Workshop on

Normal Numbers: Arithmetic, Computational and Probabilistic Aspects

Abstracts

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organized by

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Monday

Christoph Aistleitner

Introductory talk 1: Connections in normal number theory

Monday, 09:30 – 10:30

We introduce several standard notions from the theory of normal numbers. We describe the historical development of the subject, and present its relations with number theory, probability theory, harmonic analysis, uniform distribution theory and other fields. We present some classical results, and explain several generalized concepts of "normality" of mathematical objects. This talk is intended as a general introduction into the topic of the workshop.

Veronica Becher

Introductory talk 2: Constructions of normal numbers and their computation

Monday, 11:00 – 12:00

I will review some constructions of absolutely normal numbers, emphasising the computational aspects. Recent results yield a fast computation at the expense of a large discrepancy.

David H. Bailey

On the normality of Stoneham numbers, π and $\log(2)$

Monday, 14:20 – 15:00

In two previous papers, Richard Crandall and the present author: (a) demonstrated the base-2 normality of π, $\log(2)$ and some other constants can be reduced to a conjecture about the behavior of certain specific simple pseudo-random number generators; and (b) proved base-2 normality for an uncountably infinite class of explicit real constants, the simplest of which is now known as the Stoneham (2,3) constant, namely $\sum_n 1/(3^n * 2^{(3^n)})$. [This latter result was subsequently proved much more simply by applying ergodic theory techniques.] More recently, Francisco Aragon Artacho, Jonathan Borwein, Peter Borwein and the present author analyzed the normality of π, $\log(2)$, the Stoneham numbers and other constants using techniques of scientific visualization. This talk will present an overview of the earlier results, and then summarize the recent results.

Manfred Madritsch

Construction of numbers which are normal with respect to a shift-invariant measure

Monday, 15:10 – 15:50

The aim of the present talk is the construction of numbers whose distribution of blocks follows a given shift invariant measure.

Let (X, S) be a symbolic dynamical system defined over the (finite or infinite) alphabet A. Let $\mathbf{b} = b_1 \dots b_k \in A^k$ be a word, then we call

$$[\mathbf{b}] = \{(a_n)_{n\geq 1} \in A^{\mathbb{N}} \colon a_1 = b_1, \dots, a_k = b_k\}$$

the cylinder set of all sequences starting with the same letters as $\mathbf{b}$. Let μ be a probability measure μ on X. Then we call μ shift-invariant, if for each $A \subset X$ we have $\mu(S^{-1}A) = \mu(A)$. Suppose that μ is shift-invariant. Then we call $\omega \in X$ normal (generic) with respect to μ provided that for $k \geq 1$ and $\mathbf{b} \in A^k$, we have

$$\frac{1}{N} \sum_{n=0}^{N-1} \mathbb{1}_{\mathbf{b}}(T^n \omega) \xrightarrow[N\to\infty]{} \mu(C),$$

where $\mathbb{1}_{\mathbf{b}}$ denotes the indicator function of $[\mathbf{b}]$.

The aim of the present talk is to provide a Champernowne type construction which yields (under some mild conditions) a normal word with respect to any given shift invariant measure.

This is joint work with Bill Mance (Polish Academy of Sciences).

Jean-Marie De Koninck & Imre Katai

New approaches in the construction of normal numbers

Monday, 16:00 – 16:40

We expose various non traditional methods for creating large families of normal numbers. The first one is essentially based on the fact that the prime factorization of integers is locally chaotic while also being globally regular. The second one is based on a 1996 result of Bassily and Kátai regarding the distribution of consecutive digits in the q-expansion of prime numbers, allowing us to prove in particular that the real number $0.P(2)P(3)P(4)\dots$, where $P(n)$ stands for the largest prime factor of n, is a normal number, thus answering a question raised by Igor Spharlinski. We will also show how one can use the distribution of the values of shifted primes to create a variety of families of normal numbers.

Tuesday

Yann Bugeaud

Introductory talk 3: Normal numbers: recent results and open problems

Tuesday, 09:00 – 10:00

Let b be an integer greater than or equal to 2. A real number is called simply normal to base b if each digit $0, \ldots, b-1$ occurs in its b-ary expansion with the same frequency $1/b$. It is called normal to base b if it is simply normal to every base b^k, where k is a positive integer (or, equivalently, if, for every positive integer k, each block of k digits from $0, \ldots, b-1$ occurs in its b-ary expansion with the same frequency $1/b^k$). This notion was introduced in by 1909 Émile Borel, who established that almost every real number (in the sense of the Lebesgue measure) is normal to every integer base. We present classical and more recent results on normal numbers and highlight several open problems. Topic discussed include explicit construction of normal numbers, the existence of uncountably many numbers normal to base 2 but not simply normal to base 3 (a result proved independently by Cassels and Schmidt more than fifty years ago), and links between digital properties and Diophantine properties of a real number.

