"It takes a gerbe to $\sigma$-model."

The higher supergeometry of the super-$\sigma$-model

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A class of low-dimensional field theories, termed super-\(\sigma\)-models and used to model simple geometric dynamics of extended distributions of \(\mathbb{Z}/2\mathbb{Z}\)-graded charge in homogeneous spaces of Lie supergroups, shall be reviewed, with emphasis on the supersymmetries present, both global and local. A (super)geometrisation scheme for the classes in the relevant supersymmetry-invariant (Cartan–Eilenberg) cohomology of the supersymmetry group associated with the topological charge shall be presented and basic supersymmetry-invariance and -equivariance properties of the ensuing super-gerbes shall be discussed. The general discussion shall be illustrated on a number of explicit examples, whereby, in particular, asymptotic İnönü–Wigner relations between certain physically relevant curved and flat higher supergeometric structures shall be postulated as an integral guiding principle of the (super)geometrisation scheme.
Goal:

Extending the *gerbe-theoretic approach* to the bosonic two-dimensional $\sigma$-model to (super-)$\sigma$-models with *homogeneous spaces of Lie supergroups* as target (super)spaces, in a manner consistent with *rigid and local supersymmetry*.

Discussion based upon

1. arXiv:1706.05682
2. arXiv:1808.04470
The skeleton of the talk:

I Learning from life without spin, or the higher geometry of the 2d bosonic $\sigma$-model

1. The predecessor LFT: The 2d bosonic non-linear $\sigma$-model.
2. Gerbification for the sake of (pre-)QM consistency.

II Putting a spin on it, or a $\mathbb{Z}/2\mathbb{Z}$-graded higher geometry

1. Lie supergroups à la Kostant and their homogeneous spaces.
2. The sLFT of interest: The Green–Schwarz super-$\sigma$-model.
3. A supergeometrisation scheme – the super-gerbes.
4. The dual sTFT and its vacuum.
5. Higher supersymmetry, global and local.
7. Summary & Outlook.
Part I

Learning from life without spin,
or
the higher geometry
of
the 2d bosonic $\sigma$-model
The predecessor LFT: The 2d bosonic non-linear $\sigma$-model

Given a *closed* orientable 2d $m\_fold$ $\Sigma$ (the *worldsheet*) & a metric $m\_fold (M, g)$ (the *target space*) with $H \in \mathbb{Z}_{dR}^3 (M)$, consider the theory of mappings $x \in [\Sigma, M]$ determined by (the PLA for) the *Dirac–Feynman amplitudes*

$$A_{DF} \equiv \exp \left( \frac{i}{\hbar} S^{(NG)}_\sigma [\cdot] \right) : [\Sigma, M] \rightarrow U(1)$$

$$S^{(NG)}_\sigma [x] = \mu \int_\Sigma \sqrt{|\det x^* g|} + q \int_\Sigma x^* d^{-1} H^{(3)} ,$$

describing minimal embeddings deformed by Lorentz-type forces sourced by a Maxwell-type 3-form field $H^{(3)}$.

The triple $(M, g, H^{(3)})$ is called the *$\sigma$-model background*.

**Applications:** mainly the critical bosonic string (and (mem)brane) theory, but also the effective FT of (certain slow) collective excitations of spin chains
**Problem:** May need $[H]_{dR}^{(3)} \neq 0$ (e.g., for conformality), and so

$$-\exists B \in \Omega^2(M) : dB = H^{(3)}$$

E.g., $(M, g) = (G, \kappa_g \circ (\theta_L \otimes \theta_L)) \implies H^{(3)} = \lambda \kappa_g \circ (\theta_L \wedge \theta_L \wedge \theta_L)$

and the Cartan 3-form $H$ generates $H^3_{dR}(G)$ for $G$ 1-connected

But QM à la Dirac & Feynman requires that we compare amplitudes for cobordant trajectories!

**Conclusion:** Need $S_{\sigma}^{(NG)}$ with critical points (the EL eq's) as for $[H]_{dR}^{(3)} = 0$ but s.t. $A_{DF}$ is well-defined $\forall x(\Sigma) \in Z_2(M)$. This calls for the use of a **Cheeger–Simons differential character**

$\text{Hol}_{G^{(1)}} \in \text{Hom}(Z_2(M), U(1))$ s.t. $\text{Hol}_{G^{(1)}} \circ \partial_M(\cdot) = \exp\left(\frac{i}{\hbar} \int(\cdot) H^{(3)}\right)$. 
**Solution:** Fix an arbitrary good open cover $O_M = \{O_i\}_{i \in I}$ of $M$ & a tessellation $\triangle = \Sigma_2 \sqcup \Sigma_1 \sqcup \Sigma_0$ of $\Sigma$ subordinate to it for a given $x \in [\Sigma, M]$, i.e., s.t.

$$\exists \iota \in \text{Map}(\triangle, \mathcal{I}) \quad \forall \tau \in \triangle : \quad x(\tau) \subset O_{\iota \tau},$$

and pull back, along $x$, a resolution/trivialisation of $H$ over $O_M$,

$$i.e., \text{use } b = (B_i, A_{ij}, g_{ijk}) \in \Omega^2(O_i) \times \Omega^1(O_{ij}) \times U(1)_{O_{ijk}} \text{ s.t.}$$

$$(B_j - B_i)|_{O_{ij}} = dA_{ij}, \quad (A_{jk} - A_{ik} + A_{ij})|_{O_{ijk}} = i \log g_{ijk}$$

to write (for $x_{\tau} \equiv x|_{\tau}$)

$$S_{\sigma}^{(NG), \text{top}}[x] = \sum_{p \in \Sigma_2} \left[ \int_p x^*_p B_{t_p} + \sum_{e \in \partial p} \left( \int_e x^*_e A_{t_p e} - i \sum_{v \in \partial e} \varepsilon_{ev} \log g_{t_p e e v} (x(v)) \right) \right],$$

with $A_{DF}$ well-defined iff $\delta g_{ijkl} = 1$, so that $Db = (H|_{O_i}, 0)$

and $\text{Per}(H) \subset 2\pi \mathbb{Z}$ (Dirac’s quantisation of charge)
**Upshot:** As in the Clutching Theorem, the DB 2-cocycle $b^{(2)}$ geometrises as a 1-gerbe $G^{(1)}$ [Murray & Stevenson ’94-'99].

\[ \mu_L : \text{pr}_{1,2}^* L \otimes \text{pr}_{2,3}^* L \xrightarrow{\cong} \text{pr}_{1,3}^* L \]

with the (groupoid) product $\mu_L$ on fibres of $L$ associative.

The DF amplitude acquires a rigorous interpretation

\[ \mathcal{A}_{\text{DF}}^{(\text{NG}), \text{top}} [x] \equiv \text{Hol}_{G^{(1)}}(x(\Sigma)) = \iota_1([x^* G^{(1)}]) \]

for a canonical $\iota_1 : \mathcal{W}^3(\Sigma; 0) \xrightarrow{\cong} U(1)$. 
The geometrisation prescription generalises and yields a recursive definition of \( p \)-gerbes \( \mathcal{G}^{(p)} \):

\[
\delta_Y \mathcal{G}_{-1} = 1 \quad \ldots \quad \mathcal{G}^{(p-2)} : \delta_Y \mathcal{G}^{(p-1)} \cong \mathcal{I}_0^{(p-1)} \quad \mathcal{G}^{(p-1)} , \quad \text{curv} (\mathcal{G}^{(p-1)}) = \delta_Y B^{(p+1)} \quad \mathcal{I}^{(p)}_{B^{(p+1)}}
\]
la gerbe [fr.] – spray, sheaf, wreath etc…. [Giraud ’71]
Upshot & spin-off

- geometric (pre)quantisation via cohomological transgression
  [Gawędzki ’87, rrS ’11]

\[ \tau_p : H^{p+1}(M, D(p+1)^\bullet) \longrightarrow H^1(C_p M, D(1)^\bullet), \quad C_p M \equiv [C_p, M] \]

yields a (pre)quantum bundle \( \mathcal{H}_\sigma = \Gamma_{(\text{pol})}(FL_\sigma \times \mathbb{C} \times \mathbb{C}) \), where

\[ \mathbb{C} \times \longrightarrow \pi_{T^*C_p M}^* \mathcal{L}_{G(p)} \otimes \mathcal{I}_{\mathcal{L}^{(0)}_{T^*C_p M}} \equiv \mathcal{L}_\sigma, \mathcal{A}_L^{(1)} \]

\[ P_\sigma \equiv T^*C_p M, \quad \Omega_\sigma = \delta \mathcal{L}_{T^*C_p M} + \pi_{T^*C_p M}^* \mathcal{L}_{G(p+2)} = \text{curv}(\mathcal{A}_{L_\sigma}^{(1)}) \]

for \( \mathcal{L}_{G(p)} \in \tau_p([G(p)]) \), and hence – classification of \( \sigma \)-models;

- geometrisation and classification of topological defects/dualities
  [Fuchs et al. ’07, Runkel & rrS ’08, rrS ’11-’12], in particular…
... (pre)quantisable config\textsuperscript{nal} symmetries – induced from actions

\[ \lambda : G_\sigma \times M \longrightarrow M : (g, m) \mapsto \lambda_g(m) \]

of (Lie) groups \( G_\sigma \subset \text{Isom}(M, g) \) that are generalised \( H \)-hamiltonian,

\[ \forall X \in \text{Lie}(G_\sigma) \exists \kappa_X \equiv (T(e, \cdot)\lambda_X, \kappa_X) \in \Gamma(E^{(1,p)}M) : d_H \kappa_X = 0, \]

so that the \( H \text{-twisted Vinogradov bracket} \)

\[ \left[[\cdot, \cdot]\right]_H^{(p+2)} : \Gamma(E^{(1,p)}M) \times \Gamma(E^{(1,p)}M) \longrightarrow \Gamma(E^{(1,p)}M) \]

\[ \left[[\nu_1, \nu_1), (\nu_2, \nu_2)\right]_H^{(p+2)} \]

\[ = (\left[\nu_1, \nu_2\right], \mathcal{L}\nu_1 \nu_2 - \mathcal{L}\nu_2 \nu_1 - \frac{1}{2}d(\nu_1 \nu_2 - \nu_2 \nu_1) + \nu_1 \nu_2 \text{H}_{(p+2)}) \]

closes on their set \( \Gamma(E^{(1,p)}M)_{H-ham} \subset \Gamma(E^{(1,p)}M) \).
We distinguish

global/rigid symmetries
(set inequivalent field configurations in $\mathcal{A}_{DF}$-correspondence)
lift to families of $p$-gerbe 1-isomorphisms

$$\Phi_g : \lambda_g^* G^{(p)} \cong G^{(p)}$$

that transgress to automorphisms of $\mathcal{H}_\sigma$, e.g., for $p = 1$,

$$\alpha_E : \text{pr}^*_{1,3} \lambda_g^*[2]^* L \otimes \text{pr}^*_{3,4} E \xrightarrow{\cong} \text{pr}^*_{1,2} E \otimes \text{pr}^*_{2,4} L \quad \mathbb{C} \times \rightarrow E, A_E$$

$$(\alpha_E, \mu_L) \text{ comp bl } \quad \text{curv}(A_E) = \lambda_g^* B - B$$

$$\cong \lambda_g^* Y M \times_M Y M \equiv Y g M$$
\( \rightarrow \) local/gauge config\^{n}al symmetries

