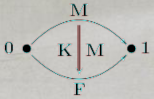
"It takes a gerbe to σ-model."

The higher supergeometry of the super-σ-model

Rafał R. Suszek

(Katedra Metod Matematycznych Fizyki,
Wydział Fizyki, Uniwersytet Warszawski)



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A class of low-dimensional field theories, termed super-σ-models and used to model simple geometric dynamics of extended distributions of $\mathbb{Z}/2\mathbb{Z}$-graded charge in homogeneous spaces of Lie supergroups, shall be reviewed, with emphasis on the supersymmetries present, both global and local. A (super)geometrisation scheme for the classes in the relevant supersymmetry-invariant (Cartan–Eilenberg) cohomology of the supersymmetry group associated with the topological charge shall be presented and basic supersymmetry-invariance and -equivariance properties of the ensuing super-gerbes shall be discussed. The general discussion shall be illustrated on a number of explicit examples, whereby, in particular, asymptotic İnönü–Wigner relations between certain physically relevant curved and flat higher supergeometric structures shall be postulated as an integral guiding principle of the (super)geometrisation scheme.

Goal:

Extending the **gerbe-theoretic approach** to the bosonic two-dimensional σ-model to (super-)σ-models with **homogeneous spaces of Lie supergroups** as target (super)spaces, in a manner consistent with **rigid and local supersymmetry**.

Discussion based upon

1. arXiv:1706.05682
2. arXiv:1808.04470
3. arXiv:1810.00856
4. arXiv:1905.05235
5. arXiv:2002.10012
6. arXiv:2010.xxxxx (in writing)

The skeleton of the talk:

I Learning from life without spin, or the higher geometry of the 2d bosonic σ-model

1. The predecessor LFT: The 2d bosonic non-linear σ-model.
2. Gerbification for the sake of (pre-)QM consistency.

II Putting a spin on it, or a $\mathbb{Z}/2\mathbb{Z}$-graded higher geometry

1. Lie supergroups à la Kostant and their homogeneous spaces.
2. The sLFT of interest: The Green–Schwarz super-σ-model.
3. A supergeometrisation scheme – the super-gerbes.
4. The dual sTFT and its vacuum.
5. Higher supersymmetry, global and local.
6. Loose ends (İnönü–Wigner contractibility, 'accidental' equivariance, ...).
7. Summary & Outlook.

Part I

Learning from life without spin,
or
the higher geometry
of
the 2d bosonic σ-model

The predecessor LFT: The 2d bosonic non-linear σ-model

Given a *closed* orientable 2d m_fold Σ (the **worldsheet**) & a metric m_fold (M, g) (the **target space**) with $\underset{(3)}{\mathrm{H}} \in Z^3_{\mathrm{dR}}(M)$, consider the theory of mappings $x \in [\Sigma, \mathcal{M}]$ determined by (the PLA for) the **Dirac–Feynman amplitudes**

$$\mathcal{A}_{\mathrm{DF}} \equiv \exp\left(\tfrac{\mathrm{i}}{\hbar}\, S_\sigma^{(\mathrm{NG})}[\cdot]\right) \;:\; [\Sigma, \mathcal{M}] \longrightarrow \mathrm{U}(1)$$

$$S_\sigma^{(\mathrm{NG})}[x] = \mu \int_\Sigma \sqrt{|\det x^*\mathrm{g}|} + q \int_\Sigma x^* \mathrm{d}^{-1} \underset{(3)}{\mathrm{H}}\,,$$

describing minimal embeddings deformed by Lorentz-type forces sourced by a Maxwell-type 3-form field $\underset{(3)}{\mathrm{H}}$.

The triple $(M, \mathrm{g}, \underset{(3)}{\mathrm{H}})$ is called the σ**-model background**.

Applications: mainly the critical bosonic string (and (mem)brane) theory, but also the effective FT of (certain slow) collective excitations of spin chains

Problem: May need $[\underset{(3)}{\mathrm{H}}]_{\mathrm{dR}} \neq 0$ (*e.g.*, for conformality), and so

$$\neg\exists_{\underset{(2)}{\mathrm{B}} \in \Omega^2(M)} \ : \ \underset{(2)}{\mathrm{dB}} = \underset{(3)}{\mathrm{H}}$$

E.g., $(M, \mathrm{g}) = \big(\mathrm{G}, \kappa_{\mathfrak{g}} \circ (\theta_{\mathrm{L}} \otimes \theta_{\mathrm{L}})\big) \implies \underset{(3)}{\mathrm{H}} = \lambda\, \kappa_{\mathfrak{g}} \circ \big(\theta_{\mathrm{L}} \wedge \theta_{\mathrm{L}} \wedge \theta_{\mathrm{L}}\big)$

and the Cartan 3-form $\underset{(3)}{\mathrm{H}}$ generates $H^3_{\mathrm{dR}}(\mathrm{G})$ for G 1-connected

But QM à la Dirac & Feynman requires that we compare amplitudes for cobordant trajectories!

Conclusion: Need $S_\sigma^{(\mathrm{NG})}$ with critical points (the EL eq^{n}s) as for $[\underset{(3)}{\mathrm{H}}]_{\mathrm{dR}} = 0$ but s.t. $\mathcal{A}_{\mathrm{DF}}$ is well-defined $\forall\, x(\Sigma) \in Z_2(M)$.

This calls for the use of a **Cheeger–Simons differential character** $\mathrm{Hol}_{\mathcal{G}^{(1)}} \in \mathrm{Hom}(Z_2(M), \mathrm{U}(1))$ s.t. $\mathrm{Hol}_{\mathcal{G}^{(1)}} \circ \partial_M(\cdot) = \exp\big(\tfrac{\mathrm{i}}{\hbar} \int_{(\cdot)} \underset{(3)}{\mathrm{H}}\big)$.

Solution: Fix an arbitrary *good* open cover $\mathcal{O}_M = \{\mathcal{O}_i\}_{i \in \mathscr{I}}$ of M & a tessellation $\triangle_\Sigma = \mathfrak{T}_2 \sqcup \mathfrak{T}_1 \sqcup \mathfrak{T}_0$ of Σ subordinate to it for a given $x \in [\Sigma, M]$, *i.e.*, s.t.

$$\exists_{\iota_\cdot \in \mathrm{Map}(\triangle_\Sigma, \mathscr{I})} \forall_{\tau \in \triangle_\Sigma} \ : \ x(\tau) \subset \mathcal{O}_{\iota_\tau} \, ,$$

and pull back, along x, a resolution/trivialisation of $\underset{(3)}{\mathrm{H}}$ over $\mathcal{O}_M$,

i.e., use $\underset{(2)}{b} = (B_i, A_{ij}, g_{ijk}) \in \Omega^2(\mathcal{O}_i) \times \Omega^1(\mathcal{O}_{ij}) \times \mathrm{U}(1)_{\mathcal{O}_{ijk}}$ s.t.

$$\underset{(3)}{\mathrm{H}} \restriction_{\mathcal{O}_i} = \mathrm{d}B_i \, , \qquad (B_j - B_i) \restriction_{\mathcal{O}_{ij}} = \mathrm{d}A_{ij} \, , \qquad (A_{jk} - A_{ik} + A_{ij}) \restriction_{\mathcal{O}_{ijk}} = \mathrm{i} \, \mathrm{d} \log g_{ijk}$$

to write (for $x_\tau \equiv x \restriction_\tau$)

$$S_\sigma^{(\mathrm{NG}),\mathrm{top}}[x] = \sum_{p \in \mathfrak{T}_2} \Big[\int_p x_p^* B_{\iota_p} + \sum_{e \in \partial p} \Big(\int_e x_e^* A_{\iota_p \iota_e} - \mathrm{i} \sum_{v \in \partial e} \varepsilon_{ev} \log g_{\iota_p \iota_e \iota_v}(x(v)) \Big) \Big] \, ,$$

with $\mathcal{A}_{\mathrm{DF}}$ well-defined iff $\check{\delta} g_{ijkl} = 1$, so that $D \underset{(2)}{b} = (\underset{(3)}{\mathrm{H}} \restriction_{\mathcal{O}_i}, 0)$

and $\mathrm{Per}(\underset{(3)}{\mathrm{H}}) \subset 2\pi\mathbb{Z}$ (Dirac's quantisation of charge)

Upshot: As in the Clutching Theorem, the DB 2-cocycle $\underset{(2)}{b}$ geometrises as a **1-gerbe** $\mathcal{G}^{(1)}$ [Murray & Stevenson '94-'99]

$$\mu_L \; : \; \mathrm{pr}_{1,2}^* L \otimes \mathrm{pr}_{2,3}^* L \xrightarrow{\cong} \mathrm{pr}_{1,3}^* L \qquad \mathbb{C}^\times \longrightarrow L, \underset{(1)}{\mathcal{A}_L} \qquad \mathcal{I}^{(1)}_{\underset{(2)}{\mathrm{B}}}$$

$$\begin{array}{ccccc} \downarrow & & \downarrow \pi_L & & \downarrow \\ \mathsf{Y}^{[3]}M & \overset{\mathrm{pr}_{1,2},\ \mathrm{pr}_{2,3},\ \mathrm{pr}_{1,3}}{\rightrightarrows} & \mathsf{Y}^{[2]}M & \overset{\mathrm{pr}_1}{\underset{\mathrm{pr}_2}{\rightrightarrows}} & \mathsf{Y}M, \underset{(2)}{\mathrm{B}} \\ & & & & \downarrow \pi_{\mathsf{Y}M} \\ & & & & M, \underset{(3)}{\mathrm{H}} \end{array}$$

$$(\mathrm{pr}_2^* - \mathrm{pr}_1^*)\underset{(2)}{\mathrm{B}} = \mathrm{curv}\big(\underset{(1)}{\mathcal{A}_L}\big) \qquad \pi_{\mathsf{Y}M}^* \underset{(3)}{\mathrm{H}} = \mathrm{d}\underset{(2)}{\mathrm{B}}$$

with the (groupoid) product μ_L on fibres of L associative.

The DF amplitude acquires a rigorous interpretation

$$\mathcal{A}_{\mathrm{DF}}^{(\mathrm{NG}),\mathrm{top}}[x] \equiv \mathrm{Hol}_{\mathcal{G}^{(1)}}(x(\Sigma)) = \iota_1([x^*\mathcal{G}^{(1)}])$$

for a canonical $\iota_1 \; : \; \mathcal{W}^3(\Sigma; 0) \xrightarrow{\cong} \mathrm{U}(1)$.

The geometrisation prescription generalises and yields a recursive definition of **p-gerbes** $\mathcal{G}^{(p)}$:

$$
\begin{array}{ccccccc}
\delta_{\mathsf{Y}}\mathcal{G}_{-1} = \mathbf{1} & \cdots & \mathcal{G}^{(p-2)} : \delta_{\mathsf{Y}}\mathcal{G}^{(p-1)} \cong \mathcal{I}_0^{(p-1)} & & \mathcal{G}^{(p-1)},\ \mathrm{curv}\,(\mathcal{G}^{(p-1)}) = \delta_{\mathsf{Y}}\underset{(p+1)}{\mathrm{B}} & & \mathcal{I}^{(p)}_{\underset{(p+1)}{\mathrm{B}}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathsf{Y}^{[p+3]}M & \cdots & \mathsf{Y}^{[3]}M & \overset{\mathrm{pr}_{ij}}{\equiv\!\!\!\Rrightarrow} & \mathsf{Y}^{[2]}M & \overset{\mathrm{pr}_i}{\rightrightarrows} & \mathsf{Y}M \\
& & & & & & \downarrow{\scriptstyle \pi_{\mathsf{Y}M}} \\
& & & & \pi_{\mathsf{Y}M}^{*}\underset{(p+2)}{\mathrm{H}} = \mathsf{d}\underset{(p+1)}{\mathrm{B}} & & M
\end{array}
$$

The Origin of Species:

la *gerbe* [fr.] – spray, sheaf, wreath *etc.*. . . [Giraud ’71]

