∞-OPERADS: AN EXAMPLE

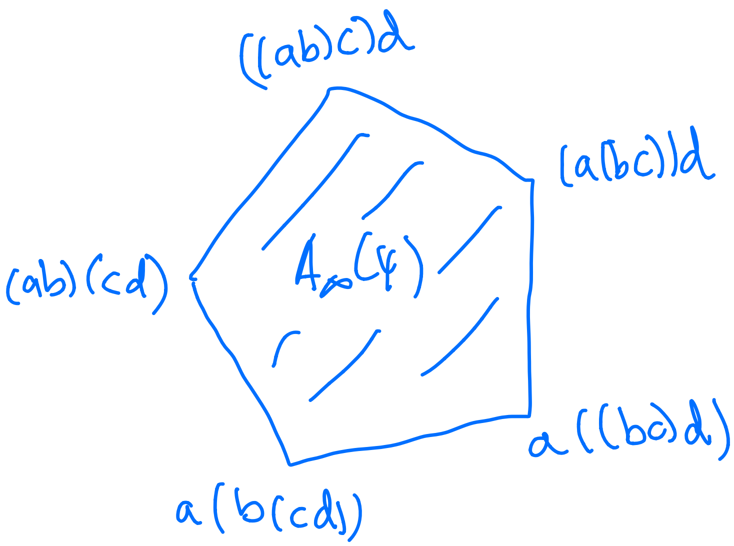
OR "HIGHER STRUCTURES OF HIGHER STRUCTURES"

j^t LUCIANA BONATTO, SAFIA CHETTIH, ABIGAIL LINTON,
SOPHIE RAYNOR & MARCY ROBERTSON

A_∞-ALG = ALGEBRA ASSOCIATIVE UP HTPY

ab $\qquad\qquad$ $(ab)c \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet\, a(bc)$

$A_\infty(2) = \{*\}$ $\qquad\qquad$ $A_\infty(3) = $ ——



ETC

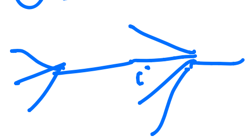
ENCODED BY AN OPERAD $A_\infty = \{A_\infty(n)\}_{n\geq 0}$

AN OPERAD $\mathcal{O} = \{\mathcal{O}(n)\}_{n\geq 0}$ + COMPOSITION LAWS

$\Sigma_n \circlearrowleft$

$\mathcal{O}(n) \times \mathcal{O}(m) \xrightarrow{\circ_i} \mathcal{O}(n+m-1)$

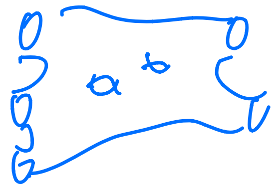
ASSOCIATIVE



∞-<u>OPERAD</u>: THESE ARE ONLY "A_∞-ASSOCIATIVE"

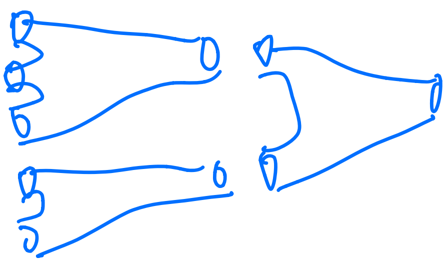
<u>QUESTION</u>: WHAT DOES THIS MEAN IN PRACTICE?

<u>MOTIVATION</u>: METRIC GRAPH MODEL OF THE COBORD. CAT $\mathcal{C}ob_2$



WHOSE COMPOSITION IS ONLY ASSOC. UP TO HTPY.

RESTRICT TO GENUS 0 "OPERADIC PART"

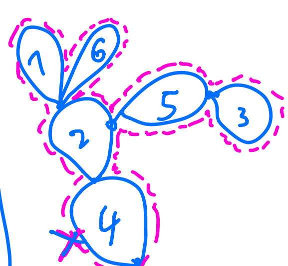


THIS IDENTIFIES WITH "<u>NORMALIZED</u> CACTI"