(\text{relate} \text{ equivalent} \text{ descriptions of a field configuration})

Gauging of \( G_\sigma \) models \text{descent to the orbispace} \( M \longrightarrow M/G_\sigma \)

\textbf{Th}^m (\textbf{Principle of Descent}) [Gawędziński, Waldorf & rrS ’10]

For \( \lambda : G_\sigma \times M \longrightarrow M \) free and proper,

\[ \mathcal{B}Grb^{(p)}(M/G_\sigma) \cong \mathcal{B}Grb^{(p)}(M)^{(G_\sigma, \varrho_\lambda = 0)}, \]

where the RHS is the (weak \((p + 1)-\))category of \( p \)-gerbes over \( M \) with a \( G_\sigma \)-equivariant structure relative to a \textit{vanishing}

\[ \varrho_\lambda \in \Omega^{p+1}(G_\sigma \times M) : \quad d\varrho_\lambda = (\lambda^* - \text{pr}_2^*)H. \]

The structure is an extension of the 0-cell \( G^{(p)} \) to a \((p + 2)\)-tuple \((G^{(p)}, \gamma^{(p)}, \gamma^{(p-1)}, \ldots, \gamma^{(0)})\) over \( N^\bullet(G_\sigma \ltimes \lambda M) \) based on a 1-isomorphism

\[ \gamma^{(p)} : \lambda^*G^{(p)} \xrightarrow{\cong} \text{pr}_2^*G^{(p)} \otimes I_{\varrho_\lambda}^{(p)}. \]
The many faces of a $G_\sigma$-equivariant structure

- an extension of the $(p+1)$-cocycle of $H^{p+1}(M, D(p+1))$ for $G_p$ to a $(p+1)$-cocycle in an extension of the Čech–de Rham bicomplex in the direction of $G_\sigma$-cohomology;

- [rrS ’12] geometric data for the topological gauge-symmetry defect of the $\sigma$-model over $\Omega_p$ (based on [Runkel & rrS ’09]).

Generically, $\varrho_\lambda$, as determined by the $\mathfrak{K}_{t_A}$ for $\text{Lie}(G_\sigma) \equiv g_\sigma = \bigoplus_{A=1}^{D} \langle t_A \rangle$, is non-vanishing, and so we need...

Universal Gauge Principle

[GW ’02–03, GW, Waldorf & rrS ’07–13, rrS ’11–13]

$A_{DF}$ admits gauging of $G_\sigma$ via ‘minimal coupling’ of $A \in \Omega^1(P_{G_\sigma}) \otimes g_\sigma$ if

1. [GW, Waldorf & rrS ’07–13, rrS ’11–13, ’19] $SGA=0$

\[ \iff \left( \bigoplus_{A=1}^{D} C^\infty(M, \mathbb{R}) \mathfrak{K}_{t_A}, [[\cdot, \cdot]]_{H^{p+2}} \right) \cong g_\sigma \ltimes_\lambda M; \]

2. [GW, Waldorf & rrS ’07–13] $LGA=0$

\[ \iff \text{exists a } G_\sigma\text{-equivariant structure on } G^{(p)} \text{ rel. to } \varrho_\lambda. \]
Applications:

- geometrisation and cohomological classification of obstructions against gauging and of inequivalent gaugings, and hence
- natural mapping of the moduli space of $\sigma$-models, with beautiful connections to TFT (explicit constructions for ‘all’ 2d RCFTs)
- reconstruction of T-duality outside the topological context...
The Higher Dogmatics: The Three+ $\mathcal{G}$-Sluagh-ghairms

$\Downarrow$ It matters iff it lifts to $\mathcal{G}$.

$\Downarrow\Downarrow$ Global symmetry is invariance of $\mathcal{G}$.

$\Downarrow\Downarrow\Downarrow$ Local config$^{\text{nal}}$ symmetry is equivariance of $\mathcal{G}$.

$\Downarrow\Downarrow\Downarrow\Downarrow$ ... (Duality/top$^{\text{al}}$ defect is a $\mathcal{G}$-bimodule etc.) ...
Part II

Putting a spin on it,
or

a $\mathbb{Z}/2\mathbb{Z}$-graded higher geometry
The goal

The higher sgeometry of a super(geometric/symmetric)-$\sigma$-model of (generalised-minimal) ‘embeddings’

$$[\Omega_p, \mathcal{M}] = ?$$

of a $(\rho + 1)$-dimensional riemannian worldvolume $\Omega_p$ ‘in’ a \textit{sm\_fold} $\mathcal{M}$ endowed with an action

$$\lambda : G \times \mathcal{M} \longrightarrow \mathcal{M} \quad (?)$$

of a \textit{supersymmetry} Lie group $G$.

Physical motivation

Understanding the (s)geometric structure (\textit{sensu largissimo}) of superstring theory-inspired & -related FTs, with view to elucidation of the deep nature of the tremendously robust yet notoriously elusive AdS/CFT correspondence.
Sm_folds \( \mathcal{M} = (|\mathcal{M}|, \mathcal{O}_{\mathcal{M}}) \) with body \(|\mathcal{M}| \in \text{Ob} \ TopMan\) and structure sheaf \( \mathcal{O}_{\mathcal{M}} : \mathcal{T}(|\mathcal{M}|)^{\text{op}} \rightarrow \text{sAlg}_{\text{scmm}}\),

\[ \mathcal{O}_{\mathcal{M}} \sim_{\text{loc}} (\mathbb{R}^m, C^\infty(\cdot, \mathbb{R}) \otimes \wedge^\bullet \mathbb{R}^n) \equiv \mathbb{R}^{m|n}, \]

form a category \( \text{sMan} \) with morphisms

\[ \varphi \equiv (|\varphi|, \varphi^*) : (|\mathcal{M}_1|, \mathcal{O}_{\mathcal{M}_1}) \longrightarrow (|\mathcal{M}_2|, \mathcal{O}_{\mathcal{M}_2}), \]

\[ |\varphi| \in \text{Hom}_{\text{TopMan}}(|\mathcal{M}_1|, |\mathcal{M}_2|), \quad \varphi^* : \mathcal{O}_{\mathcal{M}_2} \rightarrow |\varphi|^* \mathcal{O}_{\mathcal{M}_1} \]

It admits products \( \mathcal{M}_1 \times \mathcal{M}_2 = (|\mathcal{M}_1| \times |\mathcal{M}_2|, \mathcal{O}_{\mathcal{M}_1} \hat{\otimes} \mathcal{O}_{\mathcal{M}_2}). \)

By the Yoneda Lemma, \( \text{Yon.} : \text{sMan} \rightarrow \text{Presh}(\text{sMan}) \), and so

\[ \mathcal{M} \sim \text{Yon}_\mathcal{M}(-) \equiv \text{Hom}_{\text{sMan}}(-, \mathcal{M}) : \text{sMan}^{\text{op}} \rightarrow \text{Set}, \]

with \( \text{Yon}_\mathcal{M}(S) \equiv \text{Hom}_{\text{sMan}}(S, \mathcal{M}) \) the set of \( S \)-points in \( \mathcal{M} \), and

\[ \varphi \sim \text{Yon}_\varphi(-) \equiv \text{Hom}_{\text{sMan}}(-, \varphi) = \varphi^0 \]

with \( \text{Yon}_\varphi(S) : \text{Yon}_{\mathcal{M}_1}(S) \longrightarrow \text{Yon}_{\mathcal{M}_2}(S). \)
With the help of local charts \((|U_l| \in \mathcal{T}(|M_l|), l \in \{1, 2\})\)

\[\kappa_l : (|U_l|, \mathcal{O}_{M_l}|_{|U_l|}) \equiv U_l \xrightarrow{\sim} (|W_l|, \mathcal{C}^\infty(\cdot, \mathbb{R}) \otimes \wedge \cdot \mathbb{R}^{|n_l|}) \equiv W_l,\]

with the corresponding local coordinates \((x^a_l, \theta^\alpha_l)(a, \alpha) \in 1, m_l \times 1, n_l\), the above yields a \textit{local description of morphisms}

\[\varphi_{1,2} \equiv \kappa_2 \circ \varphi \circ \kappa_1^{-1} \in \text{Hom}_{s\text{Man}}(W_1, W_2) \equiv \text{Yon}_W(W_1)\]

determined (as are \textit{all} \(W_1\)-points in \(W_2\) in virtue of the LCThm) by

\[x^{a_2}_2(\theta_1, x_1) \sim \varphi^*_{1,2}(x^{a_2}_2) = \sum_{k=0}^{q_1} \theta_{1}^{\alpha_1^1} \theta_{1}^{\alpha_1^2} \cdots \theta_{1}^{\alpha_1^k} \Phi_{a_1^2 \alpha_1^2 \cdots \alpha_1^k}(x^{b_1}_1),\]

\[\theta^{a_2}_2(\theta_1, x_1) \sim \varphi^*_{1,2}(\theta^{a_2}_2) = \sum_{l=0}^{q_1} \theta_{1}^{\alpha_l^1} \theta_{1}^{\alpha_l^2} \cdots \theta_{1}^{\alpha_l^l} \Phi_{\alpha_1^2 \alpha_1^2 \cdots \alpha_1^l}(x^{b_1}_1),\]

where \(\Phi_{\alpha_1^2 \alpha_1^2 \cdots \alpha_1^{2r+1}} \equiv 0 \equiv \Phi_{\alpha_1^2 \alpha_1^2 \cdots \alpha_1^{2r}}\).

