcal{C}^\alpha \times \mathcal{C}^\beta \times \mathcal{C}^\gamma$ to $\mathcal{C}^{\alpha+\beta+\gamma}$.*

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$\rightarrow$ More correctors/commutators to deal with more general equations.

Semilinear singular SPDEs

(PAM) and intertwined paraproducts

Let u be a solution of (PAM) : $\mathscr{L} u = u\xi$
with $\mathscr{L} := \partial_t - \Delta$.

(PAM) and intertwined paraproducts

Let u be a solution of (PAM) : $\mathscr{L}u = u\xi = \mathsf{P}_u\xi + \Pi(u,\xi) + \mathsf{P}_\xi u$ with $\mathscr{L} := \partial_t - \Delta$.

(PAM) and intertwined paraproducts

Let u be a solution of (PAM) : $\mathscr{L}u = \mathsf{P}_u\xi + (2\alpha - 2)$
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Let u be a solution of (PAM) : $\mathscr{L}u = \mathsf{P}_u\xi + (2\alpha - 2)$
with $\mathscr{L} := \partial_t - \Delta$.
Using Schauder estimates, we have

$$\begin{aligned} u &= \mathscr{L}^{-1}\mathsf{P}_u\xi + \mathscr{L}^{-1}(2\alpha - 2) \\ &= \widetilde{\mathsf{P}}_u(\mathscr{L}^{-1}\xi) + (2\alpha) \end{aligned}$$

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and the remainder is

$$\begin{aligned} (2\alpha - 2) &= \Pi(u, \xi) + \mathsf{P}_\xi u \\ &= u\Pi(\mathscr{L}^{-1}\xi, \xi) + \mathsf{C}(u, \mathscr{L}^{-1}\xi, \xi) + \mathsf{P}_\xi u \end{aligned}$$

Well-defined for $\alpha + \alpha + (\alpha - 2) > 0$ if $\Pi(\mathscr{L}^{-1}\xi, \xi)$ is given.

Second order paracontrolled expansion

Consider

$$u = \widetilde{\mathsf{P}}_{u_1} Z_1 + \widetilde{\mathsf{P}}_{u_2} Z_2 + u^\sharp$$

where $Z_i \in \mathcal{C}^{i\alpha}$ and $u^\sharp \in \mathcal{C}^{3\alpha}$.

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$$\begin{aligned} \Pi(u,\xi) &= \Pi(\widetilde{\mathsf{P}}_{u_1} Z_1, \xi) + \Pi(\widetilde{\mathsf{P}}_{u_2} Z_2, \xi) + \Pi(u^\sharp, \xi) \\ &= u_1 \cdot \Pi(Z_1, \xi) + \mathsf{C}(u_1, Z_1, \xi) \\ &\quad + u_2 \Pi(Z_2, \xi) + \mathsf{C}(u_2, Z_2, \xi) \\ &\quad + \Pi(u^\sharp, \xi) \end{aligned}$$

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$\rightarrow$ need paracontrolled expansion for the u_i at an order depending on i.

Paracontrolled system

We work with **paracontrolled system** $\widehat{u} = (u_a)_{a \in \mathscr{A}}$

$$u_a = \sum_{|a|+|i| \leq n} \widetilde{\mathsf{P}}_{u_{ai}} Z_i + u_a^\sharp.$$

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For (gPAM) in dimension 3 and (gKPZ) in dimension $1+1$, we work with

$$\begin{aligned} u &= \sum_{|i|\le 3} \widetilde{\mathsf{P}}_{u_i} Z_i + u^\sharp, \\ u_i &= \sum_{|i|+|j|\le 3} \widetilde{\mathsf{P}}_{u_{ij}} Z_j + u_i^\sharp, \\ u_{ij} &= \sum_{|i|+|j|+|k|\le 3} \widetilde{\mathsf{P}}_{u_{ijk}} Z_k + u_{ij}^\sharp, \\ u_{ijk} &= u_{ijk}^\sharp. \end{aligned}$$

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- Using paracontrolled expansion and continuity results, we write the **right hand side** as

$$f(u,\xi) = \sum_{i=1}^{n} \mathsf{P}_{v_i} Y_i + v^\sharp$$

with Y_i depending on the noise $\widehat{\xi} := (\xi, Z_1, \ldots, Z_n)$ and v_i on $\widehat{u}$.

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with Y_i depending on the noise $\widehat{\xi} := (\xi, Z_1, \dots, Z_n)$ and v_i on $\widehat{u}$.
- Perform a **fixed point**

$$\begin{aligned} u &= \mathscr{P} u_0 + \mathscr{L}^{-1}\left(\sum_{i=1}^{n} \mathsf{P}_{v_i} Y_i\right) + \mathscr{L}^{-1} v^\sharp \\ &= \mathscr{P} u_0 + \sum_{i=1}^{n} \widetilde{\mathsf{P}}_{v_i}\left(\mathscr{L}^{-1} Y_i\right) + \mathscr{L}^{-1} v^\sharp \end{aligned}$$

such that we define a stable solution space.

Algebra

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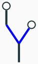
As in regularity structures, recursive definition of the object hence trees are natural to consider.

Not as regularity structures, these would be planar trees with different decorations. For example, we could set

$$\mathsf{P}_{\xi}(\mathscr{L}^{-1}\xi) =:$$  and $$\Pi(\mathscr{L}^{-1}\xi, \xi) =:$$  .