$\Uparrow$ $\boxed{\infty\text{-OPERAD}}$

GOAL TODAY

<u>CACTUS OPERAD</u> $\quad \mathcal{C}act = \{\mathcal{C}act(n)\}_{n \geq 0}$

$\mathcal{C}act(n) = \Bigg\{$  $\Bigg\}$

MODEL FOR fE_2

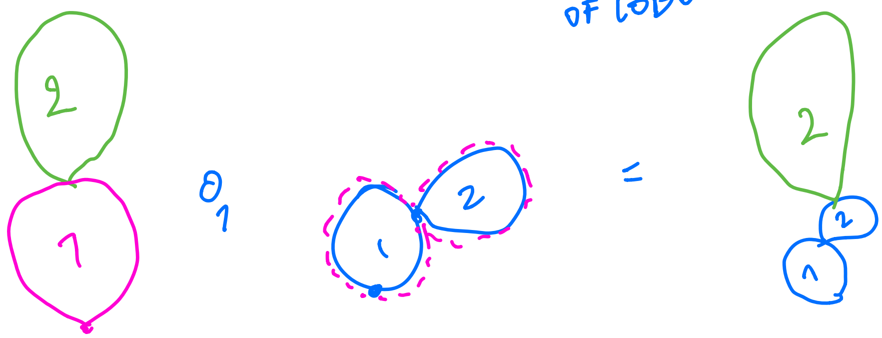
$H_*(\mathcal{C}act) = BV$

$\cup|$

$\mathcal{C}act^1(n)$ = SUBSPACE OF CACTI WITH LOBES OF LENGTH 1 EACH

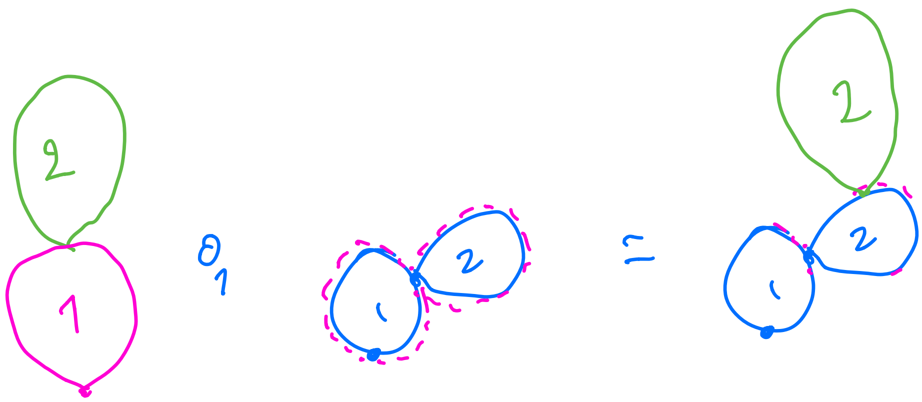
COMPOSITION = SCALE + INSERT
TO SIZE OF LOBE

(1)



DOES NOT RESPECT $\mathcal{C}act^1$

(2)



SCALE THE LOBE TO THE SIZE OF INSERTED CACTUS

$\rightsquigarrow$ DEFINES A COMP ON $\mathcal{C}act^1$

<u>FACT</u>: (1) IS ASSOCIATIVE

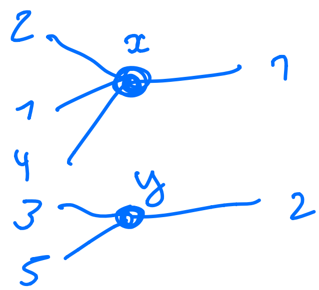
(2) IS NOT. [KAUFMANN].

IS IT "A_∞"? CAN ONE USE IT TO DEFINE AN ∞-OPERAD?

WHAT IS AN ∞-OPERAD?

<u>TODAY</u>: INNER FIBRATION $\mathcal{O} \xrightarrow{p} N\mathrm{Fin}_*$ WITH COCARTESIAN LIFTS FOR ALL INERT MORPHISMS s.t. $\mathcal{O}\langle n\rangle \xrightarrow{\simeq} \mathcal{O}\langle 1\rangle^n$ AND $\mathrm{Mor}_{\mathcal{O}}(n,m) \xrightarrow{\simeq} \prod_{i\in[m]} \mathrm{Mor}_{\mathcal{O}}(p^{-1}[i],1)$

IS: $\mathcal{O}$ ∞-CATEGORY THAT WORKS LIKE THE PROP ASSOCIATED $\mathcal{O}$



<u>RULE #1</u> OF ∞-CATEGORIES: NEVER TRY TO CONSTRUCT ONE FROM SCRATCH!

$\Downarrow$ (DENDROIDAL SPACES)

<u>50 YEARS AGO</u>: W-CONSTRUCTION (BOARDMAN-VOGT)

$\underline{P} = \mathrm{Ass}$ OPERAD OF ASS ALG $\rightsquigarrow$ $\underline{WP} \simeq A_\infty$

$\mathcal{P} = \mathcal{O}$ = OPERAD OF OPERADS

= COLORED OPERAD WHOSE ALGEBRAS ARE SYM. OPERADS.