\textbf{Upshot:} \text{Hom}_{s\text{Man}}(\Omega_p, \mathcal{M}) \text{ ruled out} \text{ as a candidate for } [\Omega_p, \mathcal{M}].
Instead [Freed ’95],

\[ [\Omega_p, \mathcal{M}] \equiv \text{Hom}_{s\text{Man}}(\Omega_p, \mathcal{M}) := \text{Hom}_{s\text{Man}}(\Omega_p \times -, \mathcal{M}) \]

\[ \in \text{Ob} \text{Presh}(s\text{Man}), \]

to be evaluated on the odd hyperplanes

\[ \mathbb{R}^{0|N} \equiv (\{\bullet\}, \mathbb{R}[\eta^1, \eta^2, \ldots, \eta^N]) \]

whereupon \( \xi \in [\Omega_p, \mathcal{M}](\mathbb{R}^{0|N}) \) decompose (locally) as

\[ \xi^* (x^a) = \xi_0^a + \xi_{i_1 i_2}^a \eta^i \eta^{j_2} + \cdots + \xi_{i_1 i_2 \cdots 2[N/2]}^a \eta^{i_1} \eta^{j_2} \cdots \eta^{j_2[N/2]}, \]

\[ \xi^* (\theta^\alpha) = \xi_i^\alpha \eta^i + \xi_{i_1 i_2 i_3}^\alpha \eta^i \eta^{j_2} \eta^{j_3} + \cdots + \xi_{i_1 i_2 \cdots 2[N-1]/2+1}^\alpha \eta^{i_1} \eta^{j_2} \cdots \eta^{j_2[N-1]/2+1} \]

& the \( (\xi_{i_1 i_2 \cdots i_k}^\alpha, \xi_{i_1 i_2 \cdots i_k}^a) \) become the (s)fields of the super-\( \sigma \)-model.
The next fundamental issue is **supersymmetry**, for which we need **Lie sgroups**, *i.e.*, group objects in **sMan**, 

\[ (G = (|G|, \mathcal{O}_G), \mu : G \times G \to G, \text{Inv} : G \circ, \varepsilon : \mathbb{R}^{0|0} \to G), \]

with body \( |G| \in \text{Ob LieGrp} \). On these, we have **LI vector fields**

\[
\mathcal{O}_G \hat{\otimes} \mathcal{O}_G \leftarrow \mu^* \mathcal{O}_G \\
L \in \Gamma(\mathcal{T}G) : \text{id}_{\mathcal{O}_G} \otimes L \quad \downarrow \quad \text{id}_{\mathcal{O}_G} \otimes L \\
\mathcal{O}_G \hat{\otimes} \mathcal{O}_G \leftarrow \mu^* \mathcal{O}_G
\]

and the dual **LI 1-forms**. The RI objects are defined analogously. The supersymmetry groups are to act on the (s)fields as *per*

\[ \lambda : G \times \mathcal{M} \to \mathcal{M}, \quad \text{and so also} \]

\[ |\lambda| : |G| \to \text{Aut}_{\text{sMan}}(\mathcal{M}) : g \mapsto \lambda \circ (\hat{g} \times \text{id}_{\mathcal{M}}) \equiv |\lambda|_g, \]

where \( \hat{g} : \mathbb{R}^{0|0} \to G \) are the topological points.
In sFT’s, Lie sgroups usually appear in disguise…

\textbf{Th}^{m}[\text{Kostant }'77] \ s\text{LieGrp} \cong \ s\text{HCp},

where \( s\text{HCp} \) is the category of \textbf{super-Harish-Chandra pairs} \( G \equiv (|G|, g, \rho) \)

\(|G| \in \text{Ob LieGrp}, \quad g = g^{(0)} \oplus g^{(1)} \in \text{Ob sLieAlg} \quad \text{s.t.} \quad g^{(0)} \cong \text{Lie} (|G|), \)

\[ \rho : |G| \longrightarrow \text{End}_{\text{sLieAlg}} (g) \quad \text{s.t.} \quad \rho(\cdot)|_{g^{(0)}} \equiv T_e \text{Ad}. \]

with morphism \( (\Phi, \phi) : (|G_1|, g_1, \rho_1) \longrightarrow (|G_2|, g_2, \rho_2) \)

\( \Phi \in \text{Hom}_{\text{LieGrp}} (|G_1|, |G_2|), \quad \phi \in \text{Hom}_{\text{sLieAlg}} (g_1, g_2) \quad \text{s.t.} \quad \phi|_{g^{(0)}} = T_e \Phi, \quad (\rho_2 \circ \Phi(\cdot)) \circ \phi = \phi \circ \rho_1(\cdot) \)

\textbf{Remark:} \( \cong \) uses the Hopf-superalgebra structure on \( U(g) \) and yields

\[ \mathcal{O}(|G|, g, \rho) = \text{Hom}_{U(g^{(0)})-\text{Mod}} (U(g), C^\infty (\text{--}, \mathbb{R})) \sim C^\infty (\text{--}, \mathbb{R}) \otimes \wedge \cdot g^{(1)*} \]
Examples of Lie sgroups:

- \( \text{sMink}(d, 1 \mid D_{d,1}) \) as an abstract Lie sgroup is

\[
\text{sMink}(d, 1\mid D_{d,1}) = (\mathbb{R}^{d+1}, C^\infty(\cdot, \mathbb{R}) \otimes \bigwedge \mathbb{R}^{D_{d,1}}), \quad D_{d,1} = \dim S_{d,1},
\]

with \( S_{d,1} \) a distinguished Majorana-spinor Cliff \((\mathbb{R}^{d,1})\)-module.

It admits global coörds \( \{ x^a, \theta^\alpha \} \) \((a, \alpha) \in \mathbb{N} \times \mathbb{N} \) and

\[
\mu^* : (x^a, \theta^\alpha) \mapsto (x^a \otimes 1 + 1 \otimes x^a - \frac{1}{2} \theta^\alpha \otimes (C \Gamma^a)_{\alpha\beta} \theta^\beta, \theta^\alpha \otimes 1 + 1 \otimes \theta^\alpha),
\]

\[
\text{Inv}^* : (x^a, \theta^\alpha) \mapsto (\bar{x}^a, -\theta^\alpha),
\]

or, equivalently, in the \( S \)-point picture,

\[
(x^a_1, \theta^\alpha_1) \cdot (x^b_2, \theta^\beta_2) = (x^a_1 + x^a_2 - \frac{1}{2} \theta_1 \bar{\Gamma}^a \theta_2, \theta^\alpha_1 + \theta^\alpha_2), \quad (x^a, \theta^\alpha)^{-1} = (-x^a, -\theta^\alpha)
\]

As a sHCp,

\[
\text{sMink}(d, 1 \mid D_{d,1}) = (\text{Mink}(d, 1), s\text{min}^\epsilon(d, 1 \mid D_{d,1}) = \bigoplus_{a=0}^{d} \langle P_a \rangle \oplus \bigoplus_{\alpha=1}^{D_{d,1}} \langle Q_\alpha \rangle, 0),
\]

\[
\{ Q_\alpha, Q_\beta \} = (C \Gamma^a)_{\alpha\beta} P_a, \quad [P_a, P_b] = 0 = [Q_\alpha, P_a].
\]
• $SU(2, 2 | 4)$ as a sHCp with the body Lie group
  \[ |SU(2, 2 | 4)| = \text{Spin}(4, 2) \times \text{Spin}(6), \]

the Lie algebra

\[
su(2, 2 | 4) = \left( \bigoplus_{a=0}^{4} \langle P_{a} \rangle \oplus \bigoplus_{a'=5}^{9} \langle P_{a'} \rangle \right) \oplus \bigoplus_{(\alpha, \alpha', l) \in \{1,4 \times 1,4 \times 1,2\}} \langle Q_{\alpha \alpha' l} \rangle \oplus \bigoplus_{a', b'=5}^{9} \langle J_{a'b'} = -J_{b'a'} \rangle \]

\[\{Q_{\alpha \alpha' l}, Q_{\beta \beta' J}\} = i (-2(\hat{C} \hat{\Gamma}^{\hat{a}} \otimes 1)_{\alpha \alpha' l \beta \beta' J} P_{\hat{a}} + (\hat{C} \hat{\Gamma}^{\hat{a} \hat{b}} \otimes \sigma_{2})_{\alpha \alpha' l \beta \beta' J} J_{\hat{a} \hat{b}}),\]

\[ [Q_{\alpha \alpha' l}, P_{\hat{a}}] = -\frac{1}{2} (\hat{\Gamma}_{\hat{a}} \otimes \sigma_{2})_{\alpha \alpha' l}^{\beta \beta' J} Q_{\beta \beta' J}, \quad [P_{\hat{a}}, P_{\hat{b}}] = \varepsilon_{\hat{a} \hat{b}} J_{\hat{a} \hat{b}}, \quad \varepsilon_{\hat{a} \hat{b}} = \begin{cases} +1 & \text{if } \hat{a}, \hat{b} \in 0,4 \\ -1 & \text{if } \hat{a}, \hat{b} \in 5,9 \\ 0 & \text{otherwise} \end{cases}, \]

\[ [J_{\hat{a} \hat{b}}, J_{\hat{c} \hat{d}}] = \eta_{\hat{a} \hat{c}} J_{\hat{b} \hat{d}} - \eta_{\hat{a} \hat{d}} J_{\hat{b} \hat{c}} + \eta_{\hat{b} \hat{c}} J_{\hat{a} \hat{d}} - \eta_{\hat{b} \hat{d}} J_{\hat{a} \hat{c}}, \]

\[ [Q_{\alpha \alpha' l}, J_{\hat{a} \hat{b}}] = -\frac{1}{2} \varepsilon_{\hat{a} \hat{b}} (\hat{\Gamma}_{\hat{a} \hat{b}} \otimes 1)_{\alpha \alpha' l}^{\beta \beta' J} Q_{\beta \beta' J}, \quad [P_{\hat{a}}, J_{\hat{b} \hat{c}}] = \eta_{\hat{a} \hat{b}} P_{\hat{c}} - \eta_{\hat{a} \hat{c}} P_{\hat{b}}.\]

and the standard spinor realisation of the former

on the Graßmann-odd component of the latter.
Sgeometric data: A Nambu–Goto sbackground

\[(\mathcal{M}, g_{(p + 2)}) \equiv \mathcal{B}_{NG}^{(p)}\]

of a super-\(\sigma\)-model consists of

- a manifold \(\mathcal{M}\) (the starget) with an action \(\lambda\) of a Lie sgroup \(G\) (the supersymmetry group), inducing fundamental vector fields

\[
\kappa_\cdot : g \equiv \Gamma(TG)^L \rightarrow T\mathcal{M} : L \mapsto -(\hat{e}^* \circ L \otimes \text{id}_{\mathcal{O}\mathcal{M}}) \circ \lambda^* ;
\]

- a G-invariant smetric \(g \in \Gamma(T^*\mathcal{M} \otimes^{\text{sym}} T^*\mathcal{M})\),

\[
\forall (g, X) \in |G| \times g : (|\lambda|^* g = g \land \mathcal{L}_{\kappa_X} g = 0) ;
\]

- a G-invariant de Rham \((p + 2)\)-scocycle \(H_{(p + 2)} \in \mathbb{Z}_{dR}^{p+2}(\mathcal{M})\),

\[
\forall (g, X) \in |G| \times g : (|\lambda|^* H_{(p + 2)} = H_{(p + 2)} \land \mathcal{L}_{\kappa_X} H_{(p + 2)} = 0) ;
\]
THE geometry: There is a large class of $\mathfrak{sB}_{\text{NG}}^{(p)}$ with G-orbits as stargets...