Mini talks & introduction of young researchers

Tuesday, 10:00 – 10:40

Theodore Slaman

Normality in Different Integer Bases

Tuesday, 11:10 – 11:50

Dong Han Kim

A new complexity function of repetition and irrationality exponents

Tuesday, 14:20 – 15:00

We introduce and study a new complexity function in combinatorics on words, which takes into account the smallest return time of a factor of an infinite word. We characterize the eventually periodic words and the Sturmian words by means of this function. Then, we establish a new result on repetitions in Sturmian words and show that it is best possible. We deduce a lower bound for the irrationality exponent of real numbers whose sequence of b-ary digits is a Sturmian sequence over $\{0, 1, , b-1\}$ and we prove that this lower bound is best possible. If the irrationality exponent of ξ

is equal to 2 or slightly greater than 2, then the b-ary expansion of ξ cannot be "too simple", in a suitable sense. Our result applies, among other classical numbers, to badly approximable numbers, non-zero rational powers of e, and $\log(1 + 1/a)$, provided that the integer a is sufficiently large. It establishes an unexpected connection between the irrationality exponent of a real number and its b-ary expansion. This is joint work with Yann Bugeaud.

Tomas Persson

Fourier dimension and equidistribution

Tuesday, 15:10 – 15:50

The Fourier dimension of a measure is the supremum of numbers s for which the Fourier transform decays with a polynomial rate with exponent $s/2$. The Fourier dimension of a set is the supremum of Fourier dimensions of measures with support in the set.

Jan Reimann

Irrationality Exponent, Hausdorff Dimension and Effectivization

Tuesday, 16:00 – 16:40

We generalize the classical theorem by Jarník and Besicovitch on the irrationality exponents of real numbers and Hausdor dimension and show that the two notions are independent. For any real number a greater than or equal to 2 and and any non- negative real b be less than or equal to $2/a$, we show that there is a Cantor-like set with Hausdor dimension equal to b such that, with respect to its uniform measure, almost all real numbers have irrationality exponent equal to a. We give an analogous result relating the irrationality exponent and the effective Hausdorff dimension of individual real numbers. We prove that there is a Cantor-like set such that, with respect to its uniform measure, almost all elements in the set have effective Hausdorff dimension equal to b and irrationality exponent equal to a. In each case, we obtain the desired set as a distinguished path in a tree of Cantor sets.

Wednesday

Olivier Carton

Finite state machines and normality

Wednesday, 09:00 – 09:40

We first recall that normality can be characterized by incompressibility by finite state machines. We show that this characterization is robust as it does not depend on the variant of finite machines considered.

We consider the preservation of normality by selection based of prefixes and suffixes. We present a notion of independence based on incompressibilty for normal sequences. This independence can, in turn, be characterized by preservation of normality in several contexts.

Joel Rivat
The Thue-Morse sequence along squares is normal

Wednesday, 09:50 – 10:30

In a joint work with M. Drmota and C. Mauduit we show a first example of an almost periodic sequence (in the sense of symbolic dynamical systems) whose subsequence along squares is a normal sequence. As an application, this provides a new method to produce normal numbers in a given base.

Joseph Vandehey
Skew-products, automata, and normality

Wednesday, 11:00 – 11:40

Suppose we have a dynamical system that gives rise to an expansion, a fibred system in the terminology of Schweiger. We can extend this to a skew-product, so that as digits are removed from the expansion, they control the action of a (usually finite) automaton. Such systems were used in the proof of Moeckel's theorem, but we wish to consider them much, much more generally. We will show how this method can be used to quickly regain many classic results, and prove many new ones, including that normality for continued fractions is preserved under non-zero rational additional and multiplication.

Thursday

Teturo Kamae
Between normal numbers and random numbers

Thursday, 09:00 – 09:40

I introduced a criterion of randomness called the Sigma function. It is a function of finite words over an alphabet defined as the sum of square of the number of occurrences of every factor of it. It measures specially the uniformity of the block distribution. An infinite word whose prefixes attain asymptotically the almost all value of it is called Sigma random. The Sigma random words are not only normal numbers, but also satisfy some recurrence properties of long blocks. Starting from any finite word, we can construct a Sigma-random word adding the letter in the alphabet attaining the smallest Sigma value in each stage. We also discuss other logic-free notions of randomness stronger than the normality.