Upshot & spin-off

- geometric (pre)quantisation *via* cohomological transgression [Gawędzki '87, rrS '11]

$$\tau_p\ :\ \mathbb{H}^{p+1}\big(M,\mathcal{D}(p+1)^\bullet\big)\longrightarrow\mathbb{H}^1\big(\mathcal{C}_pM,\mathcal{D}(1)^\bullet\big)\,,\qquad \mathcal{C}_pM\equiv[\mathcal{C}_p,M]$$

yields a (pre)quantum bundle $\mathcal{H}_\sigma=\Gamma_{(\mathrm{pol})}(\mathsf{F}\mathcal{L}_\sigma\times_{\mathbb{C}^\times}\mathbb{C})$, where

$$\mathbb{C}^\times\longrightarrow\pi^*_{\mathsf{T}^*\mathcal{C}_pM}\mathcal{L}_{\mathcal{G}^{(p)}}\otimes\mathcal{I}^{(0)}_{\vartheta_{\mathsf{T}^*\mathcal{C}_pM}}\equiv\mathcal{L}_\sigma\,,\underset{(1)}{\mathcal{A}}_{\mathcal{L}_\sigma}$$

$$\downarrow$$

$$\mathsf{P}_\sigma\equiv\mathsf{T}^*\mathcal{C}_pM\,,\ \Omega_\sigma=\delta\vartheta_{\mathsf{T}^*\mathcal{C}_pM}+\pi^*_{\mathsf{T}^*\mathcal{C}_pM}\int_{\mathcal{C}_p}\mathrm{ev}^*\underset{(p+2)}{\mathrm{H}}\equiv\mathrm{curv}(\underset{(1)}{\mathcal{A}}_{\mathcal{L}_\sigma})$$

for $\mathcal{L}_{\mathcal{G}^{(p)}}\in\tau_p([\mathcal{G}^{(p)}])$, and hence – classification of σ-models;

- geometrisation and classification of topological defects/dualities [Fuchs *et al.* '07, Runkel & rrS '08, rrS '11-'12], in particular…

…(pre)quantisable config$^{\text{nal}}$ symmetries – induced from actions

$$\lambda \;:\; \mathrm{G}_\sigma \times M \longrightarrow M \;:\; (g,m) \longmapsto \lambda_g(m)$$

of (Lie) groups $\mathrm{G}_\sigma \subset \mathrm{Isom}(M,\mathrm{g})$ that are generalised $\underset{(p+2)}{\mathrm{H}}$-hamiltonian,

$$\forall\, X \in \mathrm{Lie}(\mathrm{G}_\sigma)\; \exists\, \mathfrak{K}_X \equiv \big(\mathrm{T}_{(e,\cdot)}\lambda_X, \kappa_X\big) \in \Gamma\big(E^{(1,p)}M\big) \;:\; \mathrm{d}_{\underset{(p+2)}{\mathrm{H}}}\mathfrak{K}_X = 0\,,$$

so that the $\underset{(p+2)}{\mathrm{H}}$-twisted Vinogradov bracket

$$[[\cdot,\cdot]]_{\underset{(p+2)}{\mathrm{H}}} \;:\; \Gamma\big(E^{(1,p)}M\big) \times \Gamma\big(E^{(1,p)}M\big) \longrightarrow \Gamma\big(E^{(1,p)}M\big)$$

$$[[(\mathcal{V}_1, \upsilon_1), (\mathcal{V}_2, \upsilon_2)]]_{\underset{(p+2)}{\mathrm{H}}}$$

$$= \;\Big([\mathcal{V}_1, \mathcal{V}_2],\, \mathscr{L}_{\mathcal{V}_1}\upsilon_2 - \mathscr{L}_{\mathcal{V}_2}\upsilon_1 - \tfrac{1}{2}\,\mathrm{d}\big(\imath_{\mathcal{V}_1}\upsilon_2 - \imath_{\mathcal{V}_2}\upsilon_1\big) + \imath_{\mathcal{V}_1}\imath_{\mathcal{V}_2}\underset{(p+2)}{\mathrm{H}}\Big)$$

closes on their set $\Gamma\big(E^{(1,p)}M\big)_{\underset{(p+2)}{\mathrm{H}}-\mathrm{ham}} \subset \Gamma\big(E^{(1,p)}M\big)$.

We distinguish

$\rightarrow$ global/rigid symmetries
(set *inequivalent* field configurations in $\mathcal{A}_{\mathrm{DF}}$-correspondence)

lift to families of p-gerbe 1-isomorphisms

$$\Phi_g \; : \; \lambda_g^* \mathcal{G}^{(p)} \xrightarrow{\cong} \mathcal{G}^{(p)} \,, \qquad g \in \mathrm{G}_\sigma$$

that transgress to automorphisms of $\mathcal{H}_\sigma$, *e.g.*, for $p = 1$,

$$\alpha_E \; : \; \mathrm{pr}_{1,3}^* \widehat{\lambda}_g^{[2]\,*} L \otimes \mathrm{pr}_{3,4}^* E \xrightarrow{\cong} \mathrm{pr}_{1,2}^* E \otimes \mathrm{pr}_{2,4}^* L \qquad \mathbb{C}^\times \longrightarrow E, \underset{(1)}{\mathcal{A}_E}$$

$$\mathrm{Y}_g^{[2]} M \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} \lambda_g^* \mathrm{Y}M \times_M \mathrm{Y}M \equiv \mathrm{Y}_g M \qquad (\pi_E : E \to \mathrm{Y}_g M, \quad \pi_{\mathrm{Y}M} \circ \mathrm{pr}_2 : \mathrm{Y}_g M \to M)$$

(α_E, μ_L) comp$^{\mathrm{ble}}$ $\qquad \mathrm{curv}\big(\underset{(1)}{\mathcal{A}_E}\big) = \widehat{\lambda}_g^* \underset{(2)}{\mathrm{B}} - \underset{(2)}{\mathrm{B}}$ $\qquad M$

$\rightarrow$ local/gauge config$^{\text{nal}}$ symmetries
(relate *equivalent* descriptions of a field configuration)

Gauging of G_σ models descent to the orbispace $M \longrightarrow M/\mathrm{G}_\sigma$

Th$^{\text{m}}$ (Principle of Descent) [Gawędzki, Waldorf & rrS '10]
For $\lambda \,:\, \mathrm{G}_\sigma \times M \longrightarrow M$ free and proper,

$$\mathfrak{BGrb}^{(p)}(M/\mathrm{G}_\sigma) \cong \mathfrak{BGrb}^{(p)}(M)^{(\mathrm{G}_\sigma,\varrho_\lambda=0)}\,,$$

where the RHS is the (weak $(p+1)$-)category of p-gerbes over M with a G_σ-equivariant structure relative to a *vanishing*

$$\varrho_\lambda \in \Omega^{p+1}(\mathrm{G}_\sigma \times M) \qquad : \qquad \mathrm{d}\varrho_\lambda = \left(\lambda^* - \mathrm{pr}_2^*\right) \underset{(p+2)}{\mathrm{H}}\,.$$

The structure is an extension of the 0-cell $\mathcal{G}^{(p)}$ to a $(p+2)$-tuple $(\mathcal{G}^{(p)}, \Upsilon^{(p)}, \Upsilon^{(p-1)}, \ldots, \Upsilon^{(0)})$ over $\mathrm{N}^\bullet(\mathrm{G}_\sigma{\ltimes_\lambda}M)$ based on a 1-isomorphism

$$\Upsilon^{(p)} \;:\; \lambda^*_{\cdot}\mathcal{G}^{(p)} \xrightarrow{\;\cong\;} \mathrm{pr}_2^*\mathcal{G}^{(p)} \otimes \mathcal{I}^{(p)}_{\varrho_\lambda}\,.$$

The many faces of a G_σ-equivariant structure

- an extension of the $(p+1)$-cocycle of $\mathbb{H}^{p+1}(M, \mathcal{D}(p+1)^\bullet)$ for $\mathcal{G}_p$ to a $(p+1)$-cocycle in an extension of the Čech–de Rham bicomplex in the direction of G_σ-cohomology;
- [rrS '12] geometric data for the topological gauge-symmetry defect of the σ-model over Ω_p (based on [Runkel & rrS '09]).

Generically, ϱ_λ, as determined by the $\mathfrak{K}_{t_A}$ for $\mathrm{Lie}(\mathrm{G}_\sigma) \equiv \mathfrak{g}_\sigma = \bigoplus_{A=1}^D \langle t_A \rangle$, is *non*-vanishing, and so we need...

Universal Gauge Principle

[Gawędzki & Reis '02-'03, Gawędzki, Waldorf & rrS '07-'13, rrS '11-'13]

$\mathcal{A}_{\mathrm{DF}}$ admits **gauging** of G_σ *via* 'minimal coupling' of $\mathcal{A} \in \Omega^1(\mathrm{P}_{\mathrm{G}_\sigma}) \otimes \mathfrak{g}_\sigma$ if

1. [Gawędzki, Waldorf & rrS '07-'13, rrS '11-'13, '19] **SGA**=0
$$\Longleftrightarrow \left(\bigoplus_{A=1}^D C^\infty(M, \mathbb{R}) \mathfrak{K}_{t_A}, [[\cdot,\cdot]]_{(p+2)}^{\mathrm{H}} \right) \cong \mathfrak{g}_\sigma \ltimes_\lambda M;$$
2. [Gawędzki, Waldorf & rrS '07-'13] **LGA**=0
$\Longleftrightarrow$ exists a G_σ-equivariant structure on $\mathcal{G}^{(p)}$ rel. to ϱ_λ.

Applications:

- geometrisation and cohomological classification of obstructions against gauging and of inequivalent gaugings, and hence
- natural mapping of the moduli space of σ-models, with beautiful connections to TFT (explicit constructions for ‘all’ 2d RCFTs)
- reconstruction of T-duality outside the topological context…

The Higher Dogmatics: The Three+ $\mathcal{G}$-Sluagh-ghairms

↯ **It matters iff it lifts to $\mathcal{G}$.**

↯↯ **Global symmetry is invariance of $\mathcal{G}$.**

↯↯↯ **Local config$^{\text{nal}}$ symmetry is equivariance of $\mathcal{G}$.**

↯↯↯↯ … (Duality/top$^{\text{al}}$ defect is a $\mathcal{G}$-bimodule *etc.*) …

Part II

Putting a spin on it,
or
a $\mathbb{Z}/2\mathbb{Z}$-graded higher geometry

The goal

The higher sgeometry of a **super(geometric/symmetric)-σ-model** of (generalised-minimal) 'embeddings'

$$[\Omega_p, \mathcal{M}] = ?$$

of a $(p+1)$-dimensional riemannian worldvolume Ω_p 'in' a **sm_fold** $\mathcal{M}$ endowed with an action

$$\lambda \ : \ \mathrm{G} \times \mathcal{M} \longrightarrow \mathcal{M} \qquad (?)$$

of a **supersymmetry** Lie sgroup G.

Physical motivation

Understanding the (s)geometric structure (*sensu largissimo*) of superstring theory-inspired & -related FTs, with view to elucidation of the deep nature of the tremendously robust yet notoriously elusive **AdS/CFT correspondence**.

Sm_folds $\mathcal{M} = (|\mathcal{M}|, \mathcal{O}_\mathcal{M})$ with **body** $|\mathcal{M}| \in \mathrm{Ob}\,\mathbf{TopMan}$ and **structure sheaf** $\mathcal{O}_\mathcal{M} : \mathscr{T}(|\mathcal{M}|)^{\mathrm{op}} \longrightarrow \mathbf{sAlg_{scomm}}$,

$$\mathcal{O}_\mathcal{M} \sim_{\mathrm{loc}} \left(\mathbb{R}^m, C^\infty(\cdot, \mathbb{R}) \otimes \wedge^\bullet \mathbb{R}^n\right) \equiv \mathbb{R}^{m|n},$$

form a category **sMan** with morphisms

$$\varphi \equiv (|\varphi|, \varphi^*) : (|\mathcal{M}_1|, \mathcal{O}_{\mathcal{M}_1}) \longrightarrow (|\mathcal{M}_2|, \mathcal{O}_{\mathcal{M}_2}),$$

$$|\varphi| \in \mathrm{Hom}_{\mathbf{TopMan}}(|\mathcal{M}_1|, |\mathcal{M}_2|), \qquad \varphi^* : \mathcal{O}_{\mathcal{M}_2} \Longrightarrow |\varphi|_* \mathcal{O}_{\mathcal{M}_1}$$

It admits products $\mathcal{M}_1 \times \mathcal{M}_2 = (|\mathcal{M}_1| \times |\mathcal{M}_2|, \mathcal{O}_{\mathcal{M}_1} \widehat{\otimes} \mathcal{O}_{\mathcal{M}_2})$.