Thm [Kostant ’77, Koszul ’82, Fioresi et al. ’07]:
Let $G \in \text{Ob sLieGrp}$ and $H$ its Lie sub-sgroup with $\text{sLie } H \equiv \mathfrak{h}$.
$\exists$ an ess. unique sm_fold structure on the homogeneous space

$$G/H = (|G|/|H|, \mathcal{O}_{G/H}) \quad \text{s.t.}$$

$$\mathcal{O}_{G/H}(\cdot) = \{ f \in \mathcal{O}_G(\pi_{|G|/|H|}^{-1}(\cdot)) \mid \forall (h,J) \in |H| \times \mathfrak{h} : |\phi|^*_h(f) = f \land L_J(f) = 0 \}$$

$$\begin{align*}
G \times G & \xrightarrow{\ell} G - - - - - \rightarrow |G| \\
id_G \times \pi_{G/H} & \downarrow \quad \pi_{G/H} \downarrow \\
G \times G/H & \xrightarrow{[\ell]} G/H - - - - - \rightarrow |G|/|H| \\
\pi_{|G|/|H|} & \downarrow \\
|G|/|H| \equiv |\pi_{G/H}|.
\end{align*}$$

Actually, $(G, \pi_{G/H}, H)$ is a principal H-(s)bundle with local sections

$$\sigma_U : U \equiv (|U|, \mathcal{O}_{G/H}|_U) \rightarrow G \quad \text{with body}$$

$$|\sigma_U| : |U| \rightarrow |G|, \quad \pi_{|G|/|H|} \circ |\sigma_U| = \text{id}_{|U|}.$$
Dynamics with a nonlinear realisation of supersymmetry calls for a reductive homogeneous space: 

\[ G/H \text{ for } (G, H \subset |G|) \text{ with } s\text{Lie}(G) = \mathfrak{g} \text{ and } \text{Lie}(H) = \mathfrak{h} \text{ s.t.} \]

\[ \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{h}, \quad \mathfrak{t} = \mathfrak{t}^{(0)} \oplus \mathfrak{t}^{(1)} \equiv \bigoplus_{a=0}^{d_0} \langle P_a \rangle \oplus \bigoplus_{\alpha=1}^{d_1} \langle Q_\alpha \rangle, \quad \mathfrak{h} = \bigoplus_{\kappa=1}^{d_S} \langle J_\kappa \rangle \]

is reductive, i.e., s.t. \([\mathfrak{h}, \mathfrak{t}] \subset \mathfrak{t}\). For these, the LI \(\mathfrak{g}\)-valued Maurer–Cartan 1-sform

\[ \theta_L = \theta_L^\mu \otimes t_\mu + \theta_L^\kappa \otimes J_\kappa, \quad \bigoplus_{\mu=0}^{d_0+d_1} \langle t_\mu \rangle \equiv \mathfrak{t} \]

yields (a principal H-connection \(\Theta = \theta_L^\kappa \otimes J_\kappa\) and) H-stensors

\[ |\varphi|^* \theta_L^\mu = \rho(\cdot)^\mu_\nu \theta_L^\nu. \]

that give rise to H-basic (cov.) stensors

\[ T = \tau_{\mu_1 \mu_2 \ldots \mu_n} \theta_L^{\mu_1} \otimes \theta_L^{\mu_2} \otimes \cdots \otimes \theta_L^{\mu_n} \]

for

\[ \tau_{\mu_1 \mu_2 \ldots \mu_n} = \tau_{\nu_1 \nu_2 \ldots \nu_n} \rho(h)^{\nu_1}_{\mu_1} \rho(h)^{\nu_2}_{\mu_2} \cdots \rho(h)^{\nu_n}_{\mu_n}, \quad h \in H. \]
THE stensorial data: Model the starget $G/H$ (patchwise) by

$$\Sigma^{NG} := \bigsqcup_{i \in \mathcal{I}} \sigma_i(U_i), \quad \sigma_i : U_i \equiv (|U_i|, \mathcal{O}_{G/H}|_{U_i}) \rightarrow G$$

for an open cover $\{|U_i|\}_{i \in \mathcal{I}}$ of $|G|/H$ trivialising for the body principal $H$-bundle $(|G|, \pi_{|G|/H}, H)$, and subsequently pull back an $H$-basic LI ‘smetric’

$$g = g_{(ab)} \theta^a_L \otimes \theta^b_L, \quad g = \pi_{G/H}^* g$$

and an $H$-basic LI de Rham $(\rho + 2)$-scocycle

$$\chi_{(\rho + 2)} = \chi_{\mu_1 \mu_2 \ldots \mu_{\rho + 2}} \theta^\mu_1_L \wedge \theta^\mu_2_L \wedge \cdots \wedge \theta^\mu_{\rho + 2}_L, \quad \chi_{(\rho + 2)} = \pi_{G/H}^* H_{(\rho + 2)}$$

to $G/H$ along the $\sigma_j$, resp. use their precursors $(g, H)_{(\rho + 2)}$. 
Examples of reductive homogeneous spaces of Lie sgroups:

- $s\text{Mink}(d, 1 \mid D_{d,1}) \equiv s\text{ISO}(d, 1 \mid D_{d,1})/\text{Spin}(d, 1)$ for $s\text{ISO}(d, 1 \mid D_{d,1}) = s\text{Mink}(d, 1 \mid D_{d,1}) \rtimes L_{d,1} \oplus S_{d,1} \text{Spin}(d, 1)$, with

$$g = \eta_{ab} \theta^a_L \otimes \theta^b_L,$$

$$H^{(p + 2)} = \begin{cases} 
\theta^\alpha_L \wedge (C \Gamma_{11})_{\alpha\beta} \theta^\beta_L & (p = 0) \\
\theta^\alpha_L \wedge (C \Gamma_{a_1 a_2 ... a_p})_{\alpha\beta} \theta^\beta_L \wedge \theta^a_{L_1} \wedge \theta^a_{L_2} \wedge \ldots \wedge \theta^a_{L_p} & (1 < p < 8)
\end{cases}$$

the admissible $(d, p, N)$ filling up the ‘old brane scan’

- $s(\text{AdS}_5 \times S^5) \equiv \text{SU}(2, 2\mid 4)/(\text{Spin}(4, 1) \times \text{Spin}(5))$, with

$$g = \eta_{ab} \theta^a_L \otimes \theta^b_L + \delta_{a'b'} \theta^a_{L'} \otimes \theta^b_{L'},$$

$$H^{(3)} = \theta^\alpha_{L} \wedge (\hat{C} \Gamma_{\hat{a}} \otimes \sigma_3)_{\alpha\alpha' \beta \beta'} \theta^\beta_{L} \wedge \theta^\beta_{L} \wedge \theta^\hat{a}_{L}$$
**THE super-σ-model**

Given a *closed* orientable manifold $\Omega_p$ of $\dim \Omega_p = p + 1$, a Lie group $G$ and a closed Lie subgroup $H \subset |G|$ with $(\mathfrak{g}, \mathfrak{h})$ reductive, assume given $H$-basic LI stensors on $G$:

$$g = g_{(ab)} \theta^a_L \otimes \theta^b_L \equiv \pi^*_G/H g, \quad \chi \equiv \pi^*_G/H \in \mathbb{Z}^{p+2}_{d\mathbb{R}}(G)^G.$$ 

The **Green–Schwarz super-σ-model in the Nambu–Goto formulation** is a theory of smappings $[\Omega_p, G/H] \ni \xi$ determined by the PLA for the DF (s)amplitudes with

$$S^{(NG), (\mu_p)}_{GS}[\xi] = \mu_p \int_{\Omega_p} \sqrt{\det (\xi^* g)} + \int_{\Omega_p} \xi^* d^{-1} H_{(p + 2)},$$

where $\mu_p \in \mathbb{R}^\times$ is a parameter*.

*To be fixed in what follows.
General remarks:

The svacuum of the super-$\sigma$-model is a ‘minimal’ sembedding distorted by Lorentz-type sforces. Its ‘localisation’ effects

- a spontaneous breakdown $H \downarrow H_{\text{vac}}$ of the ‘invisible’ gauge symmetry (the isotropy group);

- a spontaneous breakdown $t^{(0)} \downarrow t^{(0)}_{\text{vac}}$ of the local translational symmetry.

Implication: A need for a mechanism of restoration of supersymmetry in the svacuum through freeze-out of the Graßmann-odd DOFs, as dictated by

$$\{ Q_\alpha, Q_\beta \} = f_{\alpha\beta}^a P_a + f_{\alpha\beta}^\kappa J_\kappa,$$

which puts us in the context of the $\kappa$-symmetry of [de Azcárraga & Lukierski ’82, Siegel ’83] –

“a ‘hidden’ symmetry, with no evident geometric interpretation”…
Physically relevant models:

(i) the original Green–Schwarz–... $p$-sbranes in  
$s\text{Mink}(d, 1|N D_{d, 1}) \equiv s\text{ISO}(d, 1|N D_{d, 1})/\text{Spin}(d, 1), \ N \in \mathbb{N}^\times$;

(ii) the Metsaev–Tseytlin sstring in  
$s(\text{AdS}_5 \times S^5) \equiv SU(2, 2|4)/(\text{Spin}(4, 1) \times \text{Spin}(5))$;

(iii) the Zhou s-0-brane and sstring in  
$s(\text{AdS}_2 \times S^2) \equiv SU(1, 1|2)_2/(\text{Spin}(1, 1) \times \text{Spin}(2))$;

(iv) the Park–Rey sstring in  
$s(\text{AdS}_3 \times S^3) \equiv SU(1, 1|2)_2^\times2/(\text{Spin}(2, 1) \times \text{Spin}(3))$;

(v) the Metsaev–Tseytlin D3-brane in  
$s(\text{AdS}_5 \times S^5) \equiv SU(2, 2|4)/(\text{Spin}(4, 1) \times \text{Spin}(5))$;

(vi) the M2-branes in $s(\text{AdS}_4 \times S^7)$ and $s(\text{AdS}_7 \times S^4)$...
Empirical facts:

(H) The $\rho$-sbrenes in $s\text{Mink}(d, 1|ND_{d, 1})$ and the 0-sbrane in $s(\text{AdS}_2 \times S^2)$ have

$$[\chi]_{dR}^{(\rho + 2)} = 0 \, , \text{ but } \quad [\chi]_{dR}^{G} \in \text{CaE}^{\rho + 2}(G) \setminus \{0\} .$$

(İW) the ssstrings in $s(\text{AdS}_q \times S^q)$, $q \in \{2, 3, 5\}$ have

$$[\chi]_{dR}^{G} = 0 \in \text{CaE}^{3}(G) ,$$

but the supersymmetric primitives
do NOT İnönü-Wigner–contract
to the sminkowskian ones.
What are the **PROBLEMS** with the empirical facts?

