Conformal Bootstrap for Liouville Conformal Field Theory

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Two faces of Quantum Field Theory

(1) Axiomatic

- Wightman, Haag-Kastler, Osterwalder-Schrader, Belavin-Polyakov-Zamolodchicov
- Algebraic, sometimes explicit formuli

(2) Constructive

- Find examples satisfying axioms (QED, ϕ^4_4, QCD...)
- Action functionals, path integrals, renormalization group
- Analytic, approximative, often perturbative

This talk: a path from (2) to (1) in **Liouville CFT**

Conformal Field Theory (CFT)

Euclidean QFT models **statistical physics**

At **critical temperature** such systems have **conformal symmetry** and the QFT is **conformal field theory**

This extra symmetry gives rise to strong constraints on correlation functions via **conformal bootstrap**

In 2 dimensions bootstrap was used by Belavin, Polyakov and Zamoldchicov (1984) to classify CFT's and find explicit expressions for the correlation functions in several cases

In more than 2 dimensions bootstrap has led to spectacular numerical predictions (e.g. 3d Ising model) by Rychkov and others.

Conformal invariance

Setup:

- **Scaling fields** $V_\alpha(x)$, $x \in \mathbb{R}^d$, e.g. Ising spin
- Expectation $\langle \cdot \rangle$

Correlation functions $\langle \prod_i V_{\alpha_i}(x_i) \rangle$ invariant under rotations and translations and under scaling

$$\langle \prod_i V_{\alpha_i}(\lambda x_i) \rangle = \prod_i \lambda^{-\Delta_{\alpha_i}} \langle \prod_i V_{\alpha_i}(x_i) \rangle$$

Δ_α scaling dimension or conformal weight

Conformal invariance: extends to conformal maps $x \to \Lambda(x)$,

E.g. in $d = 2$: $\mathbb{R}^2 \simeq \mathbb{C}$, Conformal group $= SL(2, \mathbb{C})$

$$\Lambda(z) = \frac{az + b}{cz + c} \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$$

and $\lambda^{-\Delta_{\alpha_i}} \to |\Lambda'(z)|^{-\Delta_{\alpha_i}}$.

Structure Constants

3-point functions determined up to constants

$$\langle\prod_{k=1}^{3} V_{\alpha_k}(x_k)\rangle = |x_1 - x_2|^{2\Delta_{12}}|x_2 - x_3|^{2\Delta_{23}}|x_1 - x_3|^{2\Delta_{13}} C_\gamma(\alpha_1, \alpha_2, \alpha_3)$$

with $\Delta_{12} = \Delta_{\alpha_3} - \Delta_{\alpha_1} - \Delta_{\alpha_2}$ etc.

$C(\alpha_1, \alpha_2, \alpha_3)$, the **structure constants** of the CFT.

Bootstrap

Operator Product Expansion Axiom:

$$V_{\alpha_1}(x_1)V_{\alpha_2}(x_2) = \sum_{\alpha \in \mathcal{S}} C^{\alpha}_{\alpha_1\alpha_2}(x_1, z_2, \partial_{x_2})V_{\alpha}(x_2)$$

a **convergent** sum assumed to hold when inserted to expectation:

$$\langle V_{\alpha_1}(x_1)V_{\alpha_2}(x_2)V_{\alpha_3}(x_3)\dots\rangle = \sum_{\alpha \in \mathcal{S}} C^{\alpha}_{\alpha_1\alpha_2}(x_1, x_2, \partial_{x_2})\langle V_{\alpha}(x_2)V_{\alpha_3}(x_3)\dots\rangle$$

- $C^{\alpha}_{\alpha_1\alpha_2}$ are **determined** by and **linear** in the structure constants
- $\mathcal{S}$ is called the **spectrum** of the CFT

Iterating OPE:

- All correlations are determined by $C(\alpha_1, \alpha_2, \alpha_3)$

Upshot: to “solve a CFT“ need to find its spectrum and structure constants.

Bootstrap equation for structure constants

Compute 4-point function in two ways:

$$\langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} V_{\alpha_4} \rangle = \sum_{\alpha \in \mathcal{S}} C^{\alpha}_{\alpha_1 \alpha_2} \langle V_{\alpha} V_{\alpha_3} V_{\alpha_4} \rangle = \sum_{\alpha \in \mathcal{S}} C^{\alpha}_{\alpha_1 \alpha_3} \langle V_{\alpha} V_{\alpha_2} V_{\alpha_4} \rangle$$

This becomes a **quadratic equation** for structure constants.

