Symmetries and Reduction of Multisymplectic Manifolds

Higher Structures and Field Theory

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Esi, Vienna, August 2022

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of Multi - Symplectic manifolds

Based on:

Blacker, M, Ryvkin; **Reduction of L_∞-algebra of observables on multisymplectic manifolds**; *arXiv:2206.03137*

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Geometric structure providing a prescription on how to measure the area of 2-dimensional surface elements embedded in the manifold.

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Group of transformations preserving the prescribed geometric structures.

Procedure providing another manifold of reduced dimension encoding the relevant geometrical information.

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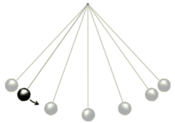

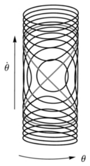

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(M, ω)

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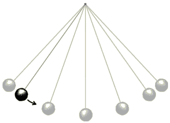

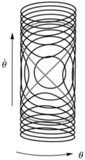

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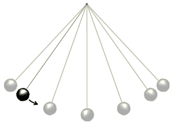

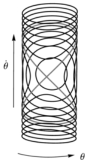

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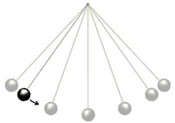

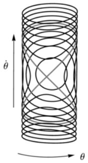

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Unital, associative, commutative algebra $C^\infty(M)$.

Symplectic geometry (mechanics flavour)

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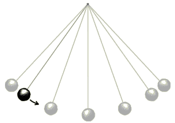

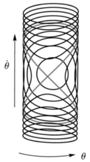

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$\upsilon_f \in \mathfrak{X}(M)$ such that:

$$\iota_{\upsilon_f}\omega = -df$$

υ_f = Ham.v.f. pertaining to $f \in C^\infty(M)$.

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$C^\infty(M)$ is a Poisson algebra with

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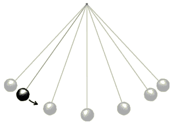

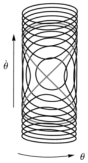

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based on the notion of "measurable quantities".

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Historical motivation

Mechanics: geometrical foundations of *(first-order)* field theories.

mechanics	geometry	
phase space	symplectic manifold	
classical observables	Poisson algebra	
symmetries	group actions admitting comoment map	

$\underbrace{\hphantom{xxxxxxxxxxxxxxxx}}_{\text{point-like particles systems}}$

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phase space	symplectic manifold	multisymplectic manifold
classical observables	Poisson algebra	L_∞-algebra
symmetries	group actions admitting comoment map	group actions admitting (homotopy) comomentum map
	point-like particles systems	field-theoretic systems

1 Introduction

2 Multisymplectic Geometry
- Multisymplectic manifolds
- Observability
- L_∞-algebra of Observables
- Leibniz-algebra of Observables

Review the basics of multisym. geometry.

3 Momentum maps and regular reduction
- Regular reduction in symplectic geometry
- Regular reduction in multisymplectic geometry

Discuss the regular reduction scheme in multisym. geometry.

4 Algebraic singular reduction
- Symplectic singular reduction
- Multisymplectic singular reduction

Discuss the singular reduction scheme in multisym. geometry.

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Def: Non-degenerate $(n+1)$-form

The $\omega^\flat$ (flat) bundle map is injective.

$$\begin{aligned} \omega^\flat \colon TM &\longrightarrow \Lambda^n T^* M \\ (x, u) &\longmapsto (x, \iota_u \omega_x) \end{aligned}$$

Examples:

- $n = 1$ $\Rightarrow$ ω is a symplectic form
- $n = (dim(M) - 1)$ $\Rightarrow$ ω is a volume form
- Let Q a smooth manifold, the multicotangent bundle $\Lambda^n T^* Q$ is naturally n-plectic. *(cfr, GIMMSY construction for classical field theories)*

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Def: Hamiltonian (n-1)−forms

$$\Omega^{n-1}_{ham} := \left\{ H \in \Omega^{n-1} \ \middle| \ \begin{array}{c} \exists X \in \mathfrak{X}_{ham} \\ : \ dH = -\iota_X \omega \end{array} \right\}$$

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Global symmetries

Def: Multisym. (Lie group) action

Smooth action $\theta : G \curvearrowright (M, \omega)$ s.t.

$$(\Phi_g)_* \omega = \omega \qquad \forall g \in G \ .$$

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(aside)

Hierarchy of conserved quantities

$\alpha \in \Omega^\bullet$ strictly conserved / globally conserved / locally conserved along $X \in \mathfrak{X}$ $\Leftrightarrow$

	strictly conserved			$\mathscr{L}_X\alpha = 0$
$\alpha \in \Omega^\bullet$	globally conserved	along $X \in \mathfrak{X}$	$\Leftrightarrow$	$\mathscr{L}_X\alpha \in B$ (exact)
	locally conserved			$\mathscr{L}_X\alpha \in Z$ (closed)

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Let be (M,ω) a n-plectic manifold.