**Ad (˙IW)** Signals potential ‘ill-definedness’ of the MT/PR/Zh super-σ-models whose construction was **based upon** the asymptotic correspondence with the GS super-σ-model. [rrS ’18]

**Ad (H)** The choice of the cohomology critical for the meaning of $\mathcal{A}_{DF}^{(\text{top})}$.

AND

Physics favours the (H-equivariant) Cartan–Eilenberg cohomology

$$\text{CaE}^\bullet(G)_{H-\text{equiv}} \equiv \mathcal{H}^\bullet_{dR}(G)^G_{H-\text{equiv}},$$

BUT

*(How) Does CaE$^\bullet(G) \setminus \mathcal{H}^\bullet(G)$ topologise?*
The Rabin-Crane-type argument/hypothesis:

Secretly, the GS super-$\sigma$-model for $[\Omega_p, G/H \equiv \mathcal{M}]$ is a theory of smappings from $[\Omega_p, \mathcal{M}/\Gamma_{KR}]$ for $\Gamma_{KR} \subset G$ s.t.

$$\mathcal{M}/\Gamma_{KR} \cong_{\text{loc.}} \mathcal{M} \land \mathcal{H}_d^d (\mathcal{M})^G \cong \mathcal{H}_d^d (\mathcal{M}/\Gamma_{KR}).$$

A working model

For $\mathcal{M} = \text{sMink}(d, 1|D_{d,1})$, the sub-sgroup was identified in [Crane & Rabin ’85] as the discrete Kostelecký-Rabin sgroup generated by integer translations

$$(x^a, \theta^\alpha) \mapsto (y^b, \varepsilon^\beta) \cdot (x^a, \theta^\alpha)$$

with $y^b_{i_1i_2...i_k}, \varepsilon^\alpha_{i_1i_2...i_k} \in \mathbb{Z}$ (in the $S$-point picture).
Field-theoretic consequences:

We must take into account the $\Gamma_{KR}$-twisted sector in $[\Omega p, G/H]$, but then the Poisson-Lie subalgebra of the Noether charges of supersymmetry of the GS super-$\sigma$-model,

$$\{h_A, h_B\}_{\Omega_\sigma} = -f_{AB}^\ C h_C + A_{AB},$$

exhibits a (classical!) wrapping anomaly [rrS ’18].

Empirical fact: Some of these extensions trivialise distinguished 2-scocycles on the supersymmetry subalgebra $g$.

Conclusion: Need to consider scentral extensions

$$0 \rightarrow \tilde{z} \rightarrow \tilde{g} \rightarrow g \rightarrow 0.$$ 

The latter is merely an (exact) (s)intuition with... a rigorous cohomology story behind it...
Towards sgeometrisation of supersymmetric de Rham scocycles...

**Thm:** $\exists$ an isomorphism

$$[\gamma] : H^{\bullet}(g, \mathbb{R}) \cong CE^{\bullet}(g, \mathbb{R}) \xrightarrow{\cong} CaE^{\bullet}(G) \cong H^{\bullet}_{dR}(G)^G.$$

**Thm:** $\exists$ a correspondence

$$\text{CE}^2(g, a) \xleftarrow{1:1} \{ \text{equivalence classes of scentral extensions of } g \text{ by } a \},$$

where

$$\tilde{g}_2 \sim \tilde{g}_1 \iff \begin{array}{c} \tilde{g}_1 \\ \downarrow \downarrow \downarrow \downarrow \downarrow \\ \tilde{g}_2 \end{array} \xrightarrow{\cong} \begin{array}{c} 0 \\ \longrightarrow \longrightarrow \longrightarrow \longrightarrow \\ a \\ \longrightarrow \longrightarrow \longrightarrow \longrightarrow \\ g \longrightarrow 0 \\ \longrightarrow \longrightarrow \longrightarrow \longrightarrow \\ \rho \end{array}.$$  

**Thm:** $0 \longrightarrow \mathbb{R} \longrightarrow \tilde{g}[\omega] \longrightarrow g \longrightarrow 0$ determined by $[\omega] \in \text{CaE}^2(G)$ integrates to $1 \longrightarrow \mathbb{C}^\times \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1$ iff $\text{Per}(\omega) \subset 2\pi \mathbb{Z}$ and $\ell : G \times (G, \omega) \longrightarrow (G, \omega)$ has a momentum map.
Idea of geometrisation – building the $p$-sgerbe $\mathcal{G}^{(p)}$:

**(Inspiration:)** extended sspacetimes of [de Azcárraga et al. ’00+]

1. Look for an LI 2-scocycle $\omega$ in

$$\left\langle \nu_{t_{\mu_1}} \nu_{t_{\mu_2}} \cdots \nu_{t_{\mu_p}} \chi \mid \mu_1, \mu_2, \ldots, \mu_p \in 0, d_0 + d_1 \right\rangle_{\mathbb{R}}.$$

2. Use $\int (\mathbf{0} \to \mathbf{a} \to \tilde{g}[\omega] \to \mathfrak{g} \to \mathbf{0}) =: (1 \to A \to \tilde{G}[\omega] \xrightarrow{\tilde{\pi}} G \to 1)$ to partially reduce $\tilde{\pi}^* \chi$ in $\text{CaE}^\bullet(\tilde{G}[\omega])$.

3. Repeat 1.-2. until complete reduction of $\tilde{\pi}^* \chi$ is obtained over an extension $\hat{G} \xrightarrow{\hat{\pi}} G$ in the corresponding $\text{CaE}^\bullet(\hat{G})$, i.e.,

$$\exists \beta \in \Omega^{p+1}(\hat{G}) : d\beta = \hat{\pi}^* \chi.$$

4. Check that $\beta$ descends to $\hat{G}/H$.

5. Use $\text{YG} := \hat{G}$ as THE surjective submersion of $\mathcal{G}^{(p)}$ & DCAF à la [Murray & Stevenson ’94-’99 et al.].
Constructive results:

Theorem I [rrS ’17(’12)] Consecutive resolution, through scentral extensions, of the various CaE 2-scocycles encountered in the analysis of the GS \((p + 2)\)-scocycles on \(sMink(d, 1|(N \cdot)D_{d, 1})\), induces a hierarchy of surjective submersions necessary for the sgeometrisation of the latter, leading to the emergence of sminkowskian Green–Schwarz \(p\)-sgerbes (explicitly for \(p \in \{0, 1, 2\}\)).

Abstraction:

An H-equivariant Cartan–Eilenberg \(p\)-sgerbe \(G^{(p)}\) of curvature \(\text{curv} (G^{(p)}) = \chi \pmod{(p + 2)}\)

\[\equiv \text{‘a } p\text{-gerbe object in } s\text{LieGrp} \text{ (with an H-equivariant structure)}‘.\]
*E.g.*, a **CaE 1-sgerbe** of curvature $\chi$, 

$\mu_L : \text{pr}_{1,2}^* L \otimes \text{pr}_{2,3}^* L \xrightarrow{\sim} \text{pr}_{1,3}^* L$

$\mathbb{C}^\times \rightarrow L, A_L$

$I_{\beta}^{(1)} \rightarrow I_{\beta}^{(2)}$

$Y^{[3]}G \xrightarrow{\text{pr}_{1,2}} Y^{[2]}G \xrightarrow{\text{pr}_{2,3}} YG, \beta$

$\pi_L \downarrow \uparrow \pi_YG \downarrow$

$G, \chi^{(3)}$

- $YG \xrightarrow{\pi_YG} G$ and $L \xrightarrow{\pi_YG} Y^{[2]}G$ are **sLieGrp** extensions;
- $\beta$ and $A_L$ are LI relative to $YG$ and $L$, respectively;
- $\mu_L$ is a **sLieGrp** isomorphism.
Constructive results – $ct^d$:

- The success of the sminkowskian sgeometrisation was repeated in [rrS ’18] in the setting of Zhou’s super-$\sigma$-model of [Zhou ’99] for the sparticle in $s(\text{AdS}_2 \times S^2)$.
- The celebrated Metsaev–Tseytlin super-$\sigma$-model of [Metsaev & Tseytlin ’98] for the sstring in $s(\text{AdS}_5 \times S^5)$, on the other hand, seems problematic. There exists an İnönü–Wigner-noncontractible trivial 1-sgerbe, and a collection of no-go theorems…
Higher supersymmetry

- Global supersymmetry built in as G-invariance.
- ‘Hidden’ gauge symmetry to be imposed as an H-equivariant structure (if we tread carefully, it is automatic – *cp* the construction of the super-σ-model).
- What about the spontaneous breakdown of (s)symmetry by the svacuum?

**Problem:** $\kappa$-symmetry mixes metric and topological DOFs.

We cannot change the nature of $\kappa$-symmetry, yet we *can* change the FT perspective...