By the Yoneda Lemma, $\mathrm{Yon}_\cdot : \mathbf{sMan} \hookrightarrow \mathbf{Presh}(\mathbf{sMan})$, and so

$$\mathcal{M} \quad \sim \quad \mathrm{Yon}_\mathcal{M}(-) \equiv \mathrm{Hom}_{\mathbf{sMan}}(-, \mathcal{M}) : \mathbf{sMan}^{\mathrm{op}} \longrightarrow \mathbf{Set},$$

with $\mathrm{Yon}_\mathcal{M}(\mathcal{S}) \equiv \mathrm{Hom}_{\mathbf{sMan}}(\mathcal{S}, \mathcal{M})$ the **set of $\mathcal{S}$-points in** $\mathcal{M}$, and

$$\varphi \quad \sim \quad \mathrm{Yon}_\varphi(-) \equiv \mathrm{Hom}_{\mathbf{sMan}}(-, \varphi) = \varphi\circ$$

$$\text{with} \quad \mathrm{Yon}_\varphi(\mathcal{S}) : \mathrm{Yon}_{\mathcal{M}_1}(\mathcal{S}) \longrightarrow \mathrm{Yon}_{\mathcal{M}_2}(\mathcal{S}).$$

With the help of local charts ($|\mathcal{U}_I| \in \mathscr{T}(|\mathcal{M}_I|),\ I \in \{1,2\}$)

$$\kappa_I\ :\ (|\mathcal{U}_I|, \mathcal{O}_{\mathcal{M}_I}\restriction_{|\mathcal{U}_I|}) \equiv \mathcal{U}_I \xrightarrow{\cong} (|\mathcal{W}_I|, C^\infty(\cdot,\mathbb{R}) \otimes \wedge^\bullet \mathbb{R}^{n_I}) \equiv \mathcal{W}_I\,,$$

with the corresponding local coordinates $(x_I^a, \theta_I^\alpha)^{(a,\alpha)\in\overline{1,m_I}\times\overline{1,n_I}}$, the above yields a local description of morphisms

$$\varphi_{1,2} \equiv \kappa_2 \circ \varphi \circ \kappa_1^{-1} \in \mathrm{Hom}_{\mathbf{sMan}}(\mathcal{W}_1, \mathcal{W}_2) \equiv \mathrm{Yon}_{\mathcal{W}_2}(\mathcal{W}_1)$$

determined (as are *all* $\mathcal{W}_1$-points in $\mathcal{W}_2$ in virtue of the LCTh$^{\mathrm{m}}$) by

$$x_2^{a_2}(\theta_1, x_1) \quad\sim\quad \varphi_{1,2}^*(x_2^{a_2}) = \sum_{k=0}^{q_1} \theta_1^{\alpha_1^1}\,\theta_1^{\alpha_1^2} \cdots \theta_1^{\alpha_1^k}\,\Phi^{a_2}_{\alpha_1^1\alpha_1^2\ldots\alpha_1^k}(x_1^{b_1})\,,$$

$$\theta_2^{\alpha_2}(\theta_1, x_1) \quad\sim\quad \varphi_{1,2}^*(\theta_2^{\alpha_2}) = \sum_{l=0}^{q_1} \theta_1^{\alpha_1^1}\,\theta_1^{\alpha_1^2} \cdots \theta_1^{\alpha_1^l}\,\Phi^{\alpha_2}_{\alpha_1^1\alpha_1^2\ldots\alpha_1^l}(x_1^{b_1})$$

where $\Phi^{a_2}_{\alpha_1^1\alpha_1^2\ldots\alpha_1^{2r+1}} \equiv 0 \equiv \Phi^{\alpha_2}_{\alpha_1^1\alpha_1^2\ldots\alpha_1^{2r}}$.

Upshot: $\mathrm{Hom}_{\mathbf{sMan}}(\Omega_p, \mathcal{M})$ ruled out as a candidate for $[\Omega_p, \mathcal{M}]$.

Instead [Freed '95],

$$[\Omega_p, \mathcal{M}] \equiv \underline{\mathrm{Hom}}_{\mathbf{sMan}}(\Omega_p, \mathcal{M}) := \mathrm{Hom}_{\mathbf{sMan}}(\Omega_p \times -, \mathcal{M})$$

$$\in \mathrm{Ob}\,\mathbf{Presh}(\mathbf{sMan}) ,$$

to be evaluated on the odd hyperplanes

$$\mathbb{R}^{0|N} \equiv (\{\bullet\}, \mathbb{R}[\eta^1, \eta^2, \ldots, \eta^N]) , \qquad N \in \mathbb{N}^\times ,$$

whereupon $\xi \in [\Omega_p, \mathcal{M}](\mathbb{R}^{0|N})$ decompose (locally) as

$$\xi^*(x^a) = \xi^a_0 + \xi^a_{i_1 i_2}\, \eta^{i_1}\, \eta^{i_2} + \cdots + \xi^a_{i_1 i_2 \ldots 2[\frac{N}{2}]}\, \eta^{i_1}\, \eta^{i_2} \cdots \eta^{i_{2[\frac{N}{2}]}} ,$$

$$\xi^*(\theta^\alpha) = \xi^\alpha_{i_1}\, \eta^{i_1} + \xi^\alpha_{i_1 i_2 i_3}\, \eta^{i_1}\, \eta^{i_2}\, \eta^{i_3} + \cdots + \xi^\alpha_{i_1 i_2 \ldots 2[\frac{N-1}{2}]+1}\, \eta^{i_1}\, \eta^{i_2} \cdots \eta^{i_{2[\frac{N-1}{2}]+1}}$$

& the $(\xi^\alpha_{i_1 i_2 \ldots i_k}, \xi^a_{i_1 i_2 \ldots i_k})$ become the (s)fields of the super-σ-model.

The next fundamental issue is **supersymmetry**, for which we need **Lie sgroups**, *i.e.*, group objects in **sMan**,

$$\left(\mathrm{G} = (|\mathrm{G}|, \mathcal{O}_{\mathrm{G}}), \mu \; : \; \mathrm{G} \times \mathrm{G} \longrightarrow \mathrm{G}, \mathrm{Inv} \; : \; \mathrm{G} \circlearrowleft, \varepsilon \; : \; \mathbb{R}^{0|0} \longrightarrow \mathrm{G}\right),$$

with body $|\mathrm{G}| \in \mathrm{Ob}\,\mathbf{LieGrp}$. On these, we have **LI vector fields**

$$L \in \Gamma(\mathcal{T}\mathrm{G}) \quad : \quad \begin{array}{ccc} \mathcal{O}_{\mathrm{G}} \widehat{\otimes} \mathcal{O}_{\mathrm{G}} & \xleftarrow{\;\mu^*\;} & \mathcal{O}_{\mathrm{G}} \\ \Big\downarrow{\scriptstyle \mathrm{id}_{\mathcal{O}_{\mathrm{G}}} \otimes L} & & \Big\downarrow{\scriptstyle L} \\ \mathcal{O}_{\mathrm{G}} \widehat{\otimes} \mathcal{O}_{\mathrm{G}} & \xleftarrow[\;\mu^*\;]{} & \mathcal{O}_{\mathrm{G}} \end{array},$$

and the dual **LI 1-forms**. The RI objects are defined analogously. The supersymmetry groups are to act on the (s)fields as *per*

$$\lambda \; : \; \mathrm{G} \times \mathcal{M} \longrightarrow \mathcal{M}, \qquad \text{and so also}$$

$$|\lambda|_{\cdot} \; : \; |\mathrm{G}| \longrightarrow \mathrm{Aut}_{\mathbf{sMan}}(\mathcal{M}) \; : \; g \longmapsto \lambda \circ \left(\widehat{g} \times \mathrm{id}_{\mathcal{M}}\right) \equiv |\lambda|_g ,$$

where $\widehat{g} \; : \; \mathbb{R}^{0|0} \longrightarrow \mathrm{G}$ are the topological points.

In sFT's, Lie sgroups usually appear in disguise…

Th$^{\underline{m}}$[Kostant '77] $\mathbf{sLieGrp} \cong \mathbf{sHCp}$,

where $\mathbf{sHCp}$ is the category of **super-Harish-Chandra pairs** $\mathrm{G} \equiv (|\mathrm{G}|, \mathfrak{g}, \rho)$

$$|\mathrm{G}| \in \mathrm{Ob}\,\mathbf{LieGrp}\,, \qquad \mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)} \in \mathrm{Ob}\,\mathbf{sLieAlg} \quad \text{s.t.} \quad \mathfrak{g}^{(0)} \equiv \mathrm{Lie}\,(|\mathrm{G}|)\,,$$

$$\rho\ :\ |\mathrm{G}| \longrightarrow \mathrm{End}_{\mathbf{sLieAlg}}(\mathfrak{g}) \quad \text{s.t.} \quad \rho(\cdot)\restriction_{\mathfrak{g}^{(0)}} \equiv \mathsf{T}_e\mathrm{Ad}.$$

with morphism $(\Phi, \phi)\ :\ (|\mathrm{G}_1|, \mathfrak{g}_1, \rho_1) \longrightarrow (|\mathrm{G}_2|, \mathfrak{g}_2, \rho_2)$

$$\Phi \in \mathrm{Hom}_{\mathbf{LieGrp}}\,(|\mathrm{G}_1|, |\mathrm{G}_2|)\,, \qquad \phi \in \mathrm{Hom}_{\mathbf{sLieAlg}}\,(\mathfrak{g}_1, \mathfrak{g}_2) \quad \text{s.t.} \quad \phi\restriction_{\mathfrak{g}^{(0)}} = \mathsf{T}_e\Phi\,,$$

$$\big(\rho_2 \circ \Phi(\cdot)\big) \circ \phi = \phi \circ \rho_1(\cdot)$$

Remark: $\cong$ uses the Hopf-superalgebra structure on $U(\mathfrak{g})$ and yields

$$\mathcal{O}_{(|\mathrm{G}|,\mathfrak{g},\rho)} = \mathrm{Hom}_{U(\mathfrak{g}^{(0)})-\mathrm{Mod}}\big(U(\mathfrak{g}), C^\infty(-,\mathbb{R})\big) \quad \sim \quad C^\infty(-,\mathbb{R}) \otimes \bigwedge^\bullet \mathfrak{g}^{(1)\,*}.$$

Examples of Lie sgroups:

- sMink$(d, 1 \,|\, D_{d,1})$ as an abstract Lie sgroup is

$$\mathrm{sMink}(d,1|D_{d,1}) = \big(\mathbb{R}^{d+1}, C^\infty(\cdot,\mathbb{R}) \otimes \bigwedge{}^\bullet \mathbb{R}^{D_{d,1}}\big)\,, \qquad D_{d,1} = \dim S_{d,1}\,,$$

with $S_{d,1}$ a distinguished Majorana-spinor Cliff$(\mathbb{R}^{d,1})$-module.
It admits global coörds $\{x^a, \theta^\alpha\}^{(a,\alpha)\in\overline{0,d}\times\overline{1,D_{d,1}}}$ and

$$\mu^* \,:\, (x^a, \theta^\alpha) \longmapsto \big(x^a \otimes \mathbf{1} + \mathbf{1} \otimes x^a - \tfrac{1}{2}\,\theta^\alpha \otimes (C\,\Gamma^a)_{\alpha\beta}\,\theta^\beta, \theta^\alpha \otimes \mathbf{1} + \mathbf{1} \otimes \theta^\alpha\big)\,,$$

$$\mathrm{Inv}^* \,:\, (x^a, \theta^\alpha) \longmapsto (-x^a, -\theta^\alpha)\,,$$

or, equivalently, in the $\mathcal{S}$-point picture,

$$(x_1^a, \theta_1^\alpha)\cdot(x_2^b, \theta_2^\beta) = \big(x_1^a + x_2^a - \tfrac{1}{2}\,\theta_1\,\overline{\Gamma}^a\,\theta_2, \theta_1^\alpha + \theta_2^\alpha\big)\,, \qquad (x^a, \theta^\alpha)^{-1} = (-x^a, -\theta^\alpha)\,.$$

As a sHCp,

$$\mathrm{sMink}(d,1\,|\,D_{d,1}) = \big(\mathrm{Mink}(d,1), \mathfrak{smink}(d,1\,|\,D_{d,1}) = \bigoplus_{a=0}^{d} \langle P_a\rangle \oplus \bigoplus_{\alpha=1}^{D_{d,1}} \langle Q_\alpha\rangle\,, 0\big)\,,$$

$$\{Q_\alpha, Q_\beta\} = (C\,\Gamma^a)_{\alpha\beta}\,P_a\,, \qquad [P_a, P_b] = 0 = [Q_\alpha, P_a]\,.$$

