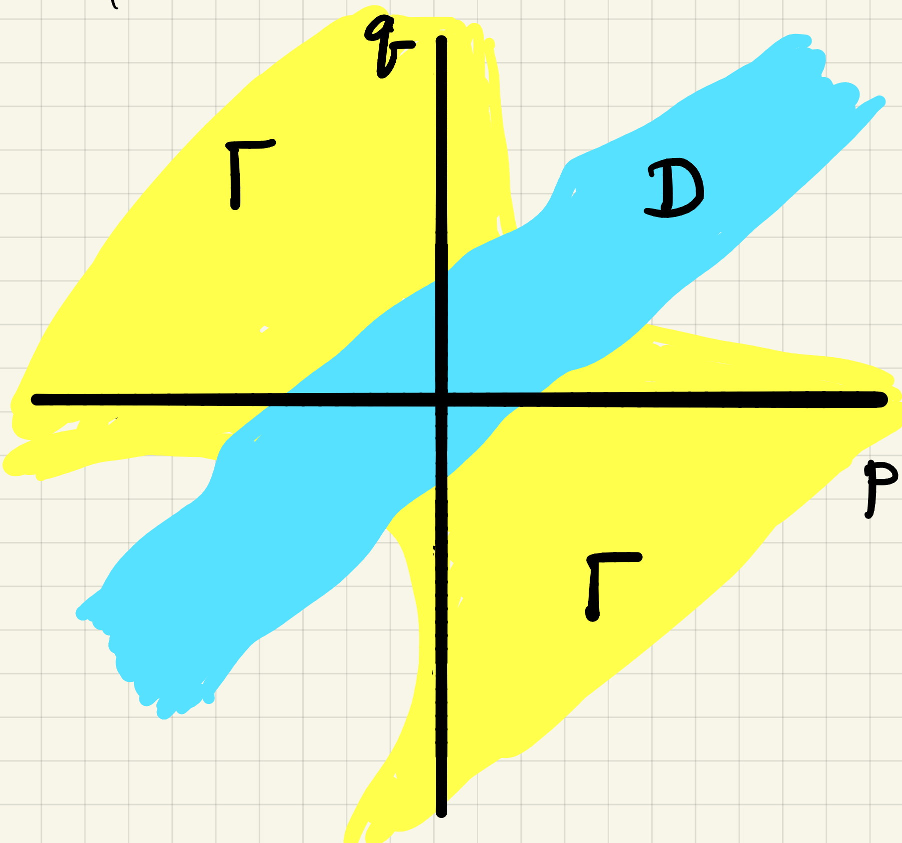
**Def** $\Gamma \subset \Lambda$ is called co-controlled if $\forall$ controlled $D \subset \Lambda^n$ $\Gamma \cap D$ is finite




$(p,q) \in \Lambda \times \Lambda$

**Def.** A coarse $n$-cochain (with values in $V$) is a function $\alpha : \underbrace{\Lambda \times \ldots \times \Lambda}_{n+1 \text{ times}} \to V$

which is

- skew-symmetric
- supported on a co-controlled subset.

Let $C^n(\Lambda; V)$ be the space of coarse $n$-cochains.

$C^n(\Lambda; V^*)$ & $C_n(\Lambda; V)$ are naturally dual:

$$\langle A, \alpha \rangle = \frac{1}{(n+1)!} \sum_{p_0 \cdots p_n} A_{p_0 \dots p_n}(\alpha(p_0, \dots, p_n))$$

$\delta : C^n(\Lambda; V) \to C^{n+1}(\Lambda; V)$ is the adjoint of $\partial$:

$$\langle \partial A, \beta \rangle = \langle A, \delta\beta \rangle$$

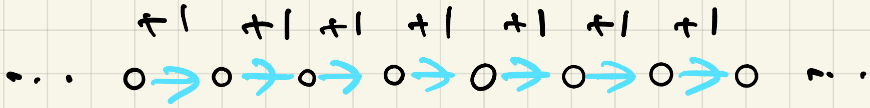
<u>Def</u>. The cohomology of $(C^\bullet(\Lambda; V), \delta)$ is called the coarse cohomology of $\Lambda$.