Def: L_∞-algebra of observables *(Rogers)* [Rog12]

$L\infty(M,\omega)$ is given by:

- a cochain-complex $(L,\{\cdot\}_1)$

$$
\begin{array}{ccccccccccccc}
0 \longrightarrow & L^{1-n} & \xrightarrow{\{\cdot\}_1} & \dots & \xrightarrow{\{\cdot\}_1} & L^{2-k} & \xrightarrow{\{\cdot\}_1} & \dots & \xrightarrow{\{\cdot\}_1} & L^{-1} & \xrightarrow{\{\cdot\}_1} & L^0 & \longrightarrow 0 \\
& \shortparallel & & & & \shortparallel & & & & \shortparallel & & \shortparallel & \\
& \Omega^0(M) & \xrightarrow{d} & \dots & \xrightarrow{d} & \Omega^{n+1-k}(M) & \xrightarrow{d} & \dots & \xrightarrow{d} & \Omega^{n-2}(M) & \xrightarrow{d} & \Omega^{n-1}_{ham}(M,\omega) &
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\end{tikzcd}$$

- with n (skew-symmetric) multibrackets $(2 \leq k \leq n+1)$

$$\begin{aligned}
\{\cdot, \dots, \cdot\}_k : \quad \left(\Omega^{n-1}_{ham}(M, \omega)\right)^{\otimes k} &\longrightarrow \Omega^{n+1-k}(M) \\
\sigma_1 \otimes \cdots \otimes \sigma_k &\longmapsto (-)^{k+1}\, \iota_{v_{\sigma_1}} \cdots \iota_{v_{\sigma_k}} \omega
\end{aligned}$$

L_∞-algebra is the notion that one obtains from a Lie algebra when one requires the Jacobi identity to be satisfied only up to a higher coherent chain homotopy.

Def: L_∞-algebra *(Lada, Markl)* [LM95]

$$\Big(L ,\ \{\mu_k\}_{k\in\mathbb{N}} \Big)$$

- $\mathbb{Z}$-Graded vector space $\quad L = \bigoplus_{i\in\mathbb{Z}} L_i$
- Family of homogenous skew-multilinear maps (*multi-brackets*) $\quad \mu_k : \wedge^k L \to L[k-2]$

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satisfying *"Higher Jacobi"* relations ($\forall m \geq 1$ and x_i homogeneous elements in L)

$$0 = \sum_{\substack{i+j=m+1 \\ \sigma\in ush(i,m-i)}} (-)^{i(j+1)} (-)^{\sigma} \epsilon(\sigma;x)\ \mu_j\Big(\mu_i\big(x_{\sigma_1},\dots,x_{\sigma_i}\big), x_{\sigma_{i+1}},\dots,x_{\sigma_m}\Big)$$

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Thm: Rogers [Rog12]

The *higher observable algebra* $L_\infty(M,\omega)$ forms an honest L_∞ algebra.

Take $\mu_1 = \mathrm{d}$, $\mu_k = \{\dots\}_k$, L is a shifted truncation of the de Rham complex.

Def: Leibniz algebra of observables

Leib(M, ω) is given by:

- the vector space
$$\Omega^{n-1}_{ham}(M,\omega)$$
- with the binary bracket
$$\begin{aligned} [\![\cdot,\dots,\cdot]\!] : \quad \left(\Omega^{n-1}_{ham}(M,\omega)\right)^{\otimes 2} &\longrightarrow \Omega^{n-1}_{ham}(M,\omega) \\ \sigma_1 \otimes \sigma_2 &\longmapsto \mathscr{L}_{v_{\sigma_1}} \sigma_2 \end{aligned}$$

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Prop:

i. $v_{[\![\alpha,\beta]\!]} = [v_\alpha, v_\beta]$

ii. $[\![\sigma,[\![\alpha,\beta]\!]]\!] = [\![[\![\sigma,\alpha]\!],\beta]\!] + [\![\alpha,[\![\sigma,\beta]\!]]\!]$

iii. $[\![\alpha,\beta]\!] + [\![\beta,\alpha]\!] = d\left(\iota_{v_\alpha}\beta + \iota_{v_\beta}\alpha\right)$

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Def: Equivariant moment map

Smooth map

$$\mu : M \to \mathfrak{g}^*$$

such that:

i. $d\langle \mu, \xi \rangle = -\iota_{\underline{\xi}}\omega \qquad , \forall \xi \in \mathfrak{g}$
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Upshot: $\widetilde{\mu}$ as a lift

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Symplectic reduction:

Procedure associating to any (suitably regular) pair of symplectic manifold and Hamiltonian action another symplectic manifold of smaller dimension.