- $\mathrm{SU}(2,2\,|\,4)$ as a sHCp with the body Lie group

$$|\mathrm{SU}(2,2\,|\,4)| = \mathrm{Spin}(4,2) \times \mathrm{Spin}(6)\,,$$

the Lie salgebra

$$\mathfrak{su}(2,2\,|\,4) = \Big(\big(\bigoplus_{a=0}^{4} \langle P_a \rangle \oplus \bigoplus_{a'=5}^{9} \langle P_{a'} \rangle \big) \oplus \bigoplus_{(\alpha,\alpha',I)\in\overline{1,4}\times\overline{1,4}\times\{1,2\}} \langle Q_{\alpha\alpha' I} \rangle \Big) \oplus \Big(\bigoplus_{a,b=0}^{4} \langle J_{ab} = -J_{ba} \rangle \oplus \bigoplus_{a',b'=5}^{9} \langle J_{a'b'} = -J_{b'a'} \rangle \Big)$$

$$\{Q_{\alpha\alpha' I}, Q_{\beta\beta' J}\} = \mathrm{i}\,\big(-2(\widehat{C}\,\widehat{\Gamma}^{\widehat{a}} \otimes \mathbf{1})_{\alpha\alpha' I \beta\beta' J}\, P_{\widehat{a}} + (\widehat{C}\,\widehat{\Gamma}^{\widehat{a}\widehat{b}} \otimes \sigma_2)_{\alpha\alpha' I\beta\beta' J}\, J_{\widehat{a}\widehat{b}}\big)\,,$$

$$[Q_{\alpha\alpha' I}, P_{\widehat{a}}] = -\tfrac{1}{2}\,(\widehat{\Gamma}_{\widehat{a}} \otimes \sigma_2)^{\beta\beta' J}_{\alpha\alpha' I}\, Q_{\beta\beta' J}\,, \qquad [P_{\widehat{a}}, P_{\widehat{b}}] = \varepsilon_{\widehat{a}\widehat{b}}\, J_{\widehat{a}\widehat{b}}\,, \quad \varepsilon_{\widehat{a}\widehat{b}} = \begin{cases} +1 & \text{if } \widehat{a},\widehat{b} \in \overline{0,4} \\ -1 & \text{if } \widehat{a},\widehat{b} \in \overline{5,9} \\ 0 & \text{otherwise} \end{cases}\,,$$

$$[J_{\widehat{a}\widehat{b}}, J_{\widehat{c}\widehat{d}}] = \eta_{\widehat{a}\widehat{d}}\, J_{\widehat{b}\widehat{c}} - \eta_{\widehat{a}\widehat{c}}\, J_{\widehat{b}\widehat{d}} + \eta_{\widehat{b}\widehat{c}}\, J_{\widehat{a}\widehat{d}} - \eta_{\widehat{b}\widehat{d}}\, J_{\widehat{a}\widehat{c}}\,,$$

$$[Q_{\alpha\alpha' I}, J_{\widehat{a}\widehat{b}}] = -\tfrac{1}{2}\,\varepsilon_{\widehat{a}\widehat{b}}\,(\widehat{\Gamma}_{\widehat{a}\widehat{b}} \otimes \mathbf{1})^{\beta\beta' J}_{\alpha\alpha' I}\, Q_{\beta\beta' J}\,, \qquad [P_{\widehat{a}}, J_{\widehat{b}\widehat{c}}] = \eta_{\widehat{a}\widehat{b}}\, P_{\widehat{c}} - \eta_{\widehat{a}\widehat{c}}\, P_{\widehat{b}}\,.$$

and the standard spinor realisation of the former on the Graßmann-odd component of the latter.

Sgeometric data: A **Nambu–Goto sbackground**

$$\big(\mathcal{M}, \mathrm{g}, \underset{(p+2)}{\mathrm{H}}\big) \equiv \mathfrak{sB}^{(p)}_{\mathrm{NG}}$$

of a **super-σ-model** consists of

- a smanifold $\mathcal{M}$ (the **starget**) with an action λ of a Lie sgroup G (the **supersymmetry group**), inducing fundamental vector fields

$$\mathcal{K}_{\cdot} \ : \ \mathfrak{g} \equiv \Gamma(\mathcal{T}\mathrm{G})^{\mathrm{L}} \longrightarrow \mathcal{T}\mathcal{M} \ : \ L \longmapsto -(\widehat{e}^{*} \circ L \otimes \mathrm{id}_{\mathcal{O}_{\mathcal{M}}}) \circ \lambda^{*} \, ;$$

- a G-invariant **smetric** $\mathrm{g} \in \Gamma(\mathcal{T}^{*}\mathcal{M} \otimes^{\mathrm{sym}} \mathcal{T}^{*}\mathcal{M})$,

$$\forall_{(g,X)\in|\mathrm{G}|\times\mathfrak{g}} \ : \ \big(|\lambda|_{g}^{*}\mathrm{g} = \mathrm{g} \quad \wedge \quad \mathscr{L}_{\mathcal{K}_X}\mathrm{g} = 0 \big) \, ;$$

- a G-invariant de Rham **$(p+2)$-scocycle** $\underset{(p+2)}{\mathrm{H}} \in Z^{p+2}_{\mathrm{dR}}(\mathcal{M})$,

$$\forall_{(g,X)\in|\mathrm{G}|\times\mathfrak{g}} \ : \ \big(|\lambda|_{g}^{*}\underset{(p+2)}{\mathrm{H}} = \underset{(p+2)}{\mathrm{H}} \quad \wedge \quad \mathscr{L}_{\mathcal{K}_X}\underset{(p+2)}{\mathrm{H}} = 0 \big) \, ;$$

THE sgeometry: There is a large class of $\mathfrak{s}\mathfrak{B}^{(p)}_{\rm NG}$ with G-orbits as stargets…

Th$^{\underline{\rm m}}$[Kostant '77, Koszul '82, Fioresi *et al.* '07]:
Let $\mathrm{G}\in\mathrm{Ob}\,\mathbf{sLieGrp}$ and H its Lie sub-sgroup with $\mathrm{sLie}\,\mathrm{H}\equiv\mathfrak{h}$. $\exists$ an ess. unique sm_fold structure on the **homogeneous space**

$$\mathrm{G}/\mathrm{H}=\big(|\mathrm{G}|/|\mathrm{H}|,\mathcal{O}_{\mathrm{G}/\mathrm{H}}\big)\qquad\text{s.t.}$$

$$\mathcal{O}_{\mathrm{G}/\mathrm{H}}(\cdot)=\big\{\,f\in\mathcal{O}_{\mathrm{G}}\big(\pi^{-1}_{|\mathrm{G}|/|\mathrm{H}|}(\cdot)\big)\quad\big|\quad\forall_{(h,J)\in|\mathrm{H}|\times\mathfrak{h}}\ :\ |\wp|^*_h(f)=f\ \wedge\ L_J(f)=0\big\},$$

$$\begin{array}{ccccc}
\mathrm{G}\times\mathrm{G} & \xrightarrow{\ \ell\ } & \mathrm{G} & \dashrightarrow & |\mathrm{G}| \\
\Big\downarrow{\scriptstyle \mathrm{id}_{\mathrm{G}}\times\pi_{\mathrm{G}/\mathrm{H}}} & & \Big\downarrow{\scriptstyle \pi_{\mathrm{G}/\mathrm{H}}} & & \Big\downarrow{\scriptstyle \pi_{|\mathrm{G}|/|\mathrm{H}|}\equiv|\pi_{\mathrm{G}/\mathrm{H}}|} \\
\mathrm{G}\times\mathrm{G}/\mathrm{H} & \xrightarrow{\ [\ell]\ } & \mathrm{G}/\mathrm{H} & \dashrightarrow & |\mathrm{G}|/|\mathrm{H}|
\end{array}\ .$$

Actually, $(\mathrm{G},\pi_{\mathrm{G}/\mathrm{H}},\mathrm{H})$ is a principal H-(s)bundle with local sections

$$\sigma_{\mathcal{U}}\ :\ \mathcal{U}\equiv\big(|\mathcal{U}|,\mathcal{O}_{\mathrm{G}/\mathrm{H}}\restriction_{|\mathcal{U}|}\big)\longrightarrow\mathrm{G}\qquad\text{with body}$$

$$|\sigma_{\mathcal{U}}|\ :\ |\mathcal{U}|\longrightarrow|\mathrm{G}|\,,\qquad \pi_{|\mathrm{G}|/|\mathrm{H}|}\circ|\sigma_{\mathcal{U}}|=\mathrm{id}_{|\mathcal{U}|}\,.$$

Dynamics with a nonlinear realisation of supersymmetry calls for a **reductive homogeneous space**:

$\mathrm{G/H}$ for $(\mathrm{G}, \mathrm{H} \subset |\mathrm{G}|)$ with $\mathrm{sLie(G)} = \mathfrak{g}$ and $\mathrm{Lie(H)} = \mathfrak{h}$ s.t.

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{h}\,, \qquad \mathfrak{t} = \mathfrak{t}^{(0)} \oplus \mathfrak{t}^{(1)} \equiv \bigoplus_{a=0}^{d_0} \langle P_a \rangle \oplus \bigoplus_{\alpha=1}^{d_1} \langle Q_\alpha \rangle\,, \qquad \mathfrak{h} = \bigoplus_{\kappa=1}^{d_S} \langle J_\kappa \rangle$$

is **reductive**, *i.e.*, s.t. $[\mathfrak{h}, \mathfrak{t}] \subset \mathfrak{t}$.

For these, the LI $\mathfrak{g}$-valued Maurer–Cartan 1-sform

$$\theta_{\mathrm{L}} = \theta_{\mathrm{L}}^{\mu} \otimes t_\mu + \theta_{\mathrm{L}}^{\kappa} \otimes J_\kappa\,, \qquad \bigoplus_{\mu=0}^{d_0+d_1} \langle t_\mu \rangle \equiv \mathfrak{t}$$

yields (a principal H-connection $\Theta = \theta_{\mathrm{L}}^{\kappa} \otimes J_\kappa$ and) H-stensors

$$|\wp|_{\cdot}^{*}\theta_{\mathrm{L}}^{\mu} = \rho(\cdot)^{\mu}{}_{\nu}\, \theta_{\mathrm{L}}^{\nu}\,.$$

that give rise to H-basic (cov.) stensors $T = \tau_{\mu_1\mu_2\ldots\mu_n}\, \theta_{\mathrm{L}}^{\mu_1} \otimes \theta_{\mathrm{L}}^{\mu_2} \otimes \cdots \otimes \theta_{\mathrm{L}}^{\mu_n}$

for $$\tau_{\mu_1\mu_2\ldots\mu_n} = \tau_{\nu_1\nu_2\ldots\nu_n}\, \rho(h)^{\nu_1}{}_{\mu_1}\, \rho(h)^{\nu_2}{}_{\mu_2} \cdots \rho(h)^{\nu_n}{}_{\mu_n}\,, \qquad h \in \mathrm{H}\,.$$

THE stensorial data: Model the starget G/H (patchwise) by

$$\Sigma^{\mathrm{NG}} := \bigsqcup_{i\in\mathscr{I}} \sigma_i(\mathcal{U}_i)\,, \qquad \sigma_i \;:\; \mathcal{U}_i \equiv \left(|\mathcal{U}_i|, \mathcal{O}_{\mathrm{G}/\mathrm{H}}\restriction_{|\mathcal{U}_i|}\right) \longrightarrow \mathrm{G}$$

for an open cover $\{|\mathcal{U}_i|\}_{i\in\mathscr{I}}$ of $|\mathrm{G}|/\mathrm{H}$ trivialising for the body principal H-bundle $(|\mathrm{G}|, \pi_{|\mathrm{G}|/\mathrm{H}}, \mathrm{H})$, and subsequently pull back an H-basic LI 'smetric'

$$\mathrm{g} = \mathrm{g}_{(ab)}\, \theta^a_{\mathrm{L}} \otimes \theta^b_{\mathrm{L}}\,, \qquad \mathrm{g} = \pi^*_{\mathrm{G}/\mathrm{H}}\underline{\mathrm{g}}$$

and an H-basic LI de Rham $(p+2)$-scocycle

$$\underset{(p+2)}{\chi} = \chi_{\mu_1\mu_2\ldots\mu_{p+2}}\,\theta^{\mu_1}_{\mathrm{L}} \wedge \theta^{\mu_2}_{\mathrm{L}} \wedge \cdots \wedge \theta^{\mu_{p+2}}_{\mathrm{L}}\,, \qquad \underset{(p+2)}{\chi} = \pi^*_{\mathrm{G}/\mathrm{H}}\underset{(p+2)}{\mathrm{H}}$$

to G/H along the σ_i, resp. use their precursors $(\underline{\mathrm{g}}, \underset{(p+2)}{\mathrm{H}})$.