If $\Lambda$ is uniformly filling, then

$$H^n(\Lambda; V) = H^n_c(\mathbb{R}^d; V) = \begin{cases} V, & \text{if } n = d \\ 0, & \text{if } n \neq d \end{cases}$$

## Ex. 1

$\Lambda = \mathbb{Z} \subset \mathbb{R}$ $\qquad H_1(\Lambda, \mathbb{R}) \cong \mathbb{R}$




$H^1(\Lambda; \mathbb{R}) = \mathbb{R}$ $\qquad$ Basis: $\alpha(p,q) = f(q) - f(p)$,

where $f(p) = 0$ for $p \gg 0$

$\dot{f}(p) = 0$ if $p \gg 0$

E.g. take $f(p) = \theta(p - \frac{1}{2})$

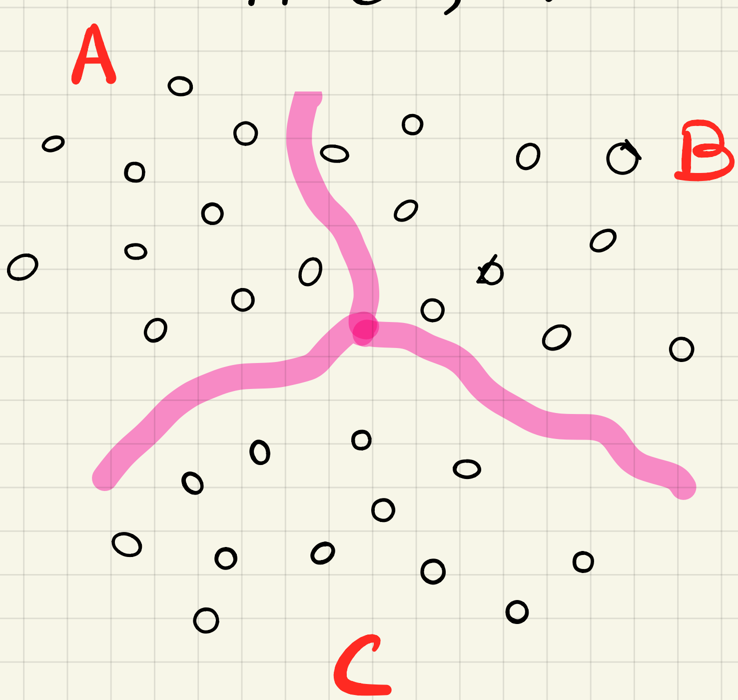
step-function

Then $\langle A, \alpha \rangle = \sum_{\substack{p < \frac{1}{2} \\ q > \frac{1}{2}}} A_{pq}$

## Ex 2

$\Lambda \subset \mathbb{R}^2$

$H^2(\Lambda, \mathbb{R}) = \mathbb{R}$




$\alpha(p,q,r) = \pm 1$

if $p, q, r$ lie in three different "sectors", zero otherwise

9. WZW forms for $d \geq 1$ and coarse geometry.

$$\Omega^{(3)}(f) = \frac{1}{2} \sum_{p,q} F^{(3)}_{pq} \left(f(q) - f(p)\right) =$$
$$= \langle F^{(3)}, \alpha \rangle$$
$$\alpha(p,q) = f(q) - f(p).$$

Technical problem:

$F^{(3)}$ is not a coarse 1-chain.

Generalization (B. Yang):

<u>Def</u>. $A : \Lambda^{n+1} \to V$ is a co-controlled-summable $n$-chain if

- $A$ is skew-symmetric
- $\forall$ co-controlled $D \subset \Lambda^{n+1}$
$$\sum_{x \in D} \| A(x) \| < \infty$$

(it is assumed that $V$ is a Banach space)

$\partial$ is still well-defined.

Dual objects are coarse co-chains which are bounded functions on $\Lambda^{n+1}$.

Luckily, for $\Lambda = \mathbb{Z} \subset \mathbb{R}$ the 1-cocycle $\alpha(p,q)$ is bounded $\Rightarrow$ $\langle F^{(3)}, \alpha \rangle$ is a well-defined 3-form on $M$.

More generally, I believe for a uniformly filling $\Lambda \subset \mathbb{R}^d$ the map

$$H^n_{\text{bounded}}(\Lambda; \mathbb{R}) \to H^n(\Lambda; \mathbb{R})$$

is an isomorphism $\forall n$.

And the map

$$H_n(\Lambda; V) \to H_n^{\substack{\text{co. contr.} \\ \text{summable}}}(\Lambda; V)$$ is

also an isomorphism.

Now we can construct a closed $(d+2)$-form $\Omega^{(d+2)}$ on the parameter space $M$ as follows.

Recall the "descent equations":

$$dF^{(2)}_q = \sum_p F^{(3)}_{pq}, \quad dF^{(3)}_{qr} = \sum_p F^{(4)}_{pqr}, \ldots$$

In our new notation,

$F^{(n+2)}_{p_0 \ldots p_n}$ is a co-controlled-summable $n$-chain with values in $(n+2)$-forms on $M$.