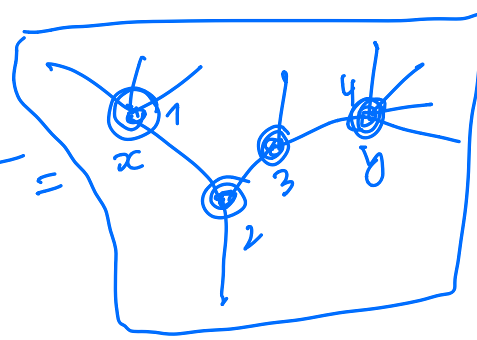
$W\mathcal{P} = W\mathcal{O}$ = OPERAD WHOSE ALG ARE OPERADS UP TO HTPY

BY [BERGER–CISINSKI–MOERDIJK ...]

$W\mathcal{O}$-ALG ARE ∞-OPERADS IN TODAY'S SENSE

PROBLEM: $W\mathcal{O}$ IS COMPLICATED!

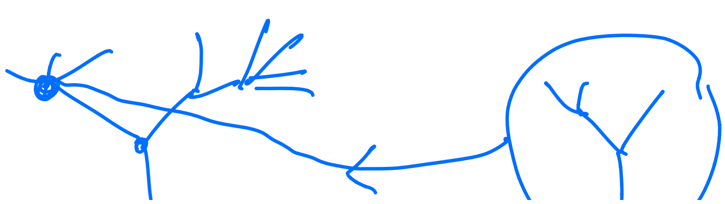
$\mathcal{O}$ IS AN $\mathbb{N}$-COLORED OPERAD WITH ELEMENTS GIVEN BY TREES

$T =$ 

$\in \mathcal{O}(3,2,2,4)$

$\mathcal{P} = \{\mathcal{P}(n)\}_{n \geq 0}$ $\quad x \in \mathcal{P}(3),\ y \in \mathcal{P}(4), \ldots$

COMPOSITION IN $\mathcal{O}$ IS BY TREE INSERTION.



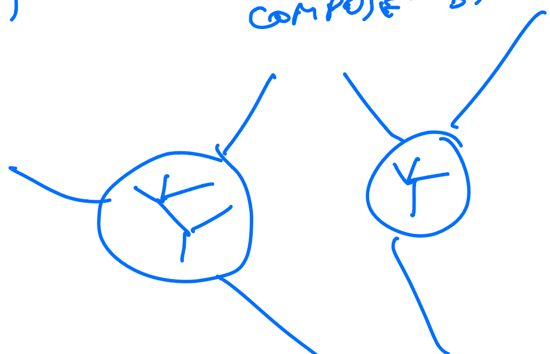
$W\mathcal{O}$ = TREES OF TREES

(FROM W-CONST)

COMPOSE BY GRAFTING

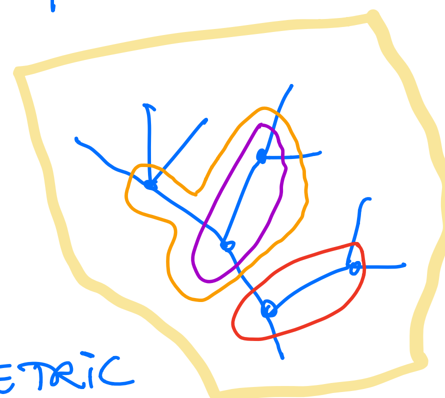
(FROM $\mathcal{O}$)

COMPOSE BY INSERTION



THM [BCLRRW] THERE IS AN OPERAD $B\mathcal{O}$

WITH $B\mathcal{O}(k_1,\dots,k_n) = \coprod_{T \in \mathcal{O}(k_1,\dots,k_n)} N_\bullet\left(\text{POSET OF BRACKETINGS IN } T\right)$

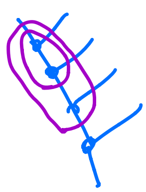


WITH OPERAD MAPS

$$W\mathcal{O} \xrightarrow{\sim} B\mathcal{O} \xrightarrow{\sim} \mathcal{O}$$

AND

- A $B\mathcal{O}$-ALG = A SYMMETRIC SEQUENCE $\{P(n)\}_{n\geq 0}$ $\circlearrowleft \Sigma_n$ + A_∞-COMPOSITION
- $\{Cact^1(n)\}_{n\geq 0}$ $\Sigma_n \circlearrowleft$ + $CACT^1$-COMP EXTENDS TO $B\mathcal{O}$-ALG STRUCTURE.



REDUCE IT TO DEFINING COMPOSITIONS VIA ~~...~~ PARAMETERS IN $\mathrm{Mon}^+(I, \partial I)$

CONVEX

REPARAMETRIZATION OF S^1