It has proven to be a very constraining condition c.f. 3d Ising model.

In **two dimensions** many explicit solutions are known.

Solutions

Compare w. harmonic analysis on compact/noncompact groups:

1. **Compact CFT's**

(a) $\mathcal{S}$ is **finite**: minimal models (e.g. Ising model)
Belavin, Polyakov, Zamolodchicov (1983)

(b) $\mathcal{S}$ is **countable**: compact G WZW models, G/H coset theories

Explicit formuli for $C(\alpha_1, \alpha_2, \alpha_3)$ in terms of Coulomb gas integrals (Dotsenko,Fateev, ...)

2. **Non-compact CFT's**

$\mathcal{S}$ is **continuous**: WZW with noncompact group, **Liouville model**, Toda CFT's

Explicit formula for $C(\alpha_1, \alpha_2, \alpha_3)$ conjectured by Dorn, Otto, Zamolodchicov, Zamolodchicov (1995) (the **DOZZ formula**).

Constructive CFT

Try to find examples satisfying the Axioms from **functional integrals** over fields $\phi : \mathbb{C} \to M$

$$\langle \prod_\alpha V_\alpha \rangle_\Sigma = \int \prod_\alpha V_\alpha(\phi) e^{-S(\phi)} D\phi$$

Minimal models $M = \mathbb{R}$ and S is (scaling limit of) $P(\phi)_2$ QFT:

$$S(\phi) = \int_{\mathbb{C}} (|\partial_z \phi(z)|^2 + P(\phi(z)))dz$$

with P, V_α polynomials in ϕ with unknown coefficients.

WZW models $M = G$ Lie Group, S explicit

Direct analysis from functional integral hard.

Liouville model

Classical Liouville action functional for $\phi : \mathbb{C} \to \mathbb{R}$

$$S_L(\phi) = \int_{\mathbb{C}} (|\partial_z \phi(z)|^2 + \mu e^{\gamma \phi(z)}) dz$$

The minimiser of S_L solves the **Liouville equation**

$$\partial_z \partial_{\bar{z}} \phi = \mu \gamma e^{\gamma \phi}$$

Solution defines a metric $e^{\gamma \phi} |dz|^2$ with constant negative curvature and was used by Picard and Poincare to uniformise Riemann surfaces.

Polyakov (81): natural probability law for Riemannian metrics:

$$\mathbb{P}(e^{\gamma \phi} |dz|^2) \propto e^{-S_L(\phi)}$$

"Quantum uniformisation",

Liouville CFT

Scaling fields are **vertex operators** $V_\alpha(z) = e^{\alpha\phi(z)}$, $\alpha \in \mathbb{C}$ and

$$\langle \prod_i V_{\alpha_i}(z_i) \rangle = \int \prod_i e^{\alpha_i \phi(z_i)} e^{-\int_{\mathbb{C}} (|\partial_z \phi(z)|^2 + \mu e^{\gamma\phi(z)}) dz} D\phi$$

- $\mu > 0$ is **not** a perturbative parameter: $\phi \to \phi + a \Leftrightarrow \mu \to e^{\gamma a}\mu$
- γ only parameter

-Polyakov (1981) Building block of **noncritical string theory**

-Kniznik-Polyakov-Zamolodchikov (1986): scaling limit of **spin systems on random surfaces** parametrized by γ.

-E.g. $\gamma = \sqrt{3}$ describes Ising model on a planar map

-Alday-Gaiotto-Tachicawa (2010): LCFT correlations $\leftrightarrow$ Nekrasov partition functions of **SuSy Yang-Mills at $d = 4$**

Spectrum and structure constants of LCFT

Curtright, Thorn (82) conjectured: **spectrum** of LCFT is **continous** given by the vertex operators

$$V_{Q+ip}(z) = e^{(Q+ip)\phi(z)}, \quad p \in \mathbb{R}, \quad Q = \frac{2}{\gamma} + \frac{\gamma}{2}.$$

What are the **structure constants**?