Examples of reductive homogeneous spaces of Lie sgroups:

- $\mathrm{sMink}(d,1\,|\,D_{d,1}) \equiv \mathrm{sISO}(d,1\,|\,D_{d,1})/\mathrm{Spin}(d,1)$ for $\mathrm{sISO}(d,1\,|\,D_{d,1}) = \mathrm{sMink}(d,1\,|\,D_{d,1}) \rtimes_{L_{d,1}\oplus S_{d,1}} \mathrm{Spin}(d,1)$, with

$$\mathrm{g} = \eta_{ab}\,\theta_{\mathrm{L}}^a \otimes \theta_{\mathrm{L}}^b\,,$$

$$\underset{(p+2)}{\mathrm{H}} = \begin{cases} \theta_{\mathrm{L}}^\alpha \wedge (C\,\Gamma_{11})_{\alpha\beta}\,\theta_{\mathrm{L}}^\beta & (p=0) \\ \theta_{\mathrm{L}}^\alpha \wedge (C\,\Gamma_{a_1 a_2 \ldots a_p})_{\alpha\beta}\,\theta_{\mathrm{L}}^\beta \wedge \theta_{\mathrm{L}}^{a_1} \wedge \theta_{\mathrm{L}}^{a_2} \wedge \cdots \wedge \theta_{\mathrm{L}}^{a_p} & (1<p<8) \end{cases}$$

  the admissible (d,p,N) filling up the 'old brane scan'

- $\mathrm{s}(\mathrm{AdS}_5 \times \mathbb{S}^5) \equiv \mathrm{SU}(2,2|4)/(\mathrm{Spin}(4,1)\times\mathrm{Spin}(5))$, with

$$\mathrm{g} = \eta_{ab}\,\theta_{\mathrm{L}}^a \otimes \theta_{\mathrm{L}}^b + \delta_{a'b'}\,\theta_{\mathrm{L}}^{a'} \otimes \theta_{\mathrm{L}}^{b'}\,,$$

$$\underset{(3)}{\mathrm{H}} = \theta_{\mathrm{L}}^{\alpha\alpha' I} \wedge \big(\widehat{C}\,\widehat{\Gamma}_{\widehat{a}} \otimes \sigma_3\big)_{\alpha\alpha' I\,\beta\beta' J}\,\theta_{\mathrm{L}}^{\beta\beta' J} \wedge \theta_{\mathrm{L}}^{\widehat{a}}$$

THE super-σ-model

Given a *closed* orientable m_fold Ω_p of $\dim \Omega_p = p+1$, a Lie sgroup G and a closed Lie subgroup $\mathrm{H} \subset |\mathrm{G}|$ with $(\mathfrak{g}, \mathfrak{h})$ reductive, assume given H-basic LI stensors on G:

$$\mathrm{g} = \mathrm{g}_{(ab)}\, \theta_{\mathrm{L}}^a \otimes \theta_{\mathrm{L}}^b \equiv \pi^*_{\mathrm{G/H}} \underline{\mathrm{g}}\,, \qquad \underset{(p+2)}{\chi} \equiv \pi^*_{\mathrm{G/H}} \underset{(p+2)}{\mathrm{H}} \in Z_{\mathrm{dR}}^{p+2}(\mathrm{G})^{\mathrm{G}}\,.$$

The **Green–Schwarz super-σ-model in the Nambu–Goto formulation** is a theory of smappings $[\Omega_p, \mathrm{G/H}] \ni \xi$ determined by the PLA for the DF (s)amplitudes with

$$S_{\mathrm{GS}}^{(\mathrm{NG}),(\mu_p)}[\xi] = \mu_p \int_{\Omega_p} \sqrt{\det\left(\xi^* \underline{\mathrm{g}}\right)} + \int_{\Omega_p} \xi^* \mathrm{d}^{-1} \underset{(p+2)}{\mathrm{H}}\,,$$

where $\mu_p \in \mathbb{R}^\times$ is a parameter*.

**To be fixed in what follows.*

General remarks:

The **svacuum** of the super-σ-model is a 'minimal' sembedding distorted by Lorentz-type sforces. Its 'localisation' effects

- a spontaneous breakdown $\mathrm{H} \searrow \mathrm{H}_{\mathrm{vac}}$ of the 'invisible' gauge symmetry (the **isotropy group**);
- a spontaneous breakdown $\mathfrak{t}^{(0)} \searrow \mathfrak{t}^{(0)}_{\mathrm{vac}}$ of the local translational symmetry.

Implication: A need for a mechanism of restoration of supersymmetry in the svacuum through freeze-out of the Graßmann-odd DOFs, as dictated by

$$\{Q_\alpha, Q_\beta\} = f_{\alpha\beta}{}^{a}\, P_a + f_{\alpha\beta}{}^{\kappa}\, J_\kappa\,,$$

which puts us in the context of the κ-symmetry of [de Azcárraga & Lukierski '82, Siegel '83] –

"a 'hidden' symmetry, with no evident geometric interpretation"...

Physically relevant models:

(i) the original Green–Schwarz–… p-sbranes in
$\mathrm{sMink}(d,1|ND_{d,1}) \equiv \mathrm{sISO}(d,1|ND_{d,1})/\mathrm{Spin}(d,1),\ N \in \mathbb{N}^{\times}$;

(ii) the Metsaev–Tseytlin sstring in
$\mathrm{s}(\mathrm{AdS}_5 \times \mathbb{S}^5) \equiv \mathrm{SU}(2,2|4)/(\mathrm{Spin}(4,1) \times \mathrm{Spin}(5))$;

(iii) the Zhou s-0-brane and sstring in
$\mathrm{s}(\mathrm{AdS}_2 \times \mathbb{S}^2) \equiv \mathrm{SU}(1,1|2)_2/(\mathrm{Spin}(1,1) \times \mathrm{Spin}(2))$;

(iv) the Park–Rey sstring in
$\mathrm{s}(\mathrm{AdS}_3 \times \mathbb{S}^3) \equiv \mathrm{SU}(1,1|2)_2^{\times 2}/(\mathrm{Spin}(2,1) \times \mathrm{Spin}(3))$;

(v) *the Metsaev–Tseytlin D3-brane in*
$\mathrm{s}(\mathrm{AdS}_5 \times \mathbb{S}^5) \equiv \mathrm{SU}(2,2|4)/(\mathrm{Spin}(4,1) \times \mathrm{Spin}(5))$;

(vi) *the M2-branes in* $\mathrm{s}(\mathrm{AdS}_4 \times \mathbb{S}^7)$ *and* $\mathrm{s}(\mathrm{AdS}_7 \times \mathbb{S}^4)$…

Empirical facts:

(H) The p-sbranes in sMink$(d,1|ND_{d,1})$ and the 0-sbrane in s$(\mathrm{AdS}_2\times\mathbb{S}^2)$ have

$$\underset{(p+2)}{[\chi]}_{\mathrm{dR}}=0\,,\quad \text{but}\quad \underset{(p+2)}{[\chi]}^{\mathrm{G}}_{\mathrm{dR}}\in\mathrm{CaE}^{p+2}(\mathrm{G})\setminus\{0\}\,.$$

(İW) the sstrings in s$(\mathrm{AdS}_q\times\mathbb{S}^q)$, $q\in\{2,3,5\}$ have

$$\underset{(3)}{[\chi]}^{\mathrm{G}}_{\mathrm{dR}}=0\in\mathrm{CaE}^3(\mathrm{G})\,,$$

but the supersymmetric primitives

do NOT İnönü-Wigner–contract

to the sminkowskian ones.

What are the **PROBLEMS** with the empirical facts?

Ad ($\dot{\rm I}$W) Signals potential 'ill-definedness' of the MT/PR/Zh super-σ-models whose construction was based upon the asymptotic correspondence with the GS super-σ-model. [rrS '18]

Ad (H) The choice of the cohomology critical for the meaning of $\mathcal{A}_{\rm DF}^{\rm (top)}$.

AND

Physics favours the (H-equivariant) Cartan–Eilenberg cohomology

$$\mathrm{CaE}^\bullet(\mathrm{G})_{\mathrm{H-equiv}} \equiv H^\bullet_{\mathrm{dR}}(\mathrm{G})^{\mathrm{G}}_{\mathrm{H-equiv}}\,,$$

BUT

(How) Does $\mathrm{CaE}^\bullet(\mathrm{G}) \setminus H^\bullet(\mathrm{G})$ topologise?

The Rabin-Crane-type argument/hypothesis:

Secretly, the GS super-σ-model for $[\Omega_p, \mathrm{G}/\mathrm{H} \equiv \mathcal{M}]$ is a theory of smappings from $[\Omega_p, \mathcal{M}/\Gamma_{\mathrm{KR}}]$ for $\Gamma_{\mathrm{KR}} \subset \mathrm{G}$ s.t.

$$\mathcal{M}/\Gamma_{\mathrm{KR}} \cong_{\mathrm{loc.}} \mathcal{M} \qquad \wedge \qquad H^\bullet_{\mathrm{dR}}(\mathcal{M})^{\mathrm{G}} \cong H^\bullet_{\mathrm{dR}}(\mathcal{M}/\Gamma_{\mathrm{KR}}) \,.$$

A working model

For $\mathcal{M} = \mathrm{sMink}(d, 1|D_{d,1})$, the sub-sgroup was identified in [Crane & Rabin '85] as the discrete Kostelecký-Rabin sgroup generated by *integer* stranslations

$$(x^a, \theta^\alpha) \longmapsto (y^b, \varepsilon^\beta) \cdot (x^a, \theta^\alpha)$$

with $y^b_{i_1 i_2 \ldots i_k}, \varepsilon^\alpha_{i_1 i_2 \ldots i_k} \in \mathbb{Z}$ (in the $\mathcal{S}$-point picture).

Field-theoretic consequences:

We must take into account the Γ_{KR}-twisted sector in $[\Omega_p, \mathrm{G}/\mathrm{H}]$, but then the Poisson-Lie salgebra of the Noether charges of supersymmetry of the GS super-σ-model,

$$\{h_A, h_B\}_{\Omega_\sigma} = -f_{AB}{}^C\, h_C + \mathcal{A}_{AB}\,,$$

exhibits a (classical!) **wrapping anomaly** [rrS '18].

Empirical fact: Some of these extensions trivialise distinguished 2-scocycles on the supersymmetry salgebra $\mathfrak{g}$.

Conclusion: Need to consider scentral extensions

$$\mathbf{0} \longrightarrow \mathfrak{z} \longrightarrow \widetilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow \mathbf{0}\,.$$

The latter is merely an (exact) (s)intuition with…
a rigorous cohomology story behind it…

Towards sgeometrisation of supersymmetric de Rham scocycles…

Th$^{\underline{\text{m}}}$: $\exists$ an isomorphism

$$[\gamma] \; : \; H^\bullet(\mathfrak{g},\mathbb{R}) \equiv \mathrm{CE}^\bullet(\mathfrak{g},\mathbb{R}) \xrightarrow{\cong} \mathrm{CaE}^\bullet(\mathrm{G}) \cong H^\bullet_{\mathrm{dR}}(\mathrm{G})^{\mathrm{G}} \,.$$

Th$^{\underline{\text{m}}}$: $\exists$ a correspondence

$$\mathrm{CE}^2(\mathfrak{g},\mathfrak{a}) \xleftrightarrow{1:1} \{ \text{ equivalence classes of scentral extensions of } \mathfrak{g} \text{ by } \mathfrak{a} \ \} \,,$$

where

$$\widetilde{\mathfrak{g}}_2 \sim \widetilde{\mathfrak{g}}_1 \quad \Longleftrightarrow \quad \begin{array}{ccccccccc} & & & & \widetilde{\mathfrak{g}}_1 & & & & \\ & & & \nearrow & & \searrow & & & \\ \mathbf{0} & \longrightarrow & \mathfrak{a} & & \downarrow\cong & & \mathfrak{g} & \longrightarrow & \mathbf{0} \;. \\ & & & \searrow & & \nearrow & & & \\ & & & & \widetilde{\mathfrak{g}}_2 & & & & \end{array}$$

Th$^{\underline{\text{m}}}$: $\mathbf{0} \longrightarrow \mathbb{R} \longrightarrow \widetilde{\mathfrak{g}}_{[\omega]} \longrightarrow \mathfrak{g} \longrightarrow \mathbf{0}$ determined by $[\omega] \in \mathrm{CaE}^2(\mathrm{G})$ integrates to $\mathbf{1} \longrightarrow \mathbb{C}^\times \longrightarrow \widetilde{\mathrm{G}} \longrightarrow \mathrm{G} \longrightarrow \mathbf{1}$ iff $\mathrm{Per}(\omega) \subset 2\pi\mathbb{Z}$ and $\ell_\cdot \; : \; \mathrm{G} \times (\mathrm{G},\omega) \longrightarrow (\mathrm{G},\omega)$ has a momentum map.

Idea of geometrisation – building the p-sgerbe $\mathcal{G}^{(p)}$:

(**Inspiration:** extended sspacetimes of [de Azcárraga *et al.* '00+])

1. Look for an LI 2-scocycle ω in

$$\left\langle \iota_{t_{\mu_1}} \iota_{t_{\mu_2}} \cdots \iota_{t_{\mu_p}} \underset{(p+2)}{\chi} \;\middle|\; \mu_1, \mu_2, \ldots, \mu_p \in \overline{0, d_0 + d_1} \right\rangle_{\mathbb{R}} .$$

2. Use $\int (\mathbf{0} \to \mathfrak{a} \to \widetilde{\mathfrak{g}}_{[\omega]} \to \mathfrak{g} \to \mathbf{0}) =: (\mathbf{1} \to A \to \widetilde{\mathrm{G}}_{[\omega]} \xrightarrow{\widetilde{\pi}} \mathrm{G} \to \mathbf{1})$ to partially reduce $\widetilde{\pi}^* \underset{(p+2)}{\chi}$ in $\mathrm{CaE}^\bullet(\widetilde{\mathrm{G}}_{[\omega]})$.