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Thm: Marsden-Weinstein reduction [MW74]

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Assume: $\phi \in \mathfrak{g}^*$ regular value of μ ($\Rightarrow \mu^{-1}(\phi) \hookrightarrow M$ smooth embedding)
$G_\phi \curvearrowright \mu^{-1}(\phi)$ free and proper ($\Rightarrow \mu^{-1}(\phi)/G_\phi$ smooth manifold)

Then: $\exists!$ symplectic structure ω_ϕ in $M_\phi := \mu^{-1}(\phi)/G_\phi$
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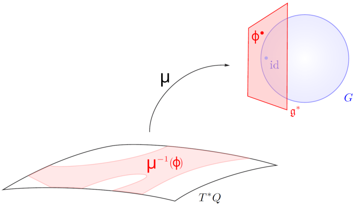

In mechanics:

it embodies the process of restricting the dynamics of the system to the level sets of the conserved quantities pertaining to the symmetry group.

(e.g. restricting to studying a point-like particle in a central potential by studying it in radial coordinates)

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Closed differential form $\phi \in \Omega^{n-1}(M,\mathfrak{g}^*)$, such that

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Thm: Multisymplectic regular reduction [Bla21]

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$G \curvearrowright M$ multisymplectic with equivariant momap. $\mu \in \Omega^{n-1}(M, \mathfrak{g}^*)$

Assume: $\phi \in \Omega^{n-1}(M, \mathfrak{g}^*)$ regular value of μ $\quad(\mu^{-1}(\phi) \hookrightarrow M$ embedding$)$
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The gist of singular reduction

- when μ is singular (i.e. $\mu^{-1}(0)$ is not a mfd.), the (geometrically) reduced space may not exist.
- a *singular reduction scheme* is a procedure to construct a "reduced" algebra of observable out of the given data
- such that it corresponds to the algebra of observable of the reduced manifold in the regular case.

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Thm: Sniatycki-Weinstein reduction [ŚW83]

Given: (M, ω) symplectic
$G \curvearrowright M$ symplectic with equivariant momap. $\mu : M \to \mathfrak{g}^*$

Then: $\left[{}^{C^\infty(M)}/{}_{I_\mu}\right]^G$ admits a Poisson algebra structure
it agrees with the M–W reduction in the regular case.

I_μ = associative ideal generated by $\widetilde{\mu}(\mathfrak{g})$

Data:

- A *constraint set* N (possibly singular),
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Strategy:

1. Define smooth fields/forms *tangent to* N,
2. define smooth fields/forms *vanishing along* N,
3. define *reducible fields* requiring the preservation of the vanishing objects,
4. define *reducible forms* requiring their conservation w.r.t. the infinitesimal action,
5. define *reducible and vanishing observables*,
6. **quotient**

Consider N closed subset of M.

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Def:

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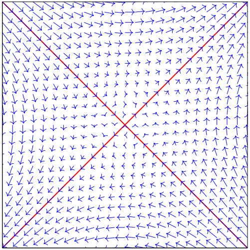
$$\mathfrak{X}_N(M) := \left\{ v \in \mathfrak{X}(M) \,\middle|\, \mathscr{L}_v(I_N) \subseteq I_N \right\}$$

Def: v.f vanishing on N

$$I_{\mathfrak{X}}(N) := \left\{ v \in \mathfrak{X}(M) \,\middle|\, \mathscr{L}_v(C^\infty(M)) \subseteq I_N \right\}$$

Vector field tangent to N



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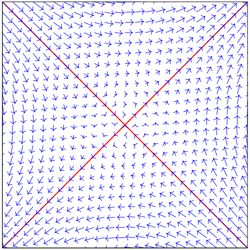

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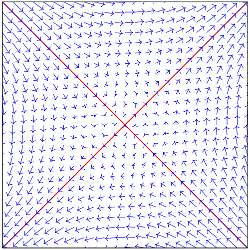

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Def: Differential form vanishing on N

$$I_{\Omega(N)} := \left\{ \alpha \in \Omega^k(M \;\middle|\; \begin{array}{l} k \geq 0, \\ \alpha(u_1, \dots, u_k) \in I_N \quad \forall u_i \in \mathfrak{X}_N(M) \end{array} \right\}$$

Consider $\mathfrak{g} \curvearrowright M$ by vector fields tangent to N

Denote by : $\underline{\mathfrak{g}} \subseteq \mathfrak{X}_N(M)$ the fundamental distribution,
$\mathfrak{X}_g$ the C^∞-module generated by $\underline{\mathfrak{g}}$.