In 1995 Zorn and Otto and Zamolodchicov and Zamolodchicov proposed a remarkable formula for the Liouville structure constants

$$C(\alpha_1, \alpha_2, \alpha_3) = \langle e^{\alpha_1 \phi(0)} e^{\alpha_2 \phi(1)} e^{\alpha_3 \phi(\infty)} \rangle$$

DOZZ formula

$$C_{DOZZ}(\alpha_1,\alpha_2,\alpha_3) = \hat{\mu}^{-s} \frac{\Upsilon'(0)\Upsilon(\alpha_1)\Upsilon(\alpha_2)\Upsilon(\alpha_3)}{\Upsilon(\frac{\alpha_1+\alpha_2+\alpha_3-2Q}{2})\Upsilon(\frac{\alpha_2+\alpha_3}{2})\Upsilon(\frac{\alpha_1+\alpha_3}{2})\Upsilon(\frac{\alpha_1+\alpha_2}{2})}$$

- $\hat{\mu} = \frac{\pi\Gamma(\frac{\gamma^2}{4})(\frac{\gamma}{2})^{\frac{4-\gamma^2}{2}}}{\Gamma(1-\frac{\gamma^2}{4})}\mu$
- Υ is an entire function on $\mathbb{C}$ related to the Barnes Gamma function

$C_{DOZZ}(\alpha_1,\alpha_2,\alpha_3)$ has simple poles in α_i on

$$\{-\frac{\gamma}{2}\mathbb{N} - \frac{2}{\gamma}\mathbb{N}\} \cup \{Q + \frac{\gamma}{2}\mathbb{N} + \frac{2}{\gamma}\mathbb{N}\}$$

Liouville Bootstrap

C_{DOZZ} solves the quadratic bootstrap equations numerically

This and the above spectrum would imply the bootstrap formula

$$\langle e^{\alpha_1\phi(0)}e^{\alpha_2\phi(z)}e^{\alpha_3\phi(1)}e^{\alpha_4\phi(\infty)}\rangle = \int_{\mathbb{R}_+} |z|^{2(\Delta_{Q+iP}-\Delta_{\alpha_1}-\Delta_{\alpha_2})}|\mathcal{F}(\alpha,p,z)|^2$$
$$\times\, C_{DOZZ}(\alpha_1,\alpha_2,Q+ip)C_{DOZZ}(\alpha_3,\alpha_4,Q-ip)dp$$

$\mathcal{F}(\alpha,p,z)$ purely representation theoretic **spherical conformal blocks** determined by c,α_i,p.

Constructive LCFT

1. Give a mathematical meaning to the functional integral

$$\langle \prod_i e^{\alpha_i \phi(z_i)} \rangle = \int \prod_i e^{\alpha_i \phi(z_i)} e^{-S_L(\phi)} D\phi$$

2. Prove

$$\langle e^{\alpha_1 \phi(0)} e^{\alpha_2 \phi(1)} e^{\alpha_3 \phi(\infty)} = C_{DOZZ}(\alpha_1, \alpha_2, \alpha_3)$$

3. Prove the bootstrap formula for the four point function

Probabilistic Liouville model

What is the mathematical meaning of the integral

$$\langle F \rangle = \int F(\phi) e^{-\int_{\mathbb{C}} (|\partial_z \phi(z)|^2 + \mu e^{\gamma \phi(z)}) dz} D\phi$$

We define it in terms of the **Gaussian Free Field** $X(z)$ on $\mathbb{C}$:

$$\mathbb{E} X(z) X(z') = \log |z - z'|^{-1} + \text{regular}$$

as

$$\phi(z) = c + X(z)$$

where $c \in \mathbb{R}$ is the constant mode of ϕ.

Gaussian Multiplicative Chaos (GMC)

The GFF X is not a function but a **distribution**:

$$\mathbb{E}X(z)^2 = \infty$$

To define $e^{\gamma\phi}$ we need to **regularize**

$$X \to \phi_\epsilon = \chi_\epsilon * X$$

and **renormalize** by Wick ordering

$$\lim_{\epsilon \to 0} e^{\gamma X_\epsilon(z) - \frac{\gamma^2}{2}\mathbb{E}X_\epsilon(z)^2} dz = M(dz) \quad \textit{almost surely}$$

M is called **Gaussian Multiplicative Chaos** measure on $\mathbb{C}$.