(after [Hughes & Polchinski ’86, Gauntlett, Itoh & Townsend ’90])
THE other sgeometry: Pick up a salgebraic model of the body of the svacuum:

\[ \bigoplus_{a=0}^{\rho} \langle P_a \rangle \equiv t_{\text{vac}}^{(0)} \subset t_{\text{vac}}^{(0)} \oplus e^{(0)} \equiv t^{(0)} , \quad \dim t_{\text{vac}}^{(0)} = \rho + 1 , \]

with an ad-isotropy algebra

\[ h_{\text{vac}} \subset h_{\text{vac}} \oplus \mathfrak{d} \equiv \mathfrak{h} , \quad \mathfrak{d} = \bigoplus_{S=1}^{T} \langle J_{\hat{S}} \rangle , \quad \mathcal{L}_{\text{Lie}} \quad H_{\text{vac}} \subset H \]

Assume reductivity of

\[ [h_{\text{vac}}, t \oplus \mathfrak{d}] \subset t \oplus \mathfrak{d} , \quad \text{with} \quad [h_{\text{vac}}, e^{(0)}] \subset e^{(0)} \supset [\mathfrak{d}, t_{\text{vac}}^{(0)}] \quad \wedge \quad [\mathfrak{d}, e^{(0)}] \subset t_{\text{vac}}^{(0)} \]

and unimodularity, or preservation of the body of the svacuum,

\[ \forall h \in H_{\text{vac}} : \det \rho(h)_{|t_{\text{vac}}^{(0)}} \equiv \det T_e \text{Ad}_h_{|t_{\text{vac}}^{(0)}} \overset{!}{=} 1 . \]

Replace the NG starget

\[ G/H \mapsto G/H_{\text{vac}} \sim \sum^{\text{NG}} = \bigsqcup_{i \in I} \sigma_i(U_i) \mapsto \sum^{\text{HP}} = \bigsqcup_{i \in \mathcal{J}} \sigma_{j_{\text{vac}}}(U_{j_{\text{vac}}}) \]

**Tranquiliser:** sPhysics only cares about \( T_e \text{Ad}_H \)-classes!
**THE other stensorial data:** \((\pi_{G/H_{\text{vac}}}: G \rightarrow G/H_{\text{vac}})\)

- the \(H_{\text{vac}}\)-basic LI svacuum-body svolume

\[
\frac{1}{(p+1)!} \epsilon_{a_0 a_1 \ldots a_p} \theta_{L}^{a_0} \wedge \theta_{L}^{a_1} \wedge \cdots \wedge \theta_{L}^{a_p} \equiv \text{Vol}(t^{(0)}_{\text{vac}}) = \pi_{G/H_{\text{vac}}}^{*} B^{H_{\text{vac}}} \]

- the H-basic LI de Rham \((p + 2)\)-scocycle

\[
\chi_{\mu_1 \mu_2 \ldots \mu_{p+2}} \theta_{L}^{\mu_1} \wedge \theta_{L}^{\mu_2} \wedge \cdots \wedge \theta_{L}^{\mu_{p+2}} \equiv \chi = \pi_{G/H_{\text{vac}}}^{*} H^{\text{vac}}.\]

**THE other sbackground:** The Hughes–Polchinski sbackground

\[
(G/H_{\text{vac}}, \lambda_{p} \, \text{dVol}(t^{(0)}_{\text{vac}}) + \chi \equiv \widehat{\chi}(\lambda_{p}) = \pi_{G/H_{\text{vac}}}^{*} \widehat{H}(\lambda_{p})) \equiv \Sigma^{(p, \lambda_{p})}_{(H_{\text{vac}})} ,
\]

with \(\lambda_{p} \in \mathbb{R}^{\times}\) to be fixed by supersymmetry...
THE other super-σ-model

Given a closed orientable m-fold $\Omega_p$ of $\dim \Omega_p = p + 1$, a Lie group $G$ and closed Lie subgroups $H_{\text{vac}} \subset H \subset |G|$ with $(g, h)$ and $(g, h_{\text{vac}})$ reductive, and the Hughes–Polchinski background $s\mathcal{B}^{(\lambda_p)}_{(\text{HP})}$, the Green–Schwarz super-σ-model in the Hughes–Polchinski formulation is a theory of mappings $[\Omega_p, G/H_{\text{vac}}] \ni \hat{\xi}$ determined by the PLA for the DF (s)amplitudes with

$$S_{\text{GS}}^{(\text{HP}), (\lambda_p)}[\xi] = \int_{\Omega_p} \hat{\xi}^* d^{-1}(\lambda_p) \equiv \sum_{\tau \in \triangle_{\Omega_p}^{(p+1)}} \int_{\tau} (\sigma_j^{\text{vac}} \circ \hat{\xi})^* d^{-1}(\lambda_p),$$

with the last equality using a tessellation $\triangle_{\Omega_p}$ of $\Omega_p$ subordinate to $\{U_j\}_{j \in \mathcal{J}}$ for a given $\hat{\xi}$.

**NB:** The above sFT is purely topological. In fact, it is...

...‘reducible to a point’...
Let \((g, h, h_{\text{vac}}, t_{\text{vac}}^{(0)})\) and \(\rho\) be constrained as above, with the following Maximal Mixing Constraint obeyed**:

\[
\langle P_{\hat{a}} \mid \exists (b, \hat{S}) \in \mathbb{0, p \times 1, T} : f_{\hat{S}} \hat{a}^b \neq 0 \rangle = e^{(0)},
\]

and suppose there exists a \(T_e \text{Ad}_H\)-invariant metric \(g\) on \(t^{(0)}\) s.t.

\[
t_{\text{vac}}^{(0)} \perp g e^{(0)}.
\]

The GS super-\(\sigma\)-model in the HP formulation for \((G/H_{\text{vac}}, \chi^{(\lambda_p)})\) becomes (class.) equivalent to the GS super-\(\sigma\)-model in the NG formulation for \((G/H, g, \chi^{(p+2)})\) for a unique value \(\mu_p^{*}(\lambda_p)\) of \(\mu_p^{(p+2)}\) upon restriction of the former FT to field configurations satisfying the Inverse Higgs Constraints

\[
(\sigma_{i_T}^{\text{vac}} \circ \hat{\xi}) * \theta_{\hat{a}}^{\frac{1}{4}} \perp 0, \quad \hat{a} \in p + 1, d_0.
\]

\(\Leftrightarrow\) the EL eqns for the Goldstone modes \(\phi^{\hat{S}}\) (in an exp gauge).

**The restriction can be relaxed.
Upshot [rrS ’20]: In the dual purely topological HP formulation, we may impose $\Delta o\gamma\mu\alpha \frac{1}{2}$ as ‘everything in sight’ sgeometrises. Indeed.

- the duality occurs ‘in’ the correspondence sdistribution

\[
\text{Corr}(s\mathcal{V}(\lambda_P)^{(\text{HP})}) = \bigcap_{\hat{a}=p+1}^d \text{Ker } \theta_L^{\hat{a}} \cap \mathcal{T}\Sigma^{\text{HP}};
\]

- supersymmetry restoration in the svacuum via restriction to

\[
\text{sSym}(s\mathcal{V}(\lambda_P^\ast)^{(\text{HP})}) = \text{Corr}(s\mathcal{V}(\lambda_P^\ast)^{(\text{HP})}) \cap \text{Ker } ((1_{d_1} - P^{(1)})^\alpha \theta_\beta^L)
\]

for $P^{(1)} = P^{(1)} \cdot P^{(1)} \in \text{End } t^{(1)}$ s.t. $\{\text{Im } P^{(1)}^T, \text{Im } P^{(1)}^T\} \subset t^{(0)}_{\text{vac}} \oplus \mathfrak{h}$;

- altogether, the EL eq^n's define*** a svacuum sdistribution

\[
\text{Vac}(s\mathcal{V}(\lambda_P^\ast)^{(\text{HP})}) = \text{sSym}(s\mathcal{V}(\lambda_P^\ast)^{(\text{HP})}) \cap \bigcap_{S=1}^T \text{Ker } \theta_L^{\hat{S}};
\]

***Under some mild assumptions, satisfied by the known super-$\sigma$-models.
Upshot [rrS ’20] – ct^d:

- geometric consistency of the svacuum $\iff$ integrability of $\text{Vac}(\mathfrak{sB}_{(\text{HP})})$ $\iff$ closure of the modelling superspace

$$\text{vac} = t^{(0)}_{\text{vac}} \oplus t^{(1)}_{\text{vac}} \oplus h_{\text{vac}}, \quad t^{(1)}_{\text{vac}} \equiv \text{Im } P^{(1)} T \subset \mathfrak{g}$$

under the sbracket into the svacuum Lie algebra (descent to the physical supertarget $G/H_{\text{vac}}$ follows);

- enhancement of gauge**** symmetry ‘in’ $\text{Corr}(\mathfrak{sB}_{(\text{HP})})$:

$$h_{\text{vac}} \mapsto t^{(1)}_{\text{vac}} \oplus \Delta^{(1)}_{\text{acc}} \oplus (h_{\text{vac}} \oplus \mathfrak{d})$$

requires further restriction to $\text{Vac}(\mathfrak{sB}_{(\text{HP})})$ for consistency, whereupon we get the $\kappa$-symmetry sdistribution

$$\kappa(\mathfrak{sB}_{(\text{HP})}) \subset \text{Vac}(\mathfrak{sB}_{(\text{HP})}) \quad \text{modelled on} \quad t^{(1)}_{\text{vac}} \oplus \Delta^{(1)}_{\text{acc}} \oplus h_{\text{vac}} \subset \text{vac}$$

**** Dependence on $\sigma^j_{\text{vac}}$ implies locality AND $\hat{\chi}(\lambda^\rho) \sim \Omega^{(\text{HP})}_{\sigma^{(\rho + 2)}}$. 

Empirical fact:

The limit $\kappa^{-\infty}(\mathfrak{g}\mathfrak{B}^{(\lambda^*_\rho)}_{(\text{HP})})$ of the weak derived flag of $\kappa(\mathfrak{g}\mathfrak{B}^{(\lambda^*_\rho)}_{(\text{HP})})$ stays within $\text{Vac}(\mathfrak{g}\mathfrak{B}^{(\lambda^*_\rho)}_{(\text{HP})})$ whenever the latter is integrable (i.e., physical), and then

$$\kappa^{-\infty}(\mathfrak{g}\mathfrak{B}^{(\lambda^*_\rho)}_{(\text{HP})}) \equiv \text{Vac}(\mathfrak{g}\mathfrak{B}^{(\lambda^*_\rho)}_{(\text{HP})}) ,$$

which is why $\kappa(\mathfrak{g}\mathfrak{B}^{(\lambda^*_\rho)}_{(\text{HP})})$ was dubbed the square root of the svacuum in [rrS ’20].

Conclusion: The Lie salgebra

$$\mathfrak{g}\mathfrak{s}_{\text{vac}} \equiv \text{vac}$$

modelling $\kappa^{-\infty}(\mathfrak{g}\mathfrak{B}^{(\lambda^*_\rho)}_{(\text{HP})})$ acquires the interpretation of the svacuum gauge-symmetry salgebra.