3. Repeat 1.-2. until complete reduction of $\widehat{\pi}^* \underset{(p+2)}{\chi}$ is obtained over an extension $\widehat{\mathrm{G}} \xrightarrow{\widehat{\pi}} \mathrm{G}$ in the corresponding $\mathrm{CaE}^\bullet(\widehat{\mathrm{G}})$, *i.e.*,

$$\exists \underset{(p+1)}{\beta} \in \Omega^{p+1}(\widehat{\mathrm{G}})^{\widehat{\mathrm{G}}} : \mathrm{d} \underset{(p+1)}{\beta} = \widehat{\pi}^* \underset{(p+2)}{\chi} .$$

4. Check that $\underset{(p+1)}{\beta}$ descends to $\widehat{\mathrm{G}}/\mathrm{H}$.

5. Use $\mathsf{YG} := \widehat{\mathrm{G}}$ as THE surjective submersion of $\mathcal{G}^{(p)}$ & DCAF à la [Murray & Stevenson '94-'99 *et al.*].

Constructive results:

Theorem I [rrS '17('12)] Consecutive resolution, through scentral extensions, of the various CaE 2-scocycles encountered in the analysis of the GS $(p+2)$-scocycles on sMink$(d,1|(N\cdot)D_{d,1})$, induces a hierarchy of surjective submersions necessary for the sgeometrisation of the latter, leading to the emergence of **sminkowskian Green–Schwarz p-sgerbes** (explicited for $p \in \{0,1,2\}$).

Abstraction:

An H-equivariant **Cartan–Eilenberg p-sgerbe** $\mathcal{G}^{(p)}$ of curvature curv $(\mathcal{G}^{(p)}) = \underset{(p+2)}{\chi}$ over G

$\equiv$ 'a p-gerbe object in **sLieGrp** (with an H-equivariant structure)'.

E.g., a **CaE 1-sgerbe** of curvature $\underset{(3)}{\chi}$,

$$\begin{array}{ccccc} \mu_L : \mathrm{pr}_{1,2}^*L \otimes \mathrm{pr}_{2,3}^*L \xrightarrow{\cong} \mathrm{pr}_{1,3}^*L & & \mathbb{C}^\times \longrightarrow \underset{(1)}{L, \mathcal{A}_L} & & \mathcal{I}^{(1)}_{\underset{(2)}{\beta}} \\ \downarrow & & \downarrow \pi_L & & \downarrow \\ \mathrm{Y}^{[3]}\mathrm{G} & \overset{\mathrm{pr}_{1,2},\ \mathrm{pr}_{2,3},\ \mathrm{pr}_{1,3}}{\Rrightarrow} & \mathrm{Y}^{[2]}\mathrm{G} & \overset{\mathrm{pr}_1}{\underset{\mathrm{pr}_2}{\rightrightarrows}} & \mathrm{YG}, \underset{(2)}{\beta} \\ & & & & \downarrow \pi_{\mathrm{YG}} \\ & & & & \mathrm{G}, \underset{(3)}{\chi} \end{array}$$

- $\mathrm{YG} \xrightarrow{\pi_{\mathrm{YG}}} \mathrm{G}$ and $L \xrightarrow{\pi_{\mathrm{YG}}} \mathrm{Y}^{[2]}\mathrm{G}$ are **sLieGrp** extensions;
- $\underset{(2)}{\beta}$ and $\underset{(1)}{\mathcal{A}_L}$ are LI relative to YG and L, respectively;
- μ_L is a **sLieGrp** isomorphism.

Constructive results – ct^{d}:

- The success of the sminkowskian sgeometrisation was repeated in [rrS '18] in the setting of Zhou's super-σ-model of [Zhou '99] for the sparticle in $\mathrm{s}(\mathrm{AdS}_2 \times \mathbb{S}^2)$.
- The celebrated Metsaev–Tseytlin super-σ-model of [Metsaev & Tseytlin '98] for the sstring in $\mathrm{s}(\mathrm{AdS}_5 \times \mathbb{S}^5)$, on the other hand, seems problematic. There exists

  an İnönü–Wigner-*non*contractible trivial 1-sgerbe,

  and a collection of no-go theorems…

Higher supersymmetry

- Global supersymmetry built in as G-invariance. ↯ ↯
- 'Hidden' gauge symmetry to be imposed as an H-equivariant structure (if we tread carefully, it is automatic – *cp* the construction of the super-σ-model). ↯ ↯ ↯
- What about the spontaneous breakdown of (s)symmetry by the svacuum? ↯

Problem: κ-symmetry mixes metric and topological DOFs.

We cannot change the nature of κ-symmetry, yet we *can* change the FT perspective…
(after [Hughes & Polchinski '86, Gauntlett, Itoh & Townsend '90])

THE *other* sgeometry: Pick up a salgebraic model of the body of the svacuum:

$$\bigoplus_{\underline{a}=0}^{p} \langle P_{\underline{a}} \rangle \equiv \mathfrak{t}_{\rm vac}^{(0)} \subset \mathfrak{t}_{\rm vac}^{(0)} \oplus \mathfrak{e}^{(0)} \equiv \mathfrak{t}^{(0)} \,, \qquad \dim \mathfrak{t}_{\rm vac}^{(0)} = p+1 \,,$$

with an ad-isotropy algebra

$$\mathfrak{h}_{\rm vac} \subset \mathfrak{h}_{\rm vac} \oplus \mathfrak{d} \equiv \mathfrak{h} \,, \qquad \mathfrak{d} = \bigoplus_{\hat{S}=1}^{T} \langle J_{\hat{S}} \rangle \,, \qquad \xleftarrow{\rm Lie} \qquad {\rm H}_{\rm vac} \subset {\rm H}$$

Assume reductivity of

$$[\mathfrak{h}_{\rm vac}, \mathfrak{t} \oplus \mathfrak{d}] \subset \mathfrak{t} \oplus \mathfrak{d} \,, \quad \text{with} \quad [\mathfrak{h}_{\rm vac}, \mathfrak{e}^{(0)}] \subset \mathfrak{e}^{(0)} \supset [\mathfrak{d}, \mathfrak{t}_{\rm vac}^{(0)}] \quad \wedge \quad [\mathfrak{d}, \mathfrak{e}^{(0)}] \subset \mathfrak{t}_{\rm vac}^{(0)}$$

and unimodularity, or preservation of the body of the svacuum,

$$\forall\, h \in {\rm H}_{\rm vac} \;:\; \det \rho(h)\restriction_{\mathfrak{t}_{\rm vac}^{(0)}} \equiv \det {\sf T}_e{\rm Ad}_h\restriction_{\mathfrak{t}_{\rm vac}^{(0)}} \stackrel{!}{=} 1 \,.$$

Replace the NG starget

$$\mathrm{G}/\mathrm{H} \longmapsto \mathrm{G}/\mathrm{H}_{\rm vac} \qquad \sim \qquad \Sigma^{\rm NG} = \bigsqcup_{i \in \mathscr{I}} \sigma_i(\mathcal{U}_i) \longmapsto \Sigma^{\rm HP} = \bigsqcup_{i \in \mathscr{J}} \sigma_j^{\rm vac}(\mathcal{U}_j^{\rm vac})$$

Tranquiliser: sPhysics only cares about ${\sf T}_e{\rm Ad}_{\rm H}$-classes!

THE *other* stensorial data: $(\pi_{\mathrm{G/H_{vac}}}\ :\ \mathrm{G}\longrightarrow\mathrm{G/H_{vac}})$

- the $\mathrm{H_{vac}}$-basic LI svacuum-body svolume

$$\frac{1}{(p+1)!}\,\epsilon_{\underline{a}_0\underline{a}_1\ldots\underline{a}_p}\,\theta_{\mathrm{L}}^{\underline{a}_0}\wedge\theta_{\mathrm{L}}^{\underline{a}_1}\wedge\cdots\wedge\theta_{\mathrm{L}}^{\underline{a}_p}\equiv\mathrm{Vol}\big(\mathrm{t}_{\mathrm{vac}}^{(0)}\big)=\pi^*_{\mathrm{G/H_{vac}}}\underset{(p+1)}{\mathrm{B}}^{\mathrm{HP}}\,;$$

- the H-basic LI de Rham $(p+2)$-scocycle

$$\chi_{\mu_1\mu_2\ldots\mu_{p+2}}\,\theta_{\mathrm{L}}^{\mu_1}\wedge\theta_{\mathrm{L}}^{\mu_2}\wedge\cdots\wedge\theta_{\mathrm{L}}^{\mu_{p+2}}\equiv\underset{(p+2)}{\chi}=\pi^*_{\mathrm{G/H_{vac}}}\underset{(p+2)}{\mathrm{H}}^{\mathrm{vac}}\,.$$

THE *other* sbackground: The Hughes–Polchinski sbackground

$$\big(\mathrm{G/H_{vac}},\lambda_p\,\mathrm{dVol}\big(\mathrm{t}_{\mathrm{vac}}^{(0)}\big)+\underset{(p+2)}{\chi}\equiv\underset{(p+2)}{\widehat{\chi}}^{(\lambda_p)}=\pi^*_{\mathrm{G/H_{vac}}}\underset{(p+2)}{\widehat{\mathrm{H}}}^{(\lambda_p)}\big)\equiv\mathfrak{sB}^{(p,\lambda_p)}_{(\mathrm{HP})}\,,$$

with a parameter $\lambda_p\in\mathbb{R}^\times$ to be fixed by supersymmetry…

THE *other* super-σ-model

Given a closed orientable m_fold Ω_p of $\dim \Omega_p = p+1$, a Lie sgroup G and closed Lie subgroups $\mathrm{H}_{\rm vac} \subset \mathrm{H} \subset |\mathrm{G}|$ with $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{g}, \mathfrak{h}_{\rm vac})$ reductive, and the Hughes–Polchinski sbackground $\mathfrak{s}\mathfrak{B}^{(\lambda_p)}_{(\rm HP)}$, the **Green–Schwarz super-σ-model in the Hughes–Polchinski formulation** is a theory of smappings $[\Omega_p, \mathrm{G}/\mathrm{H}_{\rm vac}] \ni \widehat{\xi}$ determined by the PLA for the DF (s)amplitudes with

$$S_{\rm GS}^{(\rm HP),(\lambda_p)}[\xi] = \int_{\Omega_p} \widehat{\xi}^* \mathsf{d}^{-1} \underset{(p+2)}{\widehat{\mathrm{H}}}{}^{(\lambda_p)} \equiv \sum_{\tau \in \triangle_{\Omega_p}^{(p+1)}} \int_\tau \big(\sigma_{j_\tau}^{\rm vac} \circ \widehat{\xi}\big)^* \mathsf{d}^{-1} \underset{(p+2)}{\widehat{\chi}}{}^{(\lambda_p)} ,$$

with the last equality using a tessellation $\triangle_{\Omega_p}$ of Ω_p subordinate to $\{\mathcal{U}_j\}_{j \in \mathscr{J}}$ for a given $\widehat{\xi}$.

NB: The above sFT is purely topological. In fact, it is...

... 'reducible to a point'...

Th$^{\underline{\mathrm{m}}}$[rrS '19('17)] Let $(\mathfrak{g}, \mathfrak{h}, \mathfrak{h}_{\rm vac}, \mathfrak{t}^{(0)}_{\rm vac})$ and ρ be constrained as above, with the following Maximal Mixing Constraint obeyed**:

$$\langle\, P_{\widehat{a}} \quad | \quad \exists_{(\underline{b},\widehat{S})\in\overline{0,p}\times\overline{1,T}} \ : \ f_{\widehat{S}\widehat{a}}{}^{\underline{b}} \neq 0 \,\rangle = \mathfrak{e}^{(0)} \,,$$

and suppose there exists a $\mathsf{T}_e\mathrm{Ad}_{\mathrm{H}}$-invariant metric g on $\mathfrak{t}^{(0)}$ s.t.

$$\mathfrak{t}^{(0)}_{\rm vac} \perp_{\rm g} \mathfrak{e}^{(0)} \,.$$

The GS super-σ-model in the HP formulation for $(\mathrm{G}/\mathrm{H}_{\rm vac}, \underset{(p+2)}{\widehat{\chi}^{(\lambda_p)}})$ becomes (class.) equivalent to the GS super-σ-model in the NG formulation for $(\mathrm{G}/\mathrm{H}, \mathrm{g}, \underset{(p+2)}{\chi})$ for a *unique* value $\mu_p^*(\lambda_p)$ of μ_p upon restriction of the fomer FT to field configurations satisfying the **Inverse Higgs Constraints**

$$\big(\sigma^{\rm vac}_{i_\tau} \circ \widehat{\xi}\big)^* \theta^{\widehat{a}}_{\rm L} \stackrel{!}{=} 0 \,, \qquad \widehat{a} \in \overline{p+1, d_0} \,.$$

$\Longleftrightarrow$ the EL eq$^{\underline{\mathrm{n}}}$s for the Goldstone modes $\phi^{\widehat{S}}$ (in an exp gauge).