M is a **random multifractal measure**

Surprisingly (Kahane): $M \neq 0 \Leftrightarrow \gamma < 2$

Probabilistic Liouville Theory

The functional integral is then defined by

$$\langle F(X)\rangle := \int_{\mathbb{R}} e^{2Qc}\mathbb{E}\left[F(c+X)e^{-\mu e^{\gamma c}M(\mathbb{C})}\right] dc$$

where e^{2Qc} has its roots in conformal invariance ($Q = \frac{\gamma}{2} + \frac{2}{\gamma}$).

Vertex operator correlation functions

$$\langle \prod_{i=1}^{n} V_{\alpha_i}(z_i)\rangle = \int_{\mathbb{R}} e^{2Qc}\mathbb{E}\left[\prod_{i=1}^{n} e^{\alpha_i(c+X(z_i))}e^{-\mu e^{\gamma c}M(\mathbb{C})}\right] dc$$

are defined by similar renormalisation (Wick ordering) as well.

Existence

Theorem (David, K, Rhodes, Vargas, 2015) *The Liouville correlation functions exist and are nontrivial if the* ***Seiberg bounds*** *hold:*

$$(1)\quad \alpha_i < Q \ \ \forall i, \quad \text{and} \quad (2)\ \sum_{i=1}^{n} \alpha_i > 2Q$$

V_α are primary fields with scaling dimension $\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$.

LCFT is a conformal field theory with central charge

$$c = 1 + 6Q^2$$

- (2): convergence of c-integral
- (1): regularity of GMC

Structure constants

In particular the structure constants exist and are given by

$$C(\alpha_1,\alpha_2,\alpha_3) := \langle V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)\rangle =$$

$$= \frac{2}{\gamma}\mu^{-s}\Gamma(s)\lim_{u\to\infty}|u|^{4\Delta_{\alpha_3}}\mathbb{E}\left(\int \frac{|w\vee 1|^{\gamma(\alpha_1+\alpha_2+\alpha_3)}}{|w|^{\gamma\alpha_1}|w-1|^{\gamma\alpha_2}|w-u|^{\gamma\alpha_3}}M(dw)\right)^{-s}$$

in the region

$$s := \frac{\alpha_1+\alpha_2+\alpha_3-2Q}{\gamma} > 0, \quad \alpha_i < Q$$

Similar expressions for *n*-point functions.

Integrability

Does the probabilistic expression satisfy the DOZZ formula?

Theorem (K, Rhodes, Vargas, Annals of Mathematics **191**, 81) Let α_i satisfy the Seiberg bounds. Then

$$C(\alpha_1, \alpha_2, \alpha_3) = C_{DOZZ}(\alpha_1, \alpha_2, \alpha_3)$$

Proof combines **probabilistic** analysis of GMC to derive **algebraic** identities for the structure constants that determine them uniquely.

4-point function

Möbius covariance: 4-point function depends on $z \in \mathbb{C}$:

$$G_4(z) = \left\langle V_{\alpha_1}(0) V_{\alpha_2}(z) V_{\alpha_2}(1) V_{\alpha_4}(\infty) \right\rangle$$

Probabilistic formula

$$G_4(z) = \frac{2\mu^{-s}}{\gamma}\Gamma(s) \lim_{u \to \infty} |u|^{4\Delta_{\alpha_3}} \mathbb{E}\left(\int \frac{|w \vee 1|^{\gamma(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}}{|w|^{\gamma\alpha_1}|w-z|^{\gamma\alpha_2}|w-1|^{\gamma\alpha_3}|w-u|^{\gamma\alpha_4}} M(dw) \right)^{-s}$$

Bootstrap : Can we express $G_4(z)$ in terms of 3-point functions?