Def: Reducible v.fields

$$\mathfrak{X}(M)_{[N]} := \left\{ v \in \mathfrak{X}(M) \;\middle|\; \begin{array}{l} \mathscr{L}_v(I_N) \subseteq I_N \\ \mathscr{L}_v(\mathfrak{X}_g) \subseteq \mathfrak{X}_g + I_{\mathfrak{X}} \end{array} \right\}$$

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Def: Reducible Hamiltonian forms

$$(\Omega(M)^{n-1}_{ham})_{[N]} := \left\{ \alpha \in \Omega(M)^{n-1}_{ham} \;\middle|\; \begin{array}{l} \alpha \text{ is a reducible form} \\ v_\alpha \text{ is a reducible v.field} \end{array} \right\}$$

Def/Prop: Reducible L_∞-observables

Is the L_∞-subalgebra of $L_\infty(M,\omega)$ given by

$$L_\infty(M,\omega)^k_{[N]} := \begin{cases} \Omega^{n-1-k}(M)_{[N]} & \text{(reducible forms)} & \text{if } n-1 \leq k < 0 \\ (\Omega(M)^{n-1}_{ham})_{[N]} & \text{(reducible hamiltonians)} & \text{if } k = 0 \\ 0 & & \text{if } k > 0 \end{cases}$$

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Def: Reduced L_∞-algebra of observables

Is the L_∞-quotient : $\dfrac{L_\infty(M,\omega)^k_{[N]}}{I_{L_\infty(M,\omega)}}$

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(but $\exists$ a canonical Poisson algebra morphism)

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Thank you for your attention!

Supplementary Material

Consider a smooth manifold Y,

Multicotangent bundle $\bigwedge = \bigwedge^n T^*Y$
is naturally n-plectic

$$\begin{tikzcd} \Lambda & T\Lambda \arrow[l] \arrow[d, "T\pi"] \\ \arrow[from=1-1, d, "\pi"'] Y & TY \arrow[l] \end{tikzcd}$$

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Def: Tautological n-form

$\theta \in \Omega^n(\Lambda)$ such that:

$$\begin{aligned} [\iota_{u_1\wedge\ldots\wedge u_n}\theta]_\eta &= \iota_{(T\pi)_* u_1\wedge\ldots\wedge(T\pi)_* u_n}\eta \\ &= \iota_{u_1\wedge\ldots\wedge u_n}\pi^*\eta \qquad \forall \eta\in\Lambda\,,\ \forall u_i \in T_\eta\Lambda \end{aligned}$$

Def: Tautological (multisymplectic) (n+1)-form

$$\omega := d\theta$$

Claim: ω is not degenerate.

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keywords

point-particles mechanics	$\rightsquigarrow$	classical fields mechanics
symplectic	$\rightsquigarrow$	multisymplectic
Observables (Poisson) algebra	$\rightsquigarrow$	Observables $L-\infty$ algebra
Co-moment map	$\rightsquigarrow$	Homotopy co-momentum map

Slogan: *Jacobi identity satisfied up to an higher coherent homotopy*

Higher Jacobi implies:

- Underlying chain-complex (L, μ_1) with differential $d = \mu_1$;
- $\mu_2 = [\cdot, \cdot]$ is a chain map $L^{\otimes 2} \to L$;
- $\mu_3 = j(\cdot, \cdot, \cdot)$ is a chain homotopy $\mu_2 \circ \mu_2 \Rightarrow 0$; i.e. between the usual Jacobiator $[[\cdot, \cdot], \cdot] \circ P_{\text{unsh}}$ and the 0 map
- higher analogues... e.g. μ_4, is a second order chain-homotopy between the two chain homotopies $[j(\cdot, \cdot, \cdot]), \cdot] \circ P_{\text{unsh}}$ and $j([\cdot, \cdot], \cdot, \cdot) \circ P_{\text{unsh}}$

$$
\begin{tikzcd}
\ddots & (L^{\otimes 3})_k \arrow[r, "\mu_1"] \arrow[dd, "\mu_2"] \arrow[dddl, "\mu_3"'] & (L^{\otimes 3})_{k-1} \arrow[dddl, "\mu_3"'] \\
 & & \vdots \\
(L^{\otimes 2})_{k+1} \arrow[r, "\mu_1"] \arrow[d, "\mu_2"] & (L^{\otimes 2})_k \arrow[r, "\mu_1"] \arrow[d, "\mu_2"] & (L^{\otimes 2})_{k-1} \arrow[d, "\mu_2"] \\
L_{k+1} \arrow[r, "\mu_1"] & L_k \arrow[r, "\mu_1"] & L_{k-1}
\end{tikzcd}
$$

Notation: $P_{\text{unsh}} =$ sum on all the possibile unshuffled permutation.