***The restriction can be relaxed.*

Upshot [rrS '20]: In the dual purely topological HP formulation, we may impose $\Delta o\gamma\mu\alpha$ ↯ as 'everything in sight' sgeometrises. Indeed...

- the duality occurs 'in' the **correspondence sdistribution**

$$\mathrm{Corr}\big(\mathfrak{s}\mathfrak{B}^{(\lambda_p)}_{(\mathrm{HP})}\big) = \bigcap_{\widehat{a}=p+1}^{d_0} \mathrm{Ker}\,\theta^{\widehat{a}}_{\mathrm{L}} \cap \mathcal{T}\Sigma^{\mathrm{HP}}\,;$$

- supersymmetry restoration in the svacuum *via* restriction to

$$\mathrm{sSym}\big(\mathfrak{s}\mathfrak{B}^{(\lambda^*_p)}_{(\mathrm{HP})}\big) \equiv \mathrm{Corr}\big(\mathfrak{s}\mathfrak{B}^{(\lambda^*_p)}_{(\mathrm{HP})}\big) \cap \mathrm{Ker}\,\big(\big(\mathbf{1}_{d_1} - \mathsf{P}^{(1)}\big)^{\alpha}{}_{\beta}\,\theta^{\beta}_{\mathrm{L}}\big)$$

for $\quad \mathsf{P}^{(1)} = \mathsf{P}^{(1)} \cdot \mathsf{P}^{(1)} \in \mathrm{End}\,\mathfrak{t}^{(1)} \quad$ s.t. $\quad \{\mathrm{Im}\,\mathsf{P}^{(1)\,\mathrm{T}}, \mathrm{Im}\,\mathsf{P}^{(1)\,\mathrm{T}}\} \subset \mathfrak{t}^{(0)}_{\mathrm{vac}} \oplus \mathfrak{h}\,;$

- altogether, the EL eq^{n}s define*** a **svacuum sdistribution**

$$\mathrm{Vac}\big(\mathfrak{s}\mathfrak{B}^{(\lambda^*_p)}_{(\mathrm{HP})}\big) = \mathrm{sSym}\big(\mathfrak{s}\mathfrak{B}^{(\lambda^*_p)}_{(\mathrm{HP})}\big) \cap \bigcap_{\widehat{S}=1}^{T} \mathrm{Ker}\,\theta^{\widehat{S}}_{\mathrm{L}}\,;$$

****Under some mild assumptions, satisfied by the known super-σ-models.*

Upshot [rrS '20] – ct^d:

- geometric consistency of the svacuum $\Leftrightarrow$ integrability of $\mathrm{Vac}(\mathfrak{sB}^{(\lambda_p^*)}_{(\mathrm{HP})})$ $\Leftrightarrow$ closure of the modelling superspace

$$\mathfrak{vac} = \mathfrak{t}^{(0)}_{\mathrm{vac}} \oplus \mathfrak{t}^{(1)}_{\mathrm{vac}} \oplus \mathfrak{h}_{\mathrm{vac}}\,, \qquad \mathfrak{t}^{(1)}_{\mathrm{vac}} \equiv \mathrm{Im}\,\mathsf{P}^{(1)\,\mathrm{T}} \subset \mathfrak{g}$$

  under the sbracket into the **svacuum Lie salgebra** (descent to the physical supertarget $\mathrm{G}/\mathrm{H}_{\mathrm{vac}}$ follows);

- enhancement of gauge**** symmetry 'in' $\mathrm{Corr}(\mathfrak{sB}^{(\lambda_p^*)}_{(\mathrm{HP})})$:

$$\mathfrak{h}_{\mathrm{vac}} \nearrow \mathfrak{t}^{(1)}_{\mathrm{vac}} \oplus \Delta^{(1)}_{\mathrm{acc}} \oplus \big(\mathfrak{h}_{\mathrm{vac}} \oplus \mathfrak{d}\big)\,,$$

  requires further restriction to $\mathrm{Vac}(\mathfrak{sB}^{(\lambda_p^*)}_{(\mathrm{HP})})$ for consistency, whereupon we get the **κ-symmetry sdistribution**

$$\kappa\big(\mathfrak{sB}^{(\lambda_p^*)}_{(\mathrm{HP})}\big) \subset \mathrm{Vac}\big(\mathfrak{sB}^{(\lambda_p^*)}_{(\mathrm{HP})}\big) \quad \text{modelled on} \quad \mathfrak{t}^{(1)}_{\mathrm{vac}} \oplus \Delta^{(1)}_{\mathrm{acc}} \oplus \mathfrak{h}_{\mathrm{vac}} \subset \mathfrak{vac}$$

**** Dependence on σ^{vac}_j implies locality AND $\widehat{\underset{(p+2)}{\chi}}^{(\lambda_p)} \sim \Omega^{(\mathrm{HP})}_{\sigma}$.

Empirical fact:

The limit $\kappa^{-\infty}(\mathfrak{s}\mathfrak{B}^{(\lambda_p^*)}_{(\rm HP)})$ of the weak derived flag of $\kappa(\mathfrak{s}\mathfrak{B}^{(\lambda_p^*)}_{(\rm HP)})$ stays within $\mathrm{Vac}(\mathfrak{s}\mathfrak{B}^{(\lambda_p^*)}_{(\rm HP)})$ whenever the latter is integrable (*i.e.*, physical), and then

$$\kappa^{-\infty}\big(\mathfrak{s}\mathfrak{B}^{(\lambda_p^*)}_{(\rm HP)}\big) \equiv \mathrm{Vac}\big(\mathfrak{s}\mathfrak{B}^{(\lambda_p^*)}_{(\rm HP)}\big)\,,$$

which is why $\kappa(\mathfrak{s}\mathfrak{B}^{(\lambda_p^*)}_{(\rm HP)})$ was dubbed the square root of the svacuum in [rrS '20].

Conclusion: The Lie salgebra

$$\mathfrak{gs}_{\rm vac} \equiv \mathfrak{vac}$$

modelling $\kappa^{-\infty}(\mathfrak{s}\mathfrak{B}^{(\lambda_p^*)}_{(\rm HP)})$ acquires the interpretation of the **svacuum gauge-symmetry salgebra**.

So what about $\Delta o\gamma\mu\alpha$ ↯↯↯? Benefit from topologicality!

Restrict the **extended Hughes–Polchinski p-sgerbe**

$$\widehat{\mathcal{G}}^{(p)}_{\rm HP} := \mathcal{G}^{(p)} \otimes \mathcal{I}^{(p)}_{\lambda_p^* {\rm Vol}(t^{(0)}_{\rm vac})}$$

to the sections $\sigma_j^{\rm vac}(\mathcal{U}_j^{\rm vac}) \equiv \mathcal{V}_j$, forming

$$\widehat{\mathcal{G}}^{(p)}_{\Sigma^{\rm HP}} := \bigsqcup_{j\in\mathscr{J}} \widehat{\mathcal{G}}^{(p)}_{\rm HP}\restriction_{\mathcal{V}_j} ,$$

and subsequently pull back to the **vacuum foliation**

$$\iota_{\rm vac} \ : \ \Sigma^{\rm HP}_{\rm vac} \hookrightarrow \Sigma^{\rm HP} ,$$

whereby there arises the **vacuum restriction**

$$\widehat{\mathcal{G}}^{(p)}_{\rm vac} \equiv \iota_{\rm vac}^* \widehat{\mathcal{G}}^{(p)}_{\Sigma^{\rm HP}}$$

that descends to the physical vacuum in ${\rm G}/{\rm H}_{\rm vac}$ *by construction*.

Dogmatic expectation: a $\mathfrak{gs}_{\rm vac}$-equivariant structure on $\widehat{\mathcal{G}}^{(p)}_{\rm vac}$

However, $\kappa^{-\infty}(\mathfrak{s}\mathfrak{B}^{(\lambda_p^*)}_{(\mathrm{HP})})$ envelops the vacuum, the latter being a single orbit of $\mathfrak{g}\mathfrak{s}_{\mathrm{vac}}$, resp. of the **$\kappa$-symmetry sgroup** (whenever $\exists$)

$$\int \mathfrak{g}\mathfrak{s}_{\mathrm{vac}} \equiv \mathrm{G}_{\mathrm{vac}} \,,$$

whence

Hypothesis [rrS '20]: There exists an $\mathrm{H}_{\mathrm{vac}}$-equivariant trivialisation

$$\tau_p \; : \; \widehat{\mathcal{G}}^{(p)}_{\mathrm{vac}} \xrightarrow{\cong} \mathcal{I}^{(p)}_0 \,,$$

or, equivalently, the descendant of $\mathcal{G}^{(p)}$ to $\mathrm{G}/\mathrm{H}_{\mathrm{vac}}$ (indeed, to G/H) trivialises as the volume p-sgerbe over the svacuum.

Problem: The svacuum does not possess a natural Lie-sgroup structure, hence there seems to be no room for a supersymmetric trivialisation. And yet. . .

In our formalism, we may look for a **sLieAlg shadow of τ_p**.

E.g. [rrS '20, in writing], the sminkowskian sstring trivialisation

$$\begin{array}{ccccc}
\alpha_{\mathscr{E}} \equiv \mathbf{1} \ :\ \widetilde{\mathscr{L}}_{\rm vac} \otimes {\rm pr}_2^*\mathscr{E} \xrightarrow{\ \cong\ } {\rm pr}_1^*\mathscr{E} & & \mathbb{C}^\times & \longrightarrow & \underset{(1)}{\mathscr{E}\,,\ \mathcal{A}_{\mathscr{E}}} \\
\downarrow & & & & \downarrow{\scriptstyle \pi_{\mathscr{E}}} \\
{\sf Y}^{[2]}\Sigma^{\rm HP}_{\rm vac} & & \overset{{\rm pr}_1}{\underset{{\rm pr}_2}{\rightrightarrows}} & & {\sf Y}\Sigma^{\rm HP}_{\rm vac}\,,\ \underset{(2)}{\widehat{{\sf Y}\beta}_{\rm vac}} \\
& & & & \downarrow{\scriptstyle \pi_{{\sf Y}\Sigma^{\rm HP}_{\rm vac}}} \\
& & & & \Sigma^{\rm HP}_{\rm vac}\,,\ \underset{(3)}{\widehat{\chi}_{\rm vac}} \equiv 0
\end{array}\ ,$$

has a shadow

$$\begin{array}{ccccccc}
& & \mathbf{0} & & & & \\
& & \downarrow & & & & \\
\alpha_{\widetilde{\mathfrak{e}}} \equiv \mathbf{1} \ :\ {\sf Y}^{[2]}\jmath_{\rm vac}^*\widetilde{\mathfrak{l}} \otimes {\rm pr}_2^*\mathfrak{e} \xrightarrow{\ \cong\ } {\rm pr}_1^*\mathfrak{e} & & \mathbb{R} & \longrightarrow & \mathfrak{e}\,,\ \breve{\zeta}_{\mathscr{E}} & & \\
\downarrow & & & & \downarrow{\scriptstyle \pi_{\mathfrak{e}}} & & \\
{\sf Y}^{[2]}\mathfrak{vac} & & \overset{{\rm pr}_1}{\underset{{\rm pr}_2}{\rightrightarrows}} & & {\sf Y}\mathfrak{vac}\,,\ \underset{(2)}{\widehat{{\sf Y}\beta}_{\rm vac}} & \longrightarrow & \mathbf{0} \\
& & & & \downarrow{\scriptstyle \pi_{{\sf Y}\mathfrak{vac}}} & & \\
& & & & \mathfrak{vac}\,,\ \underset{(3)}{\widehat{\chi}_{\rm vac}} \equiv 0 & &
\end{array}\ .$$

Loose ends:

- **Th^m** [rrS '19('17)] The superminkowskian GS p-sgerbes with $p \in \{0, 1\}$ are endowed with a canonical **supersymmetric** $\mathrm{Ad}_{\mathrm{sMink}(d,1|D_{d,1})}$**-equivariant structure**.

  NB: This conforms with the purely even (WZW) story.