Bootstrap

Theorem. (GKRV 2020) Let α_i satisfy Seiberg bounds with $\alpha_1+\alpha_2>Q$ and $\alpha_3+\alpha_4>Q$. Then

$$\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle = \int_{\mathbb{R}_+} |z|^{2(\Delta_{Q+iP}-\Delta_{\alpha_1}-\Delta_{\alpha_2})}|\mathcal{F}(\alpha,p,z)|^2$$
$$\times\, C_{DOZZ}(\alpha_1,\alpha_2,Q+ip)C_{DOZZ}(\alpha_3,\alpha_4,Q-ip)dp$$

$\mathcal{F}$ are purely representation theoretic **holomorphic conformal blocks**

Idea:

1. Express correlation functions as **scalar products**
2. $\mathcal{S}=$ **spectrum** of the **Hamiltonian** of the QFT
3. z-dependence from **conformal Ward identities**

Reflection positivity

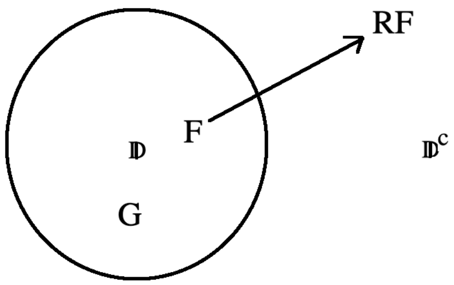
Hilbert space $\mathcal{F}_{\mathbb{D}}$ = functionals $F(\phi)$ that depend on $\phi|_{\mathbb{D}}$, $\mathbb{D}$ unit disc.

Reflection in unit circle $z \to \bar{z}^{-1}$ maps $\mathbb{D}$ to $\mathbb{D}^c$. It extends to

$$\mathcal{R} : \mathcal{F}_{\mathbb{D}} \to \mathcal{F}_{\mathbb{D}^c}$$

Scalar product $G, F \in \mathcal{F}_{\mathbb{D}} \to (G, F) := \langle G\mathcal{R}F \rangle$

Reflection positivity $\langle F\mathcal{R}F \rangle \geq 0, \quad \forall F \in \mathcal{F}_{\mathbb{D}}$.



4-point function

The four-point function

$$G_4(z) := \langle V_{\alpha_1}(0) V_{\alpha_2}(z) V_{\alpha_3}(1) V_{\alpha_4}(\infty) \rangle \quad z \in \mathbb{D}$$

can be written as a scalar product

$$G_4(z) = \big(V_{\alpha_1}(0) V_{\alpha_2}(z), V_{\alpha_3}(1) V_{\alpha_4}(0) \rangle \quad (*)$$

Bootstrap is obtained by factorising $(*)$ using the **spectral resolution** of the **Hamiltonian** of LCFT.

Hamiltonian of LCFT

Dilation $z \to e^{-t}z$ maps $\mathbb{D} \to \mathbb{D}$ and extends to a semigroup

$$e^{-tH} : \mathcal{F}_{\mathbb{D}} \to \mathcal{F}_{\mathbb{D}}$$

H is the **Hamiltonian** of the QFT

Proposition (GKRV 2020) H is a positive self adjoint operator on $\mathcal{H}$ for all $\gamma < 2$.

We find the spectral resolution of H and relate it to representation theory of the **Virasoro algebra**

Heuristic picture

$\mathcal{F}_{\mathbb{D}}$ carries a representation of two commuting Virasoro algebras $\{L_n\}$, $\{\bar{L}_n\}$ and a complete set of (generalized) eigenfunctions is

$$\bar{L}_{\mathbf{n}} L_{\mathbf{m}} V_{Q+ip}(0) = \prod_i L_{n_i} \prod_j L_{m_j} V_{Q+ip}(0)$$

Then

$$(V_{\alpha_1}(0) V_{\alpha_2}(1), V_{Q+ip}(0)) = \langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{Q+ip}(\infty)\rangle = C_{DOZZ}(\alpha_1, \alpha_2, Q+ip)$$

and using **conformal Ward identities**

$$(V_{\alpha_1}(0) V_{\alpha_2}(z), \bar{L}_{\mathbf{n}} L_{\mathbf{m}} V_{Q+ip}(0)) = f(z, \alpha_1, \alpha_2, p) C_{DOZZ}(\alpha_1, \alpha_2, Q+ip) \quad (*)$$

with explicit representation theoretic $f(z, \alpha_1, \alpha_2, p)$.

Problem: There is no local field $V_{Q+ip}(z)$!

Actual proof:

- Analytic continuation of $Q + ip \to \alpha \in \mathbb{R}$
- Probabilistic proof of the Ward identity $(*)$.