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To be closer to the semilinear setting, we rewrite the equation as

$$\partial_t u - d(u_0)\Delta u = f(u,\xi) + \Big(d(u) - d(u_0)\Big)\Delta u$$

with u_0 a smooth enough initial condition.

$\to d(u) - d(u_0)$ is expected to be small for small horizon time.

Quasilinear singular SPDEs

For technical reasons, we work with the elliptic operator

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with $V_\ell := \sqrt{d(u_0)}\partial_\ell$.

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with $V_\ell := \sqrt{d(u_0)}\partial_\ell$.The equation

$$\partial_t u - d(u_0)\Delta u = f(u,\xi) + \Big(d(u) - d(u_0)\Big)\Delta u$$

rewrites as

$$\partial_t u + Lu = f(u,\xi) + \varepsilon(u,\cdot)Lu + d_\ell(u,\cdot)V_\ell u$$

with $\varepsilon(u,\cdot)$ expected small for small horizon time.

Quasilinear singular SPDEs

We consider the **paraproducts associated** to L to solve

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Given a paracontrolled system $\widehat{u}$, we want to get a paracontrolled expression

$$\varepsilon(u,\cdot)Lu + d_\ell(u,\cdot)V_\ell u = \sum_{i=1}^{n} \mathsf{P}_{v_i} Y_i + v^\sharp$$

to build a stable solution space and perform the fixed point.

Quasilinear (PAM)

Let u be a solution of (QPAM). Then we have

$$\begin{aligned}\mathscr{L}u &= u\xi + \varepsilon(u)Lu + d_\ell(u)V_\ell u \\ &= \mathsf{P}_u\xi + \mathsf{P}_{\varepsilon(u)}Lu + (2\alpha - 2).\end{aligned}$$

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The space of functions paracontrolled by Z is **not stable**.

Quasilinear (PAM)

Larger expansion : $u = \widetilde{\mathsf{P}}_{u_1} Z + \widetilde{\mathsf{P}}_{u_2} (\mathscr{L}^{-1} L) Z + (2\alpha)$

The equation rewrites

$$\mathscr{L} u = \mathsf{P}_u \xi + \mathsf{P}_{\varepsilon(u) u_1} L Z + \mathsf{P}_{\varepsilon(u) u_2} L(\mathscr{L}^{-1} L) Z + (2\alpha - 2)'.$$

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If u is paracontrolled by a reference function Z, then u is also paracontrolled by $(\mathscr{L}^{-1} L) Z$.

(QPAM) in dimension 2

We look for a solution u of the form

$$u = \sum_{i=0}^{\infty} \widetilde{\mathsf{P}}_{u_i} Z_i + u^\sharp$$

with $Z_i := (\mathscr{L}^{-1} L)^i Z$ and $u^\sharp \in \mathcal{C}^{2\alpha}$ with some condition of convergence on $(u_i, Z_i)_i$.

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$$\begin{aligned} &\widetilde{\mathsf{P}}_{u_0} Z_0 + \widetilde{\mathsf{P}}_{u_1} Z_1 + \widetilde{\mathsf{P}}_{u_2} Z_2 + \dots \\ &= \widetilde{\mathsf{P}}_{u} Z_0 + \widetilde{\mathsf{P}}_{\varepsilon(u) u_0} (\mathscr{L}^{-1} L) Z_0 + \widetilde{\mathsf{P}}_{\varepsilon(u) u_1} (\mathscr{L}^{-1} L) Z_1 + \dots \end{aligned}$$

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The fixed point equation gives

$$u_0 = u \quad \text{and} \quad u_{i+1} = \varepsilon(u) u_i \quad \text{for } i \geq 0$$

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Finally, u is well-defined and α-Hölder since

$$\begin{aligned} \|\sum_{i\geq 0} \widetilde{\mathsf{P}}_{u_i} Z_i\|_{\mathcal{C}^\alpha} &\lesssim \sum_{i\geq 0} \|\varepsilon(u)^i u\|_{L^\infty} \|(\mathscr{L}^{-1}L)^i Z\|_{\mathcal{C}^\alpha} \\ &\lesssim \|u\|_{L^\infty} \|Z\|_{\mathcal{C}^\alpha} \sum_{i\geq 0} \Big(\|\varepsilon(u)\|_{L^\infty} \|\mathscr{L}^{-1}L\|_{\mathcal{C}^\alpha \to \mathcal{C}^\alpha} \Big)^i \end{aligned}$$

which is convergent for a small enough horizon time.

(QgPAM) in dimension 3

Same as (gPAM) but with the set of reference functions stable by $\mathscr{L}^{-1}L$. For example, (gPAM) in dimension 3 needs a term

$$\mathsf{C}(Z, Z, \xi)$$

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$\rightarrow$ only a **finite numbers of 'model terms'** hence we get a contraction for a small enough horizon time.

Algebra

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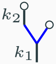

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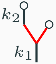
$$(\mathscr{L}^{-1}L)^{k_1}\Pi((\mathscr{L}^{-1}L)^{k_1}Z, \xi) =:$$



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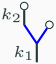

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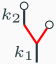

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This imposes a condition of growth with respect to this parameter in the renormalisation.

Thank you for your attention!