Consider a Lie algebra action $v : \mathfrak{g} \to \mathfrak{X}(M)$ preserving the n-plectic form ω,

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– symplectic case –

Def: Comoment map pertaining to v

Lie algebra morphism

$$f : \mathfrak{g} \to C^\infty(M)$$

such that

$$d\ f(x) = -\iota_{v_x}\omega \qquad \forall x \in \mathfrak{g}\ .$$

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— n-plectic case —

Def: Homotopy comoment map *(HCMM)*

L_∞-morphism

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– Conserved quantities –

Prop: Noether Theorem

Fixed $H \in C^\infty_{Ham}(M)$ ($\mathfrak{g}$-invariant) ,

$$\mathscr{L}_{v_H} f(x) = 0 \qquad \forall x \in \mathfrak{g}$$

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$$\mathscr{L}_{v_H} f(x) = 0 \qquad \forall x \in \mathfrak{g}$$

Prop: RWZ16 Theorem

Fixed $H \in \Omega^{n-1}_{Ham}(M)$ ($\mathfrak{g}$-invariant),

$$\mathscr{L}_{v_H} f_k(p) \in B^k(M) \qquad \forall p \in Z_k(\mathfrak{g})$$

Consider a Lie algebra action $v : \mathfrak{g} \to \mathfrak{X}(M)$ preserving the n-plectic form ω.

Def: Homotopy comomentum map *(Callies, Fregier, Rogers, Zambon)*

$$
\begin{tikzcd}
 & L_\infty(M,\omega) \\
\mathfrak{g} \arrow[ur, dashed, "(f)"] \arrow[r, "v"'] & \mathfrak{X}(M)
\end{tikzcd}
$$

HCMM is an L_∞-morphism $(f) : \mathfrak{g} \to L_\infty(M,\omega)$

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 & \boxed{\mathfrak{X}_{Ham}(M,\omega)} \arrow[d, hook] \\
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(acting via Hamiltonian vector fields!)

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Lemma: HCMM unfolded (CFRZ16)

HCMM is a sequence of (graded-skew) multilinear maps:

$$(f) = \left\{ f_k :\ \Lambda^k \mathfrak{g} \to L^{1-k} \subseteq \Omega^{n-k}(M) \;\middle|\; 0 \le k \le n+1 \right\}$$

fulfilling:

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- $df_k(p) = f_{k-1}(\boxed{\partial}\, p) - (-1)^{\frac{k(k+1)}{2}} \iota(v_p)\omega \qquad \forall p \in \Lambda^k(\mathfrak{g}),\ \forall k = 1,\dots n+1$

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Chevalley-Eilenberg boundary op.

HCMM is an L_∞-morphism $\quad (f) : \mathfrak{g} \to L_\infty(M,\omega)$

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Lemma: HCMM unfolded [CFRZ16]

HCMM is a sequence of (graded-skew) multilinear maps:

$$(f) = \left\{ f_k : \Lambda^k \mathfrak{g} \to L_{k-1} \subseteq \Omega^{n-k} \;\middle|\; 0 \le k \le n+1 \right\}$$

$$\begin{tikzcd} \bigwedge^{n+1}\mathfrak{g} \arrow[r, "\partial"] \arrow[d, "f_{n+1}"'] & \bigwedge^{n}\mathfrak{g} \arrow[r] \arrow[d, "f_n"] & \cdots \arrow[r] & \bigwedge^{k}\mathfrak{g} \arrow[r, "\partial"] \arrow[d, "f_k"'] & \bigwedge^{k-1}\mathfrak{g} \arrow[r] \arrow[d, "f_{k-1}"] & \cdots \arrow[r] & \bigwedge^{1}\mathfrak{g} \arrow[r, "\partial"] \arrow[d, "f_1"'] & \bigwedge^{0}\mathfrak{g} = \mathbb{R} \arrow[d, "f_0"] \\ 0 \arrow[d, equal] & L_{n-1} \arrow[d, equal] & & L_{k-1} \arrow[d, equal] & L_{k-2} \arrow[d, equal] & & L_0 = \Omega^{n-1}_{ham} \arrow[d, hook] & 0 \arrow[d, hook] \\ 0 \arrow[r, "i"'] & \Omega^0 \arrow[r, "d"'] & \cdots \arrow[r] & \Omega^{n-k} \arrow[r, "d"'] & \Omega^{n-k+1} & \cdots \arrow[r] & \Omega^{n-1} \arrow[r, "d"'] & \Omega^n \end{tikzcd}$$

fulfilling:

- $f_0 = 0,\ f_{n+1} = 0$

TAM

Practically a HCMM is given by several multilinear maps

$$f_i = \Lambda^i \mathfrak{g} \to L_{i-1}$$

satisfying:

1. $df_1(\xi) = -\iota_{v_\xi}\omega$
2. $\sum \ldots$

HCMM is an L_∞-morphism $\quad (f) : \mathfrak{g} \to L_\infty(M, \omega)$

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$$\begin{tikzcd}
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HCMM is a sequence of (graded-skew) multilinear maps:

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\bigwedge^{n+1}\mathfrak{g} \arrow[r, "\partial"] \arrow[d, "f_{n+1}"'] \arrow[ddr, "\iota^{n+1}_{\mathfrak{g}}\omega"] & \bigwedge^{n}\mathfrak{g} \arrow[r] \arrow[d, "f_n"] & \cdots \arrow[r] & \bigwedge^{k}\mathfrak{g} \arrow[r, "\partial"] \arrow[d, "f_k"'] \arrow[ddr, "\iota^{k}_{\mathfrak{g}}\omega"] & \bigwedge^{k-1}\mathfrak{g} \arrow[r] \arrow[d, "f_{k-1}"] & \cdots \arrow[r] & \bigwedge^{1}\mathfrak{g} \arrow[r, "\partial"] \arrow[d, "f_1"'] \arrow[ddr, "\iota^{1}_{\mathfrak{g}}\omega"] & \bigwedge^{0}\mathfrak{g} = \mathbb{R} \arrow[d, "f_0"] \\
0 \arrow[d, equal] & L_{n-1} \arrow[d, equal] & & L_{k-1} \arrow[d, equal] & L_{k-2} \arrow[d, equal] & & L_0 = \Omega^{n-1}_{ham} \arrow[d, hook] & 0 \arrow[d, hook] \\
0 \arrow[r, "i"'] & \Omega^0 \arrow[r, "d"'] & \cdots \arrow[r] & \Omega^{n-k} \arrow[r, "d"'] & \Omega^{n-k+1} & \cdots \arrow[r] & \Omega^{n-1} \arrow[r, "d"'] & \Omega^n
\end{tikzcd}$$

fulfilling:

- $f_0 = 0, \ f_{n+1} = 0$
- $df_k(p) = f_{k-1}(\partial p) - (-1)^{\frac{k(k+1)}{2}} \iota(v_p)\omega \qquad \forall p \in \Lambda^k(\mathfrak{g}), \ \forall k = 1, \dots n+1$

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2. $\sum \dots$

EXTRA SLIDES on regular reduction

All the credit for the next slides is to C. Blacker

Based on:

B., Reduction of multisymplectic manifolds, *Lett. Math. Phys.*, 2021

See also:

Reduction of multisymplectic manifolds (slides) at Good Morning SFARS, 7 June 2021.

Reduction is a procedure that takes a space and returns a "smaller" space

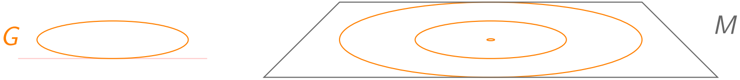
> *Reduction theory is by no means completed. . . . Only a few instances and examples of multisymplectic reduction are really well understood. . . so one can expect to see more activity in this area as well.*

— J. Marsden and A. Weinstein, 2001, *Comments on the history, theory, and applications of symplectic reduction*

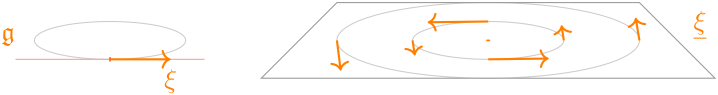
> *One of the most interesting problems in multisymplectic geometry is how to extend the well-known Marsden–Weinstein reduction scheme for symplectic manifolds to the case of multisymplectic structures.*

— M. de León, 2018, *Review of "Remarks on multisymplectic reduction" by Echeverría-Enríquez, Muñoz-Lecanda, and Román-Roy*

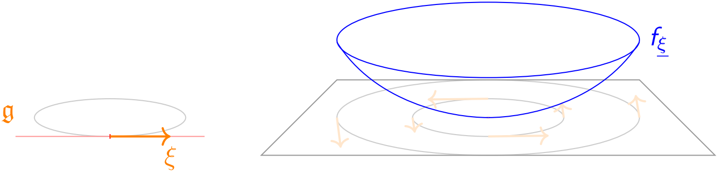
To specify a symplectic action $G \curvearrowright M$ ...



we could describe the induced map $\xi \mapsto \underline{\xi}$ ...