H as a Schrödinger operator

Write $\phi(e^{i\theta}) = c + \varphi(\theta) = c + \sum_{n\neq 0} \varphi_n e^{in\theta}$.

Hilbert space $\to$ wave functions $\psi(c,\varphi) \in L^2(dc \times \mathbb{P}(d\varphi))$.

Feynman-Kac formula gives

$$H = H_0 + \mu V$$

$$H_0 = \tfrac{1}{2}\Big(-\tfrac{d^2}{dc^2} + Q^2 + \sum_{n=1}^{\infty}(a_n^* a_n + \bar{a}_n^* \bar{a}_n)\Big)$$

$$V(c,\varphi) = e^{\gamma c} \int_0^{2\pi} e^{\gamma\varphi(\theta) - \frac{\gamma^2}{2}\mathbb{E}\varphi(\theta)^2} d\theta$$

where $a_n = i\frac{\partial}{\partial \varphi_n}$ etc.

We need to find a complete set of eigenfunctions $\psi(c,\varphi)$ of H:

$$(H_0 + \mu V)\psi = E\psi$$

They are obtained by **scattering theory**.

Toy Liouville

Keep only c variable:

$$H = \tfrac{1}{2}\left(-\frac{d^2}{dc^2} + Q^2\right) + \mu e^{\gamma c}$$

Schrödinger operator on $L^2(\mathbb{R}, dc)$ with a wall potential

$$V(c) = e^{\gamma c} \to \begin{cases} 0 & \text{if } c \to -\infty \\ \infty & \text{if } c \to \infty \end{cases}$$

Scattering theory: Generalized eigenfunctions

$$\psi_p(c) \sim \begin{cases} e^{ipc} + R(p)e^{-ipc} & c \to -\infty \\ 0 & c \to \infty \end{cases}$$

with $p \in \mathbb{R}_+$ and eigenvalue $\frac{1}{2}(Q^2 + p^2) = 2\Delta_{Q+ip}$.

LCFT

Spectrum of H_0

$H_0 = \frac{1}{2}(-\frac{d^2}{dc^2} + Q^2 + \sum_{n=1}^{\infty}(a_n^* a_n + \bar{a}_n^* \bar{a}_n))$ on $L^2(dc \times \mathbb{P}(d\varphi)$.

$L^2(dc \times \mathbb{P}(d\varphi)$ carries a representation of $Vir \oplus \overline{V}ir$

Highest weight states $\psi_p(c, \varphi) = e^{ipc}$.

Basis of generalized eigenstates

$$\psi_{p,\mathbf{n},\mathbf{m}} = \bar{L}_{\mathbf{n}} L_{\mathbf{m}} \psi_p = e^{ipc} h_{\mathbf{n},\mathbf{m}}(\varphi)$$

$h_{\mathbf{n},\mathbf{m}}(\varphi)$ polynomials on the φ_k's.

$$H_0 \psi_{p,\mathbf{n},\mathbf{m}} = E_{p,\mathbf{n},\mathbf{m}} \psi_{p,\mathbf{n},\mathbf{m}}$$

Spectrum of LCFT

Theorem (GKRV 2020). H has a basis of generalized eigenstates with the **same** eigenvalues

$$H\Psi_{p,\mathbf{n},\mathbf{m}} = E_{p,\mathbf{n},\mathbf{m}}\Psi_{p,\mathbf{n},\mathbf{m}}$$
$$\Psi_{p,\mathbf{n},\mathbf{m}}(c,\varphi) \sim \psi_{p,\mathbf{n},\mathbf{m}}(c,\varphi) + \text{reflected waves} \quad \text{as } c \to -\infty$$

Corollary. Plancharel identity holds

$$G_4(z) = \sum_{\mathbf{n},\mathbf{n}',\mathbf{m},\mathbf{m}'} \int_{\mathbb{R}_+} (V_{\alpha_1}(0)V_{\alpha_2}(z), \Psi_{p,\mathbf{n},\mathbf{m}})(\Psi_{p,\mathbf{n}',\mathbf{m}'}, V_{\alpha_3}(1)V_{\alpha_4}(0)) \times \mathcal{F}(p)_{\mathbf{m},\mathbf{m}'}\mathcal{F}(p)_{\mathbf{n},\mathbf{n}'}\,dp$$

with explicit Gram matrix $\mathcal{F}(p)$.