Ai-Hua Fan

Martingale method in the study of the almost convergence of ergodic series

Thursday, 09:50 – 10:30

We study the almost everywhere convergence of ergodic series of the form $\sum a_n f(T^n x)$ where (a_n) is assumed square-summable and f has zero mean value. Extra-conditions are necessary to ensure the convergence. By using a martingale decomposition, we find some sufficient conditions, sometimes necessary too. The method applies to other situations, including lacunary dilated series in harmonic analysis, for which we find the optimal regularity condition (Gaposhkins critical value $1/2$). This is a joint work with Christophe Cuny.

El Houcein El Abdalaoui

On the Erdős flat polynomials problem

Thursday, 11:00 – 11:40

In this talk I will present my recent contribution on the Erdős
at polynomials problems and its connections. I will further present some ingredients and ideas of the proof of the following fact:
There are no square L^2-flat sequences of polynomials of the type

$$\frac{1}{\sqrt{q}}(\epsilon_0 + \epsilon_1 z + \epsilon_2 z^2 + \cdots + \epsilon_{q-2} z^{q-2} + \epsilon_{q-1} z^{q-1}),$$

where for each $j,\ 0 \leq j \leq q-1,\ \epsilon_j = \pm 1$ As a consequence, we obtain that the Erdős–Newman conjectures on Littlewood polynomials holds. This further gives that Turyn–Golay's conjecture is true. Therefore, by appealing to Downarowicz-Lacroix result on square L^2-flat polynomials, we conclude that

- The special conjecture of Turyn-Golay's conjecture on the Barker sequences hold, that is, there are only finitely many Barker sequences.
- Moreover, the spectrum of any continuous Morse sequences is singular. This answer one of the problem in the list of problems asked by M. Keane in his famous 1968's paper on the generalized Morse sequences.

I will further discuss some open problems related to Erdős flat polynomials.

Mario Neumüller

Metrical star discrepancy bounds for subsequences of digital Kronecker-sequences

Thursday, 14:20 – 15:00

By a well known result of Wall a number α is normal to base λ if and only if the sequence $(\{\alpha\lambda^n\})_{n\geq 0}$ is uniformly distributed. Moreover, Gal and Gal showed that for almost all α and every integer base λ we have that

$$D^*(N, (\{\alpha\lambda^n\})_{n\geq 0}) = \mathcal{O}\left(\sqrt{\frac{\log\log N}{N}}\right).$$

In this talk we consider a similar situation for the d-dimensional case. More precisely, we investigate certain subsequences of digital Kronecker sequences $\mathcal{S}(\boldsymbol{f}) = (\boldsymbol{x}_n)_{n\geq 1}$ in the unit cube $[0,1)^d$, whose construction is based on the field of formal Laurent series over the finite field $\mathbb{F}_q$ of prime order q. For $\boldsymbol{f} = (f_1, \ldots, f_d) \in (\mathbb{F}_q((t^{-1})))^d$ we define $\boldsymbol{x}_n = (\{t^{n-1}f_1\}_{|t=q}, \ldots, \{t^{n-1}f_d\}_{|t=q})$. We are able to prove that for almost all $\boldsymbol{f}$

$$D_N^*(\mathcal{S}(\boldsymbol{f})) \leq C(\varepsilon)\sqrt{\frac{d\log d}{N}}$$

with probability $1-\varepsilon$ and $C(\varepsilon)$ is independent of N and d and is of order $C(\varepsilon) \asymp \log\varepsilon^{-1}$.

Similar results for classical Kronecker sequences have been recently shown by Th. Löbbe. This is joint work with Friedrich Pillichshammer.

Anne Bertrand-Mathis

Normal Numbers : what we know about Pisot bases and dynamical systems

Thursday, 15:10 – 15:50

Let $\beta > 1$ be a real number and call $B(\beta)$ the set of "normal numbers in base β", i.e. the set of numbers such that the sequence $(x\beta^n)_{n\geq 1}$ is uniformly distributed modulo one; Weyl proved hundred years ago that almost all x belongs to $B(\beta)$; in the same way given an algebraic integer β of degree d, call $T(\beta)$ the set of x such that the sequence $(x\beta^n, \ldots, x\beta^{n+d-1})_{n\geq 1}$ is uniformly distributed in the d-dimensional torus: almost all x belong to $T(\beta)$. We shall see that $B(\beta)$ is of course included in $T(\beta)$ but also that the converse is never true.