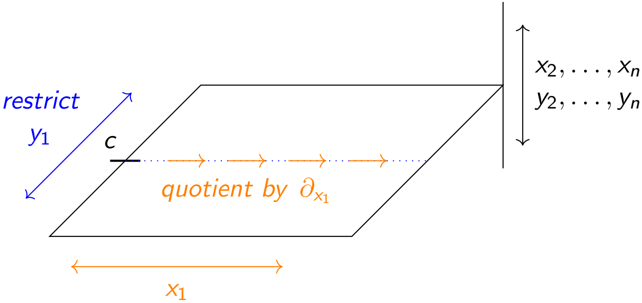
or an assignment of Hamiltonian functions $\xi \mapsto f_{\underline{\xi}}$.



When this is possible[1], the action is called **Hamiltonian**.

[1] *and $\xi \mapsto f_{\underline{\xi}}$ is a homomorphism of Lie algebras

$$\omega = \underbrace{\mathrm{d}x_1 \wedge \mathrm{d}y_1}_{\text{to be removed}} + \mathrm{d}x_2 \wedge \mathrm{d}y_2 + \cdots + \mathrm{d}x_n \wedge \mathrm{d}y_n$$



Restrict and quotient conjugate degrees of freedom.

$$\begin{tikzcd}
\boxed{\text{Action Descent}} \arrow[dr, "\substack{\textit{existence}\\ \textit{uniqueness}\\ \textit{closedness}}"'] & & \boxed{\text{Linear Symplectic Reduction}} \arrow[dl, "\textit{nondegeneracy}"] \\
& \boxed{\text{Symplectic Reduction}} &
\end{tikzcd}$$

1. Apply the *Action Descent Lemma* to $G_\lambda \curvearrowright \mu^{-1}(\lambda)$ and $i^*\omega$.

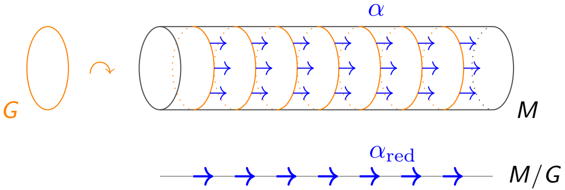
$$\begin{tikzcd}
i^*\omega & \mu^{-1}(\lambda) \arrow[d, "\pi"'] \\
\omega_\lambda & M_\lambda
\end{tikzcd}$$

2. Use *Linear Symplectic Reduction* to conclude that ω_λ is nondegenerate.

Lemma:

If

- $G \curvearrowright M$ free and proper,
- $\alpha \in \Omega^*(M)$ *invariant* and *horizontal* ($\iota_{\underline{\mathfrak{g}}}\alpha = 0$),



then

- $\exists!\, \alpha_{\mathrm{red}} \in \Omega^*(M/G)$ such that $\alpha = \pi^*\alpha_{\mathrm{red}}$,
- $\mathrm{d}\alpha = 0 \implies \mathrm{d}\alpha_{\mathrm{red}} = 0$.

- (M, ω, G, μ) Hamiltonian G-space
- $\phi \in \Omega^{k-1}(M, \mathfrak{g}^*)$

Question: When is $\mu + \phi$ a moment map?

- $\mathrm{d}\phi = 0$, since

$$\mathrm{d}(\mu + \phi)_\xi = \iota_\xi \omega \iff \mathrm{d}\phi_\xi = 0.$$

- $\mathscr{L}_\xi \phi_\zeta = \phi_{[\xi,\zeta]}$, as

$$\mathscr{L}_\xi(\mu + \phi) = (\mu + \phi)_{[\xi,\zeta]} \iff \mathscr{L}_\xi \phi = \phi_{[\xi,\zeta]}.$$

i.e. ϕ is a moment map for the trivial action $G \curvearrowright M$.

The space of moment maps is an affine space modeled on $\{\phi \in \Omega^{k-1}(M, \mathfrak{g}^*) \mid \mathrm{d}\phi = 0,\ G_\phi = G\}$.