So what about $\Delta \sigma \gamma \mu \alpha \frac{\dd}{\dd t}$? Benefit from topologicality!
Restrict the extended Hughes–Polchinski $\rho$-sgerbe

$$\hat{G}^{(p)}_{\text{HP}} := G^{(p)} \otimes I^{(p)} \lambda^*_p \text{Vol}(t^{(0)}_{\text{vac}})$$

to the sections $\sigma_j^{\text{vac}}(U_j^{\text{vac}}) \equiv \mathcal{V}_j$, forming

$$\hat{G}^{(p)}_{\Sigma_{\text{HP}}} := \bigsqcup_{j \in J} \hat{G}^{(p)}_{\text{HP}} \rvert_{\mathcal{V}_j},$$

and subsequently pull back to the vacuum foliation

$$\iota_{\text{vac}} : \Sigma^{\text{HP}}_{\text{vac}} \hookrightarrow \Sigma^{\text{HP}},$$

whereby there arises the vacuum restriction

$$\hat{G}^{(p)}_{\text{vac}} \equiv \iota^{*}_{\text{vac}} \hat{G}^{(p)}_{\Sigma_{\text{HP}}}$$

that descends to the physical vacuum in $G/H_{\text{vac}}$ by construction.

**Dogmatic expectation:** a $\mathfrak{g}_{\text{vac}}$-equivariant structure on $\hat{G}^{(p)}_{\text{vac}}$
However, $\kappa^{-\infty}(s\mathcal{B}_{(HP)}^{(\lambda^*_p)})$ envelops the vacuum, the latter being a single orbit of $\mathfrak{g}_{\text{vac}}$, resp. of the $\kappa$-symmetry sgroup (whenever $\exists$)

$$\int \mathfrak{g}_{\text{vac}} \equiv G_{\text{vac}},$$

whence

**Hypothesis [rrS ’20]:** There exists an $H_{\text{vac}}$-equivariant trivialisation

$$\tau_p : \widehat{G}_{\text{vac}} \xrightarrow{\cong} \mathcal{I}_0^{(p)},$$

or, equivalently, the descendant of $G^{(p)}$ to $G/H_{\text{vac}}$ (indeed, to $G/H$) trivialises as the volume $p$-gerbe over the svacuum.

**Problem:** The svacuum does not possess a natural Lie-sgroup structure, hence there seems to be no room for a supersymmetric trivialisation. And yet...

In our formalism, we may look for a $\text{sLieAlg}$ shadow of $\tau_p$. 
E.g. [rrS ’20, in writing], the sminkowskian sstring trivialisation

\[ \alpha_\mathcal{E} \equiv 1 : \mathcal{L}_{\text{vac}} \otimes \text{pr}_2^*\mathcal{E} \xrightarrow{\sim} \text{pr}_1^*\mathcal{E} \]

\[ \mathbb{C}^\times \xrightarrow{\pi_{\mathcal{E}}} \mathcal{E}, A_\mathcal{E} \]

\[ \sum_{\text{vac}}^{\Sigma_{\text{HP}}} \xrightarrow{\pi_{\Sigma_{\text{HP}}}} \hat{\mathcal{Y}}_{\text{vac}}^{\beta} \]

\[ \alpha_{\tilde{e}} \equiv 1 : \mathcal{Y}_{\text{vac}}^{[2]} \tilde{l} \otimes \text{pr}_2^*\tilde{e} \xrightarrow{\sim} \text{pr}_1^*\tilde{e} \]

\[ \mathbb{R} \xrightarrow{\pi_{\tilde{e}}} \epsilon, \zeta_{\mathcal{E}} \]

\[ \sum_{\text{vac}}^{\Sigma_{\text{HP}}} \xrightarrow{\pi_{\Sigma_{\text{vac}}}} \hat{\chi}_{\text{vac}} = 0 \]

\[ \mathcal{Y}_{\text{vac}}^{[2]} \xrightarrow{\pi_{\mathcal{Y}_{\text{vac}}}} \mathcal{Y}_{\text{vac}}, \hat{\mathcal{Y}}_{\text{vac}}^{\beta} \]

\[ \hat{\chi}_{\text{vac}} = 0 \]

has a shadow

\[ \alpha_{\tilde{e}} \equiv 1 : \mathcal{Y}_{\text{vac}}^{[2]} \tilde{l} \otimes \text{pr}_2^*\tilde{e} \xrightarrow{\sim} \text{pr}_1^*\tilde{e} \]
Loose ends:

- **Thm [rrS ’19(’17)]** The superminkowskian GS p-sgerbes with $p \in \{0, 1\}$ are endowed with a canonical **supersymmetric** $\text{Ad}_{s\text{Mink}(d,1|D_{d,1})}$-equivariant structure. 
  
  **NB:** This conforms with the purely even (WZW) story.

- The GS super-σ-models with curved stargets $s(\text{AdS}_q \times \mathbb{S}^q)$ (MT, PR, Zh) are constructed on the basis of an asymptotic correspondence with their superminkowskian counterparts,

$$s(\text{AdS}_q \times \mathbb{S}^q) \rightarrow s(\text{AdS}_p(R) \times \mathbb{S}^q(R)) \xrightarrow{R \rightarrow \infty} s\text{Mink}(2q - 1, 1|D_{2q-1,1}).$$

It is natural to gerbify the underlying İnönü–Wigner contractions

$$g^q_{\text{curv}} \rightarrow g^q_{\text{curv}(R)} \xrightarrow{R \rightarrow \infty} s\text{mink}(2q - 1, 1|D_{2q-1,1})$$

by requiring that they **lift to sLieAlg shadows** of Murray diagrams, & turn it into an organising principle on the moduli space of super-σ-models.

**Outcome: Problems** with the definition of the stringy super-σ-models.
Conclusions:

1. The physically relevant CaE \((p+2)\)-scocycles on supersymmetry Lie supergroups geometrise – in an interplay of CaE & CE cohomology – for a large class of sbackgrounds as the H-equivariant CaE \(p\)-sgerbes of [rrS ’17, ’18].

2. The CaE \(p\)-sgerbes are global supersymmetry-invariant and endowed with (the expected and) natural equivariant structures with respect to the supersymmetries of the relevant super-\(\sigma\)-models amenable to gauging, in conformity with the underlying physics and the bosonic intuition. [rrS ’19]

3. \(\kappa\)-symmetry demystified, geometrised & gerbified in the dual HP formulation of the GS super-\(\sigma\)-model. [rrS ’19, ’20]

4. The construction generalises to physically relevant curved homogeneous spaces of supersymmetry Lie supergroups, and sometimes suggests – via gerbification of the \(\dot{I}W\) contraction – corrections to the existing sFT results. [rrS ’18]
Outlook:

- Uniqueness of the construction and its relation to the approach of Huerta, Baez, Schreiber et al. ($\kappa$-symm., H-equiv., ÍW-contr.)? Reconstruction of the (weak) $(p + 1)$-categories of $p$-sgerbes.
- The relevance of the ÍW-contractibility & the ultimate fate of the curved sbackgrounds?
- The higher sgeometry and salgebra ($s\text{LieAlg}$ shadows) of supersymmetric defects (incl. boundary states) & their fusion.
- Relation to the worldvolume supersymmetry, possibly \textit{via} Sorokin’s Superembedding Formalism.
- Relation to the $\text{String}$-structure.
- The bosonisation/fermionisation defect.
- T-duality \textit{via} the HP formulation, also in the bosonic setting.
- The gauging of the $\text{Ad}_{s\text{Mink}}(d,1|D_{d,1})$-supersymmetry and the ensuing CS-type sTFT.
- SUSY NCG \textit{etc}...
(Ceci n’est pas) La Fin...
Part III

super-Xtras
1. A **Lie algebroid** is a quintuple \((\mathcal{V}, \pi_\mathcal{V}, M, \alpha_T M, [\cdot, \cdot]_\mathcal{V})\) composed of

- a smooth manifold \(M\), termed the **base**;
- a smooth vector bundle \(\pi_\mathcal{V} : \mathcal{V} \rightarrow M\);
- a smooth vector-bundle morphism \(\alpha_T M : \mathcal{V} \rightarrow T M\), termed the **anchor map**;
- a Lie bracket \([\cdot, \cdot]_\mathcal{V} : \Gamma(\mathcal{V}) \times \mathcal{V} \rightarrow \Gamma(\mathcal{V})\) on the vector space \(\Gamma(\mathcal{V})\) of sections of \(\mathcal{V}\),

with the following properties:

- the induced map \(\Gamma\alpha_T M : (\Gamma(\mathcal{V}), [\cdot, \cdot]_\mathcal{V}) \rightarrow (\Gamma(T M), [\cdot, \cdot])\) is a Lie-algebra homomorphism;
- the Lie bracket \([\cdot, \cdot]_\mathcal{V}\) obeys the Leibniz identity

\[
\forall (X, Y, f) \in \Gamma(\mathcal{V}) \times \mathcal{V} \times C^\infty(M, \mathbb{R}) : [X, f \triangleright Y]_\mathcal{V} = f \triangleright [X, Y]_\mathcal{V} + \alpha_T M(X)(f) \triangleright Y.
\]
2. The Lie supergroup of the Metsaev-Tseytlin super-$\sigma$-model:

$$SU(2, 2 \mid 4)$$ with the body

$$|SU(2, 2 \mid 4)| = SO(4, 2) \times SO(6)$$

and the Lie superalgebra ($R$-rescaled, for $R \in \mathbb{R}$)

$$su(2, 2 \mid 4)^{(R)} = \left( \bigoplus_{a=0}^{4} \langle P_a \rangle \oplus \bigoplus_{a'=5}^{9} \langle P_{a'} \rangle \right) \oplus \bigoplus_{(\alpha, \alpha', I) \in \{1, 4\} \times \{1, 4\} \times \{1, 2\}} \langle Q_{\alpha \alpha' I} \rangle$$

$$\oplus \left( \bigoplus_{a,b=0}^{4} \langle J_{ab} = -J_{ba} \rangle \oplus \bigoplus_{a', b'=5}^{9} \langle J_{a'b'} = -J_{b'a'} \rangle \right)$$

$$\{Q_{\alpha \alpha' I}, Q_{\beta \beta' J}\} = i \left( -2(\hat{\Gamma}^{a} \otimes \mathbf{1})_{\alpha \alpha' I \beta \beta' J} P_{\hat{a}} + \frac{1}{R^2} (\hat{C} \hat{\Gamma}^{a \hat{b}} \otimes \sigma_2)_{\alpha \alpha' I \beta \beta' J} J_{\hat{a} \hat{b}} \right) ,$$

$$[Q_{\alpha \alpha' I}, P_{\hat{a}}] = -\frac{1}{2R} (\hat{\Gamma}_{\hat{a}} \otimes \sigma_2)_{\alpha \alpha' I \beta \beta' J} Q_{\beta \beta' J} ,$$

$$[P_a, P_b] = \frac{1}{R^2} \varepsilon_{ab} J_{\hat{a} \hat{b}} , \quad \varepsilon_{ab} = \begin{cases} +1 & \text{if } \hat{a}, \hat{b} \in \{0, 4\} \\ -1 & \text{if } \hat{a}, \hat{b} \in \{5, 9\} \\ 0 & \text{otherwise} \end{cases}$$

$$[J_{\hat{a} \hat{b}}, J_{\hat{c} \hat{d}}] = \eta_{\hat{a} \hat{c}} J_{\hat{b} \hat{d}} - \eta_{\hat{a} \hat{d}} J_{\hat{b} \hat{c}} + \eta_{\hat{b} \hat{c}} J_{\hat{a} \hat{d}} - \eta_{\hat{b} \hat{d}} J_{\hat{a} \hat{c}} ,$$

$$[Q_{\alpha \alpha' I}, J_{\hat{a} \hat{b}}] = -\frac{1}{2} \varepsilon_{ab} (\hat{\Gamma}_{\hat{a} \hat{b}} \otimes \mathbf{1})_{\alpha \alpha' I \beta \beta' J} Q_{\beta \beta' J} ,$$

$$[P_{\hat{a}}, J_{\hat{b} \hat{c}}] = \eta_{\hat{a} \hat{c}} P_{\hat{b}} - \eta_{\hat{a} \hat{b}} P_{\hat{c}} .$$

with the İnönü-Wigner asymptote $$su(2, 2 \mid 4)^{(R)} \xrightarrow{R \to \infty} \mathfrak{s} \text{mink}(9, 1 \mid 32)$$
3. Some Lie-superalgebra cohomology . . .