Remains to connect $(V_{\alpha_1}(0)V_{\alpha_2}(z), \Psi_{p,\mathbf{n},\mathbf{m}})$ to structure constants.

Bootstrap

Theorem (GKRV2020)

$$(V_{\alpha_1}(0)V_{\alpha_2}(z), \Psi_{p,\mathbf{n},\mathbf{m}}) = C(\alpha_1, \alpha_2, Q + ip) \times \text{ explicit factor}$$

Proof:

- Analytic continuation of $\Psi_{p,\mathbf{n},\mathbf{m}}$ in $Q + ip \to \alpha \in \mathbb{R}$
- $\mathbf{n}, \mathbf{m}$ dependence by Ward identity of LCFT.

Corollary. Bootstrap formula holds:

$$\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle = \int_{\mathbb{R}_+} |z|^{2(\Delta_{Q+iP}-\Delta_{\alpha_1}-\Delta_{\alpha_2})}|\mathcal{F}(\alpha, p, z)|^2$$
$$\times C_{DOZZ}(\alpha_1, \alpha_2, Q + ip)C_{DOZZ}(\alpha_3, \alpha_4, Q - ip)dp$$

Remarks

1. There is **no local field** $V_{Q+ip}(z)$. The spectral state $\psi_{p,0}$ is an **analytic continuation** of the state $V_\alpha(0)$. It is a **macroscopic state**.

The **microscopic state** $V_\alpha(0)$ is **not** in the Hilbert space. This has been emphasized before by Seiberg and Teschner.

2. The Liouville potential

$$V(c,\varphi) = e^{\gamma c} \int_0^{2\pi} e^{\gamma\varphi(\theta) - \frac{\gamma^2}{2}\mathbb{E}\varphi(\theta)^2} d\theta$$

is a well defined multiplication operator if $\gamma < \sqrt{2}$ but it **vanishes** identically if $\gamma \geq \sqrt{2}$!. It has to be defined as a measure in the Hilbert space if $\gamma \in [\sqrt{2}, 2)$. Then the Hamiltonian exists as a Friedrichs extension.

Prospects

Bootstrap for LCFT on 2d torus (in progress) and genus ≥ 2.

Toda CFT's

Other noncompact CFT: $G^{\mathbb{C}}/G$ WZW model, 2d black hole?

Thank you!

Proof ideas

1. **Analyticity.** $C(\alpha_1, \alpha_2, \alpha_3)$ are analytic in a neighborhood of $\alpha_1 + \alpha_2 + \alpha_3 > 2Q$, $\alpha_i < Q$.

2. **Reflection.** $C(\alpha_1, \alpha_2, \alpha_3)$ has analytic continuation beyond $\alpha_i \in (0, Q)$ which satisfies

$$C(\alpha_1, \alpha_2, \alpha_3) = R(\alpha_1)C(2Q - \alpha_1, \alpha_2, \alpha_3)$$

3. **Periodicity.** Let $\alpha = \frac{\gamma}{2}$ or $\alpha = \frac{2}{\gamma}$. Then for all $\alpha_1 \in \mathbb{R}$:

$$C(\alpha_1 - \alpha, \alpha_2, \alpha_3) = D(\alpha, \alpha_1, \alpha_2, \alpha_3)C(\alpha_1 + \alpha, \alpha_2, \alpha_3)$$

For $\gamma^2 \notin \mathbb{Q}$ this determines $C = C_{DOZZ}$. Continuity in $\gamma \implies \square$.