- $\phi \in \Omega^*(M, \mathfrak{g}^*)$
- $\xi \in \mathfrak{g}$

$$\begin{aligned}
\forall \zeta \in \mathfrak{g}: \quad \mathscr{L}_\xi \phi_\zeta = \phi_{[\xi,\zeta]} \quad &\iff \forall \zeta \in \mathfrak{g}: & 0 &= \mathscr{L}_\xi \phi_\zeta - \phi_{[\xi,\zeta]} \\
& & &= \mathscr{L}_\xi \phi_\zeta + \langle \mathrm{ad}^*_\xi \phi, \zeta \rangle \\
& & &= \langle \mathscr{L}_\xi \phi + \mathrm{ad}^*_\xi \phi, \zeta \rangle \\
&\iff & 0 &= \left(\mathscr{L}_\xi + \mathrm{ad}^*_\xi \right) \phi \\
&\iff & \xi &\in \mathfrak{g}_\phi
\end{aligned}$$

in terms of the induced action $G \curvearrowright \Omega^*(M, \mathfrak{g}^*)$. Thus,

$$\forall \xi, \zeta \in \mathfrak{g}: \quad \mathscr{L}_\xi \phi_\zeta = \phi_{[\xi,\zeta]} \quad \iff \quad G \cdot \phi = \phi$$

Rather than:

- family of moment maps $\{\mu - \phi \,|\, \mathrm{d}\phi = 0,\ G_\phi = G\}$
- reduction at $\mu - \phi = 0$

We instead consider:

- fixed moment map μ
- family of levels $\{\phi \,|\, \mathrm{d}\phi = 0,\ \cancel{G_\phi = G}\}$
- reduction at $\mu = \phi$

ϕ-level set:

$$\mu^{-1}(\phi) := \{\mu = \phi\}$$

$$\begin{tikzcd}[column sep=small] \boxed{\text{Action Descent}} \arrow[dr, "\substack{\textit{existence}\\ \textit{uniqueness}\\ \textit{closedness}}"'] & & \boxed{\cancel{\text{Linear Multisymplectic Reduction}}} \arrow[dl, "\cancel{\textit{nondegeneracy}}"] \\ & \boxed{\text{Multisymplectic Reduction}} & \end{tikzcd}$$

1. Apply the *Action Descent Lemma* to $G_\phi \curvearrowright \mu^{-1}(\phi)$ and $i^*\omega$.

$$\begin{tikzcd} i^*\omega & \mu^{-1}(\phi) \arrow[d, "\pi"'] \\ \omega_\phi & M_\phi \end{tikzcd}$$

2. ~~Use *Linear Multisymplectic Reduction* to conclude that ω_ϕ is nondegenerate.~~

Two steps:

1. $G_\phi \curvearrowright M$ preserves $\mu^{-1}(\phi)$,
2. $i^*\omega$ is invariant and horizontal.

1. $G_\phi \curvearrowright M$ preserves $\mu^{-1}(\phi)$.

- $\mu^{-1}(\phi) = \{\mu - \phi = 0\}$
- $\forall\, \xi, \zeta \in \mathfrak{g}_\phi$,

$$\begin{aligned} \mathscr{L}_\xi\,(\mu - \phi)_\zeta &= (\mu - \phi)_{[\xi,\zeta]}, && \text{by the Leibniz condition,} \\ &= 0 && \text{on } \mu^{-1}(\phi). \end{aligned}$$

2. $i^*\omega$ is invariant and horizontal.

- **invariant:** Hamiltonian actions are multisymplectic.
- **horizontal:** For $\xi \in \mathfrak{g}_\phi$,

$$\begin{aligned}
\iota_\xi i^*\omega &= i^*\iota_\xi\omega, && \text{since } G_\phi \text{ preserves } \mu^{-1}(\phi), \\
&= i^*\mathrm{d}\mu_\xi, && \text{by the Hamiltonian condition,} \\
&= i^*\mathrm{d}\phi_\xi, && \text{since } \mu = \phi \text{ on } \mu^{-1}(\phi), \\
&= 0, && \text{as } \phi \text{ is closed.}
\end{aligned}$$

$\square$

1. The proof makes no use of the nondegeneracy or homogeneity of $\omega \in \Omega^{k+1}(M)$.
2. Extends naturally to a reduction scheme for closed forms.

$$\omega \in \Omega^*(M) \text{ closed}$$
$$\phi \in \Omega^*(M, \mathfrak{g}^*) \text{ closed}$$

$$\frac{\mu \in \Omega^*(M, \mathfrak{g}^*)}{\begin{gathered} \mathrm{d}\mu_\xi = \iota_\xi \omega \\ \mathscr{L}_\xi \mu_\zeta = \mu_{[\xi,\zeta]} \end{gathered}}$$

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- MW reduction as "restriction to level sets", C. Lessig, arXiv:1206.3302
- all slides from 7 in Appendix are due to C. Blacker, Reduction of multisymplectic manifolds (slides) at Good Morning SFARS, 7 June 2021.