Bi-grading:
- form degree
- chain degree

Two differentials: $d$ & $\partial$

The descent equations are

$$dF^{(n+2)} = \partial F^{(n+3)}, \quad n = 0, 1, \ldots$$

(Kitaev, unpublished)

Let $\alpha \in C^d(\Lambda; \mathbb{R})$ be a (bounded) coarse $d$-cocycle.

Let

$$\Omega^{(d+2)}(\alpha) = \langle F^{(d+2)}, \alpha \rangle.$$

Then:

$$d\Omega^{(d+2)}(\alpha) = \langle dF^{(d+2)}, \alpha \rangle = \langle F^{(d+3)}, \delta\alpha \rangle$$
$$= 0.$$

Also:

$$\Omega^{d+2}(\alpha + \delta\beta) - \Omega^{d+2}(\alpha) = \langle F^{(d+2)}, \delta\beta \rangle$$
$$= \langle \partial F^{(d+2)}, \beta \rangle = d \langle F^{(d+1)}, \beta \rangle$$

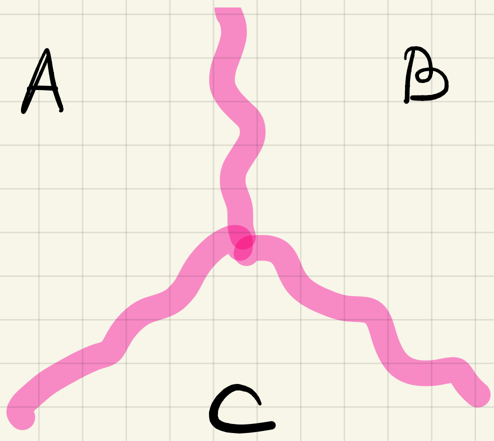
(where $\beta \in C^{d-1}(\Lambda, \mathbb{R})$)

Thus $[\Omega^{d+2}(\alpha)]$ is independent of $\alpha$.

See books by John Roe.

Kitaev, talk at Dan Freed's birthday conference.

Example $d=2$.




$$\Omega^{(4)} \sim \sum_{\substack{p\in A \\ q\in B \\ r\in C}} F^{(4)}_{pqr}$$

- Are periods of $\Omega^{(d+2)}$ quantized?

  Not in general, but for SRE phases expect quantization.

- If so, is there a $d$-gerbe associated to a family of $d$-dimensional SRE systems?

  ( 0-gerbe = line bundle )

  1-gerbe = gerbe

  2-gerbe = ?

## 10. Examples of topologically non-trivial families of gapped systems.

① Free fermions on a 1d lattice

$$H = \sum_{j,i\in\mathbb{Z}} a_i^+ h^i_j a^j \quad , \quad h^i_j = 0 \text{ if } |i-j|>N$$

Specialize ~~to~~ $h^i_j = h(i-j)$ (translationally-invariant case).

Fourier transform:

$$H(\lambda) = \int \frac{dp}{2\pi} a^+(p) a(p) \tilde{h}(p,\lambda).$$

Can choose $\tilde{h}(p,\lambda)$ so that there is a gap in the spectrum $\forall p \in S^1, \forall \lambda \in M$.

Say, $0 \in$ gap.

Then projector to negative eigenvalues defines a bundle $E_-$ over $S^1 \times M$

<u>Proposition</u> (Spodyneiko + A.K.)

$$[\Omega^{(3)}] \sim \int_{S^1} ch(E_-) \sim \int_{S^1} Tr\left(\frac{F}{2\pi} \wedge \frac{F}{2\pi}\right)$$

$F$ = curvature of the Berry-Bloch connection

② Continuum model of free fermions in 1+1 d.

Two complex fermions $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$

$$\mathcal{L} = -\bar{\Psi} i \partial\!\!\!/ \Psi - i \bar{\Psi} (M^0 + i \gamma^{01} \vec{M} \cdot \vec{\sigma}) \Psi$$

$(M^0, \vec{M}) \in S^3 \subset \mathbb{R}^4$.

( Abanov-Wiegmann model)

$$S_{eff} = \int_{\mathbb{R}^2} \phi^* \omega^{(2)} + \ldots$$

$$\Omega^{(3)} = d\omega^{(2)} = \frac{1}{6\pi} \epsilon^{abcd} M^a dM^b dM^c dM^d .$$

N.B. $\int_{S^3} \Omega^{(3)} \neq 0 \Rightarrow$ gapless point at $M^a = 0$ is robust w.r.to arbitrary interactions.

"Diabolical point" in 1+1 d.

See 2004.10758 ( Hsin + Thorngren + AK)