Reflection and Periodicity

DOZZ formula satisfies reflection and periodicity with

$$D(\alpha,\alpha_1,\alpha_2,\alpha_3) = -\frac{1}{\pi\mu}\frac{\Gamma(-\alpha^2)\Gamma(-\alpha\alpha_1)\Gamma(-\alpha\alpha_1-\alpha^2)\Gamma(\frac{\alpha}{2}(2\alpha_1-\bar{\alpha}))}{\Gamma(\frac{\alpha}{2}(2Q-\bar{\alpha}))\Gamma(\frac{\alpha}{2}(2\alpha_3-\bar{\alpha}))\Gamma(\frac{\alpha}{2}(2\alpha_2-\bar{\alpha}))}$$
$$\times\frac{\Gamma(1+\frac{\alpha}{2}(\bar{\alpha}-2Q))\Gamma(1+\frac{\alpha}{2}(\bar{\alpha}-2\alpha_3))\Gamma(1+\frac{\alpha}{2}(\bar{\alpha}-2\alpha_2))}{\Gamma(1+\alpha^2)\Gamma(1+\alpha\alpha_1)\Gamma(1+\alpha\alpha_1+\alpha^2)\Gamma(1+\frac{\alpha}{2}(\bar{\alpha}-2\alpha_1))}$$
$$R(\alpha) = -((\frac{\gamma}{2})^{\frac{\gamma^2}{2}-2}\tilde{\mu})^{\frac{2(Q-\alpha)}{\gamma}}\frac{\Gamma(\frac{\gamma}{2}(\alpha-Q))\Gamma(\frac{2}{\gamma}(\alpha-Q))}{\Gamma(\frac{\gamma}{2}(Q-\alpha))\Gamma(\frac{2}{\gamma}(Q-\alpha))}.$$

In particular the **reflection relation** has been a mystery:

$$e^{\alpha\phi} = R(\alpha)e^{(2Q-\alpha)\phi}$$

In our proof

- Coefficients R and D follow from asymptotic analysis of multiplicative chaos integrals
- The **reflection coefficient** $R(\alpha)$ has a probabilistic origin in tail behaviour of multiplicative chaos.

Ward identity

Theorem. (GKRV 2020) For an explicit function $\mathcal{T}_{\alpha,\beta,p}(\mathbf{n})$

$$(V_{\alpha_1}(0)V_{\alpha_2}(z), \Psi_{p,\mathbf{n},\mathbf{m}})_{\mathbb{D}} = \mathcal{T}_{\alpha_1,\alpha_2,p}(\mathbf{n})\mathcal{T}_{\alpha_1,\alpha_2,p}(\mathbf{m})C_{DOZZ}(\alpha_1,\alpha_2,Q+ip) \tag{1}$$

Heuristic explanation: $\Psi_{p,\mathbf{n},\mathbf{m}} = \tilde{L}_{\mathbf{n}}L_{\mathbf{m}}V_{Q+ip}(0)$ and

$$\begin{aligned}(V_{\alpha_1}(0)V_{\alpha_2}(z), \Psi_{p,0,0}) &= (V_{\alpha_1}(0)V_{\alpha_2}(z), V_{Q+ip}(0)) \\ &= \langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{Q+ip}(\infty)\rangle_{S^2} \\ &= C_{DOZZ}(\alpha_1,\alpha_2,Q+ip)\end{aligned}$$

$\mathcal{T}$ factors are produced by $\tilde{L}_{\mathbf{n}}L_{\mathbf{m}}$ via **conformal Ward identities**.

Problem: There is no local field $V_{Q+ip}(z)$!

Actual proof:

- Analytic continuation of $\psi_{p,\mathbf{n},\mathbf{m}}$: $Q+ip \to \alpha \in \mathbb{R}$
- Probabilistic proof of the Ward identity (1)

Bootstrap

Corollary. (GKRV) Bootstrap formula holds:

$$\langle e^{\alpha_1\phi(0)} e^{\alpha_2\phi(z)} e^{\alpha_3\phi(1)} e^{\alpha_4\phi(\infty)} \rangle_{S^2} =$$

$$= \int_{\mathbb{R}_+} C_{DOZZ}(\alpha_1, \alpha_2, Q + ip) C_{DOZZ}(\alpha_3, \alpha_4, Q + ip) |\mathcal{F}(\alpha, p, z)|^2 dp$$

where $\mathcal{F}$ are spherical holomorphic conformal blocks given by

$$\mathcal{F}(\alpha, p, z) := \sum_{k=0}^{\infty} \beta_k z^k$$

The sum converges in $|z| < 1$ for almost all p and

$$\beta_k := \sum_{|\mathbf{n}|, |\mathbf{n}'| = k} \mathcal{T}_{\alpha_1, \alpha_2, p}(\mathbf{n}) \mathcal{F}(p)^{-1}_{\mathbf{n}, \mathbf{n}'} \mathcal{T}_{\alpha_3, \alpha_4, p}(\mathbf{n}').$$