Given a real number $\beta > 1$, Rényi explained how to expand $x \in [0,1]$ in base β: $x = \sum_{k\geq 1}\frac{x_k}{\beta^k}$ where x_k is a positive integer and where the rest $\sum_{k\geq k_0}\frac{x_k}{\beta^k}$ is smaller than $\frac{1}{\beta^{k_0}}$. The set of sequences $(y_n)_{n\in\mathbb{Z}}$ where all subsequences $y_h y_{h+1}\ldots$ are the expansion of a number $y \in [0,1]$ is a symbolical dynamical system called β-shift and denoted X_β. The β-shift admits a maximal invariant measure μ_β called Parry Measure; for almost all x the digits in the expansion of x are distributed according to the measure μ_β and we call $D(\beta)$ the set of these numbers. If β is a Pisot number of degree d

then $D(\beta) = T(\beta) \subset B(\beta)$. So by example the Champernowne number give a simple example of normal number in a Pisot base.

In order to prove this result and use the possibilities given by Ergodic Theory we define a map f_1 between the β-shift X_β and the d-dimensional torus $(\mathbb{R}/\mathbb{Z})^d$:

$$(x_n)_{n\in\mathbb{Z}} \mapsto \left(\sum_{n\in\mathbb{Z}} x_n\beta^{-n}, \ldots, \sum_{n\in\mathbb{Z}} x_n\beta^{-n+d-1}\right) \qquad \text{modulo one;}$$

this series are absolutely convergent and the map is onto; but we shall prefer the map

$$f : (x_n)_{n\in\mathbb{Z}} \mapsto \left(\sum_{n\in\mathbb{Z}} x_n \frac{\beta^{-n}}{g'(\beta)}, \ldots, \sum_{n\in\mathbb{Z}} x_n \frac{\beta^{-n+d-1}}{g'(\beta)}\right) \qquad \text{modulo one,}$$

where g design the minimal polynomial of β because this choice minimize the volume of the image $f(X_\beta)$; after many peripeties the last developments of Rauzy fractale Theory says that f is “almost bijective”. So the torus endowed with the companion map of β becomes an almost one to one factor of the β-shift. As the β-shift is a Bernoulli shift we can apply to $D(\beta)$ the theory of disjonction of processus: if add a number y with zero entropy to a number x of $D(\beta)$, $x + y$ is still in $D(\beta)$ and deterministic subsequences of the Champernowne number are still normal numbers. Entropy theory also apply to the set $X_\beta/T(\beta)$ but study this difference set is more difficult than study $T(\beta)$ itself.

Elvira Mayordomo & Jack Lutz

Efficient Computation of Absolutely Normal Numbers

Thursday, 16:00 – 16:40

We discuss recent progress on efficient algorithms for computing real numbers that are Borel normal in every base.

Friday

Hajime Kaneko

Algebraic independence of real numbers related to beta expansion and beta representation

Friday, 09:00 – 09:40

Borel conjectured that any algebraic irrational numbers are normal in each integral base $b \geq 2$. If Borel’s conjecture is true, then the average frequencies of nonzero digits in the base-b expansions of algebraic irrational numbers tend to $(b-1)/b$. Consequently, any irrational numbers whose average frequencies of nonzero digits in the base-b expansions tend to 0 are transcendental, which is still not proven.

Giving lower bounds for the numbers of nonzero digits in the binary expansions of algebraic irrational numbers, Bailey, Borwein, Crandall, and Pomerance gave a criterion for the transcendence of real numbers, which gives partial results of Borel's conjecture. Improving their method, we study algebraic independence of real numbers related to β-expansions and β-representations, where β is a Pisot or Salem number. Moreover, we also investigate criteria for the algebraic independence of the values of power series whose coefficients are unbounded.

Adrian Scheerer

(Absolute) Normality for continued fractions and beta expansions

Friday, 09:50 – 10:30

I will define normality in the setting of an ergodic dynamical system and say a few words about immediate consequences from ergodic theory. In the case of β-expansions and continued fraction expansions, by using large deviation theorems for sums of dependent random variables, I will show how to make certain constants completely explicit whose existence can be derived using methods from ergodic theory but are as such ineffective. This can be used to give computable constructions of absolutely normal numbers that are also normal to every real base that is a Pisot number, or also have a normal continued fraction expansion. This is based on modifications of the algorithms of Becher, Heiber and Slaman, and of Sierpinski.
If there is no significant overlap with talks by other participants, I shall also say something about the complexity and speed of convergence to normality of algorithms that produce absolutely normal numbers by outputting the digits of such a number to some base one after the other.

Bill Mance

Normality of different orders for the Cantor series expansions

Friday, 11:00 – 11:40

Let $\mathbb{N} = A \cup B$ be a partition of the natural numbers. We prove that if A is sufficiently close to being a subsemigroup, then there exists a basic sequence Q where the set of numbers that are Q-normal of all orders in A and not Q-normal of any orders in B is non empty. Furthermore, these sets will have full Hausdorff dimension under some additional conditions. This stands in sharp contrast to the fact that if a real number is normal of order k for the b-ary expansion, then it is also normal of all orders less than k.
