Discussion: L. Foissy - Cointeracting bialgebras

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ESI - Higher Structures Emerging from Renormalisationn
Nov 17th '21

As users of (Hopf) algebra we are accustomed to endowing objects with algebraic structure that we might, a priori, not be interested in.

Example: if one is investigating a commutative (connected, graded) algebra and one happens to find a compatible coproduct, then the algebra is automatically free. (Milnor-Moore)

Cointeraction is another such structure.

(B, δ) coalgebra, $\mathcal{C}$ the category of (right) B-comodules.
In general it is not monoidal: let $(M_1, \rho_1), (M_2, \rho_2)$ be two right comodules. Then

$$(\rho_1 \otimes \rho_2) : M_1 \otimes M_2 \to (M_1 \otimes B) \otimes (M_2 \otimes B).$$

To turn this into a coaction we need a multiplication m_B in B.
Then

$$m^{1,3,24} \circ (\rho_1 \otimes \rho_2) : M_1 \otimes M_2 \to (M_1 \otimes M_2) \otimes B,$$

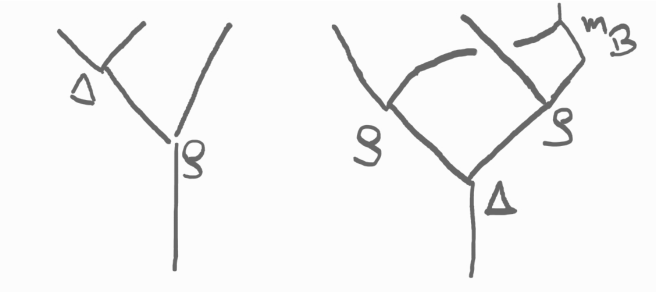
where

$$m^{1,3,24}(m_1 \otimes b_1 \otimes m_2 \otimes b_2) := m_1 \otimes m_2 \otimes m_B(b_1 \otimes b_2).$$

So, if (B, δ, m_B) is a bialgebra, the tensor product turns $\mathcal{C}$ monoidal.

A **bimonoid** in this category is a (right) comodule A with coaction ρ such that A is endowed with a bialgebra structure (A, Δ, m_A) satisfying[1]

$$(\Delta \otimes \mathrm{id}_B) \circ \rho = m^{1,3,24} \circ (\rho \otimes \rho) \circ \Delta.$$



[1].. and $\rho(xy) = \rho(x)\rho(y)$, and condition on counit ..

Example: QSym

The comodule:

$$A = \mathbb{R}[\text{words in the alphabet } \mathbb{N}_{\geq 1}]$$

$$m_A = \text{ quasi-shuffle } \overset{q}{\sqcup\!\sqcup}$$

e.g. $21 \overset{q}{\sqcup\!\sqcup} 5 = 215 + 26 + 251 + 71 + 521$

$$\Delta = \text{ deconcatenation}$$

e.g. $\Delta 215 = 215 \otimes \mathrm{e} + 21 \otimes 5 + 2 \otimes 15 + \mathrm{e} \otimes 215$.

over

$$B = A$$

$$m_B = B$$

$$\delta = \text{ 'inner' coproduct}$$

e.g. $\delta 215 = 8 \otimes 215 + 35 \otimes (21 \overset{q}{\sqcup\!\sqcup} 5) + 26 \otimes (2 \overset{q}{\sqcup\!\sqcup} 15) + 215 \otimes (2 \overset{q}{\sqcup\!\sqcup} 1 \overset{q}{\sqcup\!\sqcup} 5)$.

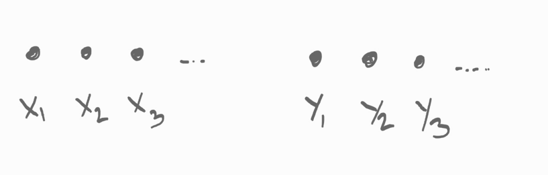
the coaction is $\rho = \delta$.

Interpretation deconcatenation

Associate a quasisymmetric function to a word

$$215 \mapsto Q_{215}(X) := \sum_{i_1 < i_2 < i_3} X_{i_1}^2 X_{i_2} X_{i_3}^5.$$

Then, with the variables $X + Y$ ordered such:

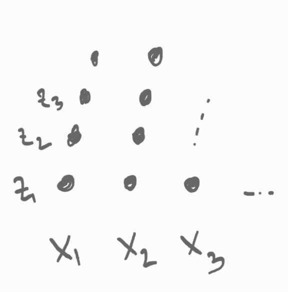


then

$$Q_{215}(X+Y) = Q_{215}(X)Q_e(Y) + Q_{21}(X)Q_5(Y) + Q_2(X)Q_{15}(Y) + Q_e(X)Q_{215}(Y).$$

Interpretation inner coproduct (the coaction)

With the variables $X \cdot Z$ ordered such:



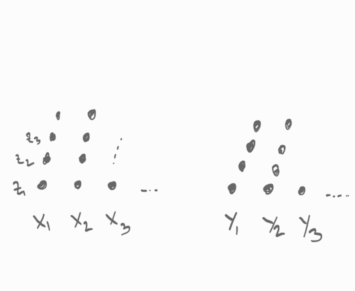
then

$$Q_{215}(X \cdot Z) = Q_8(X)Q_{215}(Z) + Q_{35}(X)Q_{21 \stackrel{q}{\sqcup\!\sqcup} 5}(Z) + Q_{26}(X)Q_{2 \stackrel{q}{\sqcup\!\sqcup} 15}(Z) + Q_{215}(X)Q_{2 \stackrel{q}{\sqcup\!\sqcup} 1 \stackrel{q}{\sqcup\!\sqcup} 5}(Z).$$

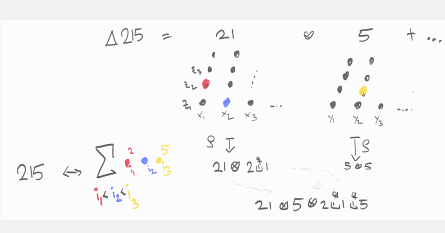
Now $(\Delta \otimes \mathrm{id}) \otimes \rho(215)$ corresponds to

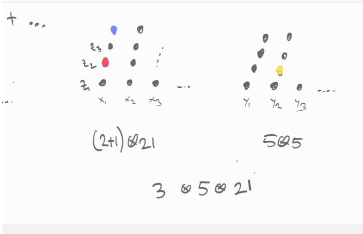
$$Q_{215}((X+Y)\cdot Z) = \sum Q_{w^{(1)}}(X) Q_{w^{(2)}}(Y) Q_{w^{(3)}}(Z),$$

where the variables are ordered such:



What is $m^{1,3,24} \circ (\rho \otimes \rho) \circ \Delta$?



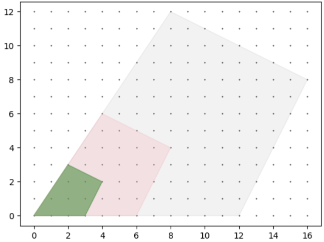


So, the cointeraction corresponds to a certain distributivity

$$Q((X+Y)\cdot Z) = Q(XZ^{(1)} + XZ^{(2)}).$$

Ehrhart polynomials

For $\mathrm{P} \subset \mathbb{R}^d$ a lattice polynomial (all vertices are in $\mathbb{Z}^d$).
Count the integral points in scaled versions:



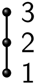
Ehrhart '67[2] showed that there is a polynomial ehr_P such that

$$\mathrm{ehr}_P(n) = \#(n\mathrm{P} \cap \mathbb{Z}^d), \quad n \in \mathbb{N}.$$

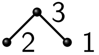
In the above picture it is $7\ X^2 + 3\ X + 1$.

[2] He finished his PhD 20 years later, at the age of 60.

Loïc investigates polytopes induced by quasi posets[3]



$$\mapsto \left\{(x_1, x_2, x_3) \in [0,1]^3 \mid x_1 < x_2 < x_3\right\}$$

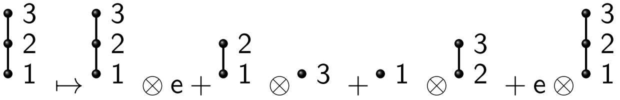


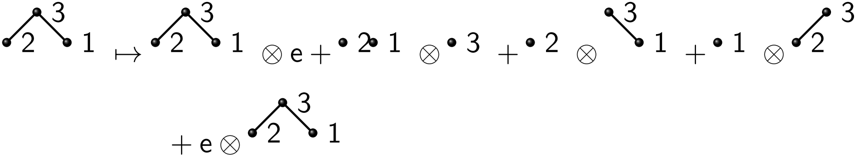
$$\mapsto \left\{(x_1, x_2, x_3) \in [0,1]^3 \mid x_2 < x_3 \wedge x_1 < x_3\right\}$$

The linear span of quasi posets are turned into an **algebra** using disjoint union. (Up to taking an orbit over permuations, this corresponds to the product of the corresponding polytopes.)

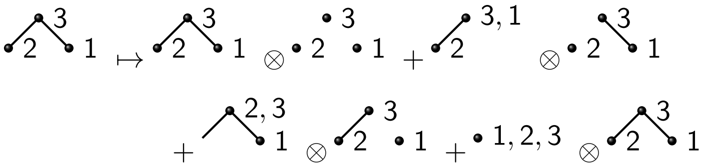
[3] poset without the condition of antisymmetry

The first coproduct Δ can be thought of as "deconcatenation of downsets/upsets"





The second coproduct δ is a contraction / restriction operation



Loïc shows that Δ is a comodule over δ having the cointeraction property, the Ehrhart polynomial is a bialgebra morphism (on both bialgebra structures) into $\mathbb{R}[X]$ and it is unique with this property.

QUESTIONS

- Compatibility with dendriform structure?
- Is there maybe a better name ('cointeraction' sounds symmetric ..)?