**Defn:** A *(left)* $\hat{g}$-module of an LSA $\hat{g}$ is a pair $(\hat{V}, \ell.)$ composed of a $K$-linear superspace $\hat{V} = \hat{V}^{(0)} \oplus \hat{V}^{(1)}$ and a left $\hat{g}$-action

$$\ell. : \hat{g} \times \hat{V} \longrightarrow \hat{V} : (X, v) \longmapsto X \triangleright v$$

consistent with the $\mathbb{Z}/2\mathbb{Z}$-gradings, $X \triangleright v = \tilde{X} + \tilde{v}$, and such that for any two homogeneous elements $X_1, X_2 \in \mathfrak{g}$ and $v \in \hat{V}$,

$$[X_1, X_2] \triangleright v = X_1 \triangleright (X_2 \triangleright v) - (-1)^{\tilde{X}_1 \cdot \tilde{X}_2} X_2 \triangleright (X_1 \triangleright v).$$

and the fundamental . . .
**Def**: Let \((\hat{g}, [\cdot, \cdot])\) be an LSA over field \(\mathbb{K}\) and let \((\hat{V}, \ell.)\) be a \(\hat{g}\)-module. A \(p\)-cochain on \(\hat{g}\) with values in \(\hat{V}\) is a \(p\)-linear map \(\varphi : \hat{g}^\times p \longrightarrow \hat{V}\) that is totally super-skewsymmetric,

\[
\varphi(X_1, X_2, \ldots, X_{i-1}, X_{i+1}, X_i, X_{i+2}, X_{i+3}, \ldots, X_p) = -(-1)^{\tilde{x}_i \tilde{x}_{i+1}} \varphi(X_1, X_2, \ldots, X_p).
\]

They form a \(\mathbb{Z}_2\)-graded group of \(p\)-cochains on \(\hat{g}\) valued in \(\hat{V}\),

\[
C^p(\hat{g}, \hat{V}) = C^p_0(\hat{g}, \hat{V}) \oplus C^p_1(\hat{g}, \hat{V}),
\]

with \(\varphi(X_1, X_2, \ldots, X_p) \in \hat{V}\sum_{i=1}^p \tilde{x}_i + n\) for \(\varphi \in C^p_n(\hat{g}, \hat{V})\), composed of **even** \((n = 0)\) and **odd** \((n = 1)\) \(p\)-cochains.

These groups form a semi-bounded complex

\[
C^*(\hat{g}, \hat{V}) : C^0(\hat{g}, \hat{V}) \xrightarrow{\delta^{(0)}_\hat{g}} C^1(\hat{g}, \hat{V}) \xrightarrow{\delta^{(1)}_\hat{g}} \cdots \xrightarrow{\delta^{(p-1)}_\hat{g}} C^p(\hat{g}, \hat{V}) \xrightarrow{\delta^{(p)}_\hat{g}} \cdots
\]
The coboundary operators

\[ \delta^{(p)}_g : C^n_p(g, V) \longrightarrow C^{n+1}_p(g, V) \]

evaluate on the homogeneous \( X_i \in g, \; i \in 0, p + 1, \varphi \in C^p(g, V) \)
as

\[ (\delta^{(0)}_g \varphi)(X) := (-1)^{i} X_0 \triangleright \varphi, \]

\[ (\delta^{(p)}_g \varphi)(X_1, X_2, \ldots, X_{p+1}) := \sum_{j=1}^{p+1} (-1)^{i} X_j \triangleright \varphi(X_1, X_2, \ldots, X_{p+1}) \]

\[ + \sum_{1 \leq j < k \leq p+1} (-1)^{S(X_j)+S(X_k)+|X_j||X_k|} \varphi([X_j, X_k], X_1, X_2, \ldots, X_{p+1}), \]

\[ S(X_i) := |X_i| \cdot \sum_{j=1}^{i-1} |X_j| + i - 1. \]

The \( \mathbb{Z}/2\mathbb{Z} \)-graded \( V \)-valued cohomology groups of \( g \) are

\[ H^p(g, V) := H^p_0(g, V) \oplus H^p_1(g, V), \quad H^p_n(g, V) := \frac{\ker \delta^{(p)}_g |_{C^n_p(g, V)}}{\text{im} \delta^{(p-1)}_g |_{C^{n-1}_p(g, V)}}. \]
Defn: Let $(\widehat{g}, [\cdot, \cdot]_{\widehat{g}})$ and $(\widehat{a}, [\cdot, \cdot]_{\widehat{a}})$ be LSAs over field $\mathbb{K}$. A **supercentral extension of** $\widehat{g}$ **by** $\widehat{a}$ is an LSA $(\tilde{g}, [\cdot, \cdot]_{\tilde{g}})$ over $\mathbb{K}$ that determines a short exact sequence of LSAs

$$0 \longrightarrow \alpha \xrightarrow{j_{\widehat{a}}} \tilde{g} \xrightarrow{\pi_{\tilde{g}}} g \longrightarrow 0,$$

written in terms of an LSA mono $j_{\widehat{a}}$ and of an LSA epi $\pi_{\tilde{g}}$, and s.t. $j_{\widehat{a}}(\widehat{a}) \subset \mathfrak{z}(\tilde{g})$ (the supercentre of $\tilde{g}$). Whenever $\pi_{\tilde{g}}$ admits an LSA section, i.e., there exists

$$\sigma \in \text{Hom}_{s\text{Lie}}(\widehat{g}, \tilde{g}), \quad \pi_{\tilde{g}} \circ \sigma = \text{id}_{\tilde{g}},$$

the supercentral extension is said to **split**.

An **equivalence of supercentral extensions** $\tilde{g}_\alpha, \alpha \in \{1, 2\}$ of $\widehat{g}$ by $\widehat{a}$ is represented by a commutative diagram of LSAs

$$
\begin{array}{ccc}
0 & \longrightarrow & \widehat{a} & \xrightarrow{\iota} & \tilde{g} & \xrightarrow{\pi_{\tilde{g}}} & g & \longrightarrow & 0 \\
& & \, \downarrow \, & \, \downarrow \iota & \, \downarrow \, & \, \downarrow \pi_{\tilde{g}} & \, \downarrow \, & \, \downarrow \, & \\
& & \tilde{g}_1 & \, \xrightarrow{j_{\tilde{g}_1}} & \tilde{g} & \, \xrightarrow{\pi_{\tilde{g}}} & 0 \\
& & \tilde{g}_2 & \, \xrightarrow{j_{\tilde{g}_2}} & \tilde{g} & \, \xrightarrow{\pi_{\tilde{g}}} & 0
\end{array}
$$
The relevant one-way ticket:

Given an LSA \((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})\) and its supercommutative module \(\mathfrak{a}\), as well as a representative \(\Theta \in Z^2_0(\mathfrak{g}, \mathfrak{a})\) of a class in \(H^2_0(\mathfrak{g}, \mathfrak{a})\), we define

\[
\tilde{\mathfrak{g}} := \mathfrak{a} \oplus \mathfrak{g}
\]

and put on it the Lie superbracket

\[
\{\cdot, \cdot\}_\Theta : \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}
\]

\[
: ((A_1, X_1), (A_2, X_2)) \mapsto (\Theta(X_1, X_2), [X_1, X_2]_{\mathfrak{g}}).
\]
4. A **Lie 2-algebra** is a quintuple \((V_0, V_1, \delta, [-, -], \text{Jac})\) composed of

- vector spaces \(V_0\) and \(V_1\);
- a linear map \(\delta : V_1 \to V_0\) (the **differential**);
- a **skew** bilinear map \([ -, -] : (V_0 \oplus V_1)^\times 2 \to V_0 \oplus V_1\);
- a **skew** trilinear map \(\text{Jac} : V_0^\times 3 \to V_1\) (the **jacobiator**),

with the following properties, written for \(R, S \in V_1, x, y, z \in V_0\),

- \([V_i, V_j] \subset V_{i+2j}\);
- \([R, S] = 0\);
- \(\delta[x, R] = [x, \delta R]\);
- \([\delta R, S] = [R, \delta S]\);
- \([[x, y], z] + [[z, x], y] + [[y, z], x] = \delta\text{Jac}(x, y, z)\)
- \(\text{Jac}(\delta R, x, y) = -[[x, y], R] + [[x, R], y] + [x, [y, R]]\);
- ‘septagonal’ coherence for \(\text{Jac}\) and \([-, -]\).
There exists a one-to-one correspondence between isomorphism classes of skeletal Lie 1-algebras (with $\delta = 0$) and equivalence classes of quadruples $(g, V, \rho, \chi)$ composed of a Lie algebra $g$, a vector space $V$, a representation $\rho : g \to \text{End}(V)$, and a $V$-valued 3-cocycle $\chi$ on $g$.

In particular, to a quadruple $(g, V, \rho, \chi)$ as above, we associate a Lie 2-algebra with $(V_0, V_1, \text{Jac}) = (g, V, \chi)$ and $[-, -]$ defined as

$$[X, Y] = [X, Y]_g, \quad [X, \nu] = \rho_X(\nu), \quad [\nu, \omega] = 0$$

for arbitrary $X, Y \in g$ and $\nu, \omega \in V$.

The correspondence was subsequently generalised to higher slim $L_\infty$ algebras and Lie-algebra modules with higher cocycles, and supersised by Baez & Huerta in [Baez & Huerta ’11].