- The GS super-σ-models with curved stargets $\mathrm{s}(\mathrm{AdS}_q \times \mathbb{S}^q)$ (MT, PR, Zh) are constructed on the basis of an asymptotic correspondence with their superminkowskian counterparts,

$$\mathrm{s}\big(\mathrm{AdS}_q \times \mathbb{S}^q\big) \longrightarrow \mathrm{s}\big(\mathrm{AdS}_p(R) \times \mathbb{S}^q(R)\big) \xrightarrow{R\to\infty} \mathrm{sMink}(2q-1,1|D_{2q-1,1})\,.$$

It is natural to gerbify the underlying İnönü–Wigner contractions

$$\mathfrak{g}^q_{\mathrm{curv}} \longrightarrow \mathfrak{g}^q_{\mathrm{curv}}(R) \xrightarrow{R\to\infty} \mathfrak{smink}(2q-1,1|D_{2q-1,1})$$

by requiring that they lift to **sLieAlg** shadows of Murray diagrams, & turn it into an organising principle on the moduli space of super-σ-models.

Outcome: Problems with the definition of the stringy super-σ-models.

Conclusions:

1. The physically relevant CaE $(p+2)$-scocycles on supersymmetry Lie sgroups geometrise – in an interplay of CaE & CE cohomology – for a large class of sbackgrounds as the H-equivariant CaE p-sgerbes of [rrS '17, '18].
2. The CaE p-sgerbes are global supersymmetry-invariant and endowed with (the expected and) natural equivariant structures with respect to the supersymmetries of the relevant super-σ-models amenable to gauging, in conformity with the underlying physics and the bosonic intuition. [rrS '19]
3. κ-symmetry demystified, geometrised & gerbified in the dual HP formulation of the GS super-σ-model. [rrS '19, '20]
4. The construction generalises to physically relevant curved homogeneous spaces of supersymmetry Lie supergroups, and sometimes suggests – *via* gerbification of the İW contraction – corrections to the existing sFT results. [rrS '18]

Outlook:

- Uniqueness of the construction and its relation to the approach of Huerta, Baez, Schreiber *et al.* (κ-symm., H-equiv., $\dot{\rm I}$W-contr.)? Reconstruction of the (weak) $(p+1)$-categories of p-sgerbes.
- The relevance of the $\dot{\rm I}$W-contractibility & the ultimate fate of the curved sbackgrounds?
- The higher sgeometry and salgebra (**sLieAlg** shadows) of supersymmetric defects (incl. boundary states) & their fusion.
- Relation to the worldvolume supersymmetry, possibly *via* Sorokin's Superembedding Formalism.
- Relation to the $\mathfrak{String}$-structure.
- The bosonisation/fermionisation defect.
- T-duality *via* the HP formulation, also in the bosonic setting.
- The gauging of the $\mathrm{Ad}_{\mathrm{sMink}(d,1|D_{d,1})}$-supersymmetry and the ensuing CS-type sTFT.
- SUSY NCG *etc...*

(Ceci n'est pas) **La Fin**. . .

Part III

super-Xtras

1. A **Lie algebroid** is a quintuple $(\mathbb{V}, \pi_{\mathbb{V}}, M, \alpha_{\mathsf{T}M}, [\cdot,\cdot]_{\mathbb{V}})$ composed of

- a smooth manifold M, termed the **base**;
- a smooth vector bundle $\pi_{\mathbb{V}} : \mathbb{V} \longrightarrow M$;
- a smooth vector-bundle morphism $\alpha_{\mathsf{T}M} : \mathbb{V} \longrightarrow \mathsf{T}M$, termed the **anchor map**;
- a Lie bracket $[\cdot,\cdot]_{\mathbb{V}} : \Gamma(\mathbb{V})^{\times 2} \longrightarrow \Gamma(\mathbb{V})$ on the vector space $\Gamma(\mathbb{V})$ of sections of $\mathbb{V}$,

with the following properties:

- the induced map $\Gamma\alpha_{\mathsf{T}M} : (\Gamma(\mathbb{V}), [\cdot,\cdot]_{\mathbb{V}}) \longrightarrow (\Gamma(\mathsf{T}M), [\cdot,\cdot])$ is a Lie-algebra homomorphism;
- the Lie bracket $[\cdot,\cdot]_{\mathbb{V}}$ obeys the Leibniz identity

$$\forall_{(X,Y,f)\in\Gamma(\mathbb{V})^{\times 2}\times C^\infty(M,\mathbb{R})} : [X, f \triangleright Y]_{\mathbb{V}} = f \triangleright [X,Y]_{\mathbb{V}} + \alpha_{\mathsf{T}M}(X)(f) \triangleright Y .$$

2. The Lie supergroup of the Metsaev-Tseytlin super-σ-model:

$\mathrm{SU}(2,2\,|\,4)$ with the body

$$|\mathrm{SU}(2,2\,|\,4)| = \mathrm{SO}(4,2)\times\mathrm{SO}(6)$$

and the Lie superalgebra (R-rescaled, for $R\in\mathbb{R}$)

$$\mathfrak{su}(2,2\,|\,4)^{(R)} = \Big(\big(\bigoplus_{a=0}^{4}\langle P_a\rangle \oplus \bigoplus_{a'=5}^{9}\langle P_{a'}\rangle\big)\oplus \bigoplus_{(\alpha,\alpha',I)\in\overline{1,4}\times\overline{1,4}\times\{1,2\}}\langle Q_{\alpha\alpha' I}\rangle\Big) \oplus\Big(\bigoplus_{a,b=0}^{4}\langle J_{ab}=-J_{ba}\rangle\oplus\bigoplus_{a',b'=5}^{9}\langle J_{a'b'}=-J_{b'a'}\rangle\Big)$$

$$\{Q_{\alpha\alpha' I},Q_{\beta\beta' J}\} = \mathrm{i}\,\big(-2(\widehat{C}\,\widehat{\Gamma}^{\widehat{a}}\otimes\mathbf{1})_{\alpha\alpha' I\beta\beta' J}\,P_{\widehat{a}} + \tfrac{1}{R^2}\,(\widehat{C}\,\widehat{\Gamma}^{\widehat{a}\widehat{b}}\otimes\sigma_2)_{\alpha\alpha' I\beta\beta' J}\,J_{\widehat{a}\widehat{b}}\big)\,,$$

$$[Q_{\alpha\alpha' I},P_{\widehat{a}}] = -\tfrac{1}{2R}\,(\widehat{\Gamma}_{\widehat{a}}\otimes\sigma_2)^{\beta\beta' J}{}_{\alpha\alpha' I}\,Q_{\beta\beta' J}\,,\qquad [P_{\widehat{a}},P_{\widehat{b}}] = \tfrac{1}{R^2}\,\varepsilon_{\widehat{a}\widehat{b}}\,J_{\widehat{a}\widehat{b}}\,,\quad \varepsilon_{\widehat{a}\widehat{b}} = \begin{cases} +1 & \text{if } \widehat{a},\widehat{b}\in\overline{0,4} \\ -1 & \text{if } \widehat{a},\widehat{b}\in\overline{5,9} \\ 0 & \text{otherwise}\end{cases}\,,$$

$$[J_{\widehat{a}\widehat{b}},J_{\widehat{c}\widehat{d}}] = \eta_{\widehat{a}\widehat{d}}\,J_{\widehat{b}\widehat{c}} - \eta_{\widehat{a}\widehat{c}}\,J_{\widehat{b}\widehat{d}} + \eta_{\widehat{b}\widehat{c}}\,J_{\widehat{a}\widehat{d}} - \eta_{\widehat{b}\widehat{d}}\,J_{\widehat{a}\widehat{c}}\,,$$

$$[Q_{\alpha\alpha' I},J_{\widehat{a}\widehat{b}}] = -\tfrac{1}{2}\,\varepsilon_{\widehat{a}\widehat{b}}\,(\widehat{\Gamma}_{\widehat{a}\widehat{b}}\otimes\mathbf{1})^{\beta\beta' J}{}_{\alpha\alpha' I}\,Q_{\beta\beta' J}\,,\qquad [P_{\widehat{a}},J_{\widehat{b}\widehat{c}}] = \eta_{\widehat{a}\widehat{b}}\,P_{\widehat{c}} - \eta_{\widehat{a}\widehat{c}}\,P_{\widehat{b}}\,.$$

with the İnönü-Wigner asymptote $\mathfrak{su}(2,2\,|\,4)^{(R)} \xrightarrow{R\to\infty} \mathfrak{smink}(9,1\,|\,32)$.

3. Some Lie-superalgebra cohomology…

Def$^{\underline{n}}$: A (**left**) $\widehat{\mathfrak{g}}$**-module** of an LSA $\widehat{\mathfrak{g}}$ is a pair $(\widehat{V}, \ell.)$ composed of a $\mathbb{K}$-linear superspace $\widehat{V} = \widehat{V}^{(0)} \oplus \widehat{V}^{(1)}$ and a left $\widehat{\mathfrak{g}}$-action

$$\ell. \, : \widehat{\mathfrak{g}} \times \widehat{V} \longrightarrow \widehat{V} \, : \, (X, v) \longmapsto X \triangleright v$$

consistent with the $\mathbb{Z}/2\mathbb{Z}$-gradings, $\widetilde{X \triangleright v} = \widetilde{X} + \widetilde{v}$, and such that for any two homogeneous elements $X_1, X_2 \in \mathfrak{g}$ and $v \in \widehat{V}$,

$$[X_1, X_2\} \triangleright v = X_1 \triangleright (X_2 \triangleright v) - (-1)^{\widetilde{X_1} \cdot \widetilde{X_2}} X_2 \triangleright (X_1 \triangleright v)\,.$$

and the fundamental…

Def$^{\underline{n}}$: Let $(\widehat{\mathfrak{g}}, [\cdot, \cdot\})$ be an LSA over field $\mathbb{K}$ and let $(\widehat{V}, \ell.)$ be a $\widehat{\mathfrak{g}}$-module. A **p-cochain on $\widehat{\mathfrak{g}}$ with values in $\widehat{V}$** is a p-linear map $\underset{(p)}{\varphi} : \widehat{\mathfrak{g}}^{\times p} \longrightarrow \widehat{V}$ that is totally super-skewsymmetric,

$$\begin{aligned} &\underset{(p)}{\varphi}(X_1, X_2, \ldots, X_{i-1}, X_{i+1}, X_i, X_{i+2}, X_{i+3}, \ldots, X_p) \\ = \quad &-(-1)^{\widetilde{X_i}\widetilde{X_{i+1}}} \underset{(p)}{\varphi}(X_1, X_2, \ldots, X_p) . \end{aligned}$$

They form a $\mathbb{Z}_2$-graded **group of p-cochains on $\widehat{\mathfrak{g}}$ valued in $\widehat{V}$**,

$$C^p(\widehat{\mathfrak{g}}, \widehat{V}) = C^p_0(\widehat{\mathfrak{g}}, \widehat{V}) \oplus C^p_1(\widehat{\mathfrak{g}}, \widehat{V}) ,$$

with $\underset{(p)}{\varphi}(X_1, X_2, \ldots, X_p) \in \widehat{V}_{\sum_{i=1}^p \widetilde{X_i}+n}$ for $\underset{(p)}{\varphi} \in C^p_n(\widehat{\mathfrak{g}}, \widehat{V})$, composed of **even** $(n = 0)$ and **odd** $(n = 1)$ p-cochains.

These groups form a semi-bounded complex

$$C^\bullet(\widehat{\mathfrak{g}}, \widehat{V}) \; : \; C^0(\widehat{\mathfrak{g}}, \widehat{V}) \xrightarrow{\delta^{(0)}_{\widehat{\mathfrak{g}}}} C^1(\widehat{\mathfrak{g}}, \widehat{V}) \xrightarrow{\delta^{(1)}_{\widehat{\mathfrak{g}}}} \cdots \xrightarrow{\delta^{(p-1)}_{\widehat{\mathfrak{g}}}} C^p(\widehat{\mathfrak{g}}, \widehat{V}) \xrightarrow{\delta^{(p)}_{\widehat{\mathfrak{g}}}} \cdots$$

The coboundary operators

$$\delta_{\mathfrak{g}}^{(p)} \; : \; C_n^p(\mathfrak{g}, V) \longrightarrow C_n^{p+1}(\mathfrak{g}, V)$$

evaluate on the homogeneous $X_i \in \mathfrak{g}, \; i \in \overline{0, p+1}, \; \underset{(p)}{\varphi} \in C^p(\mathfrak{g}, V)$

as

$$(\delta_{\mathfrak{g}}^{(0)} \underset{(0)}{\varphi})(X) := (-1)^{|X_0| \cdot |\underset{(0)}{\varphi}|} X_0 \rhd \underset{(0)}{\varphi} \,,$$

$$(\delta_{\mathfrak{g}}^{(p)} \underset{(p)}{\varphi})(X_1, X_2, \ldots, X_{p+1}) := \sum_{j=1}^{p+1} (-1)^{|X_j| \cdot |\underset{(p)}{\varphi}| + S(X_j)} X_j \rhd \underset{(p)}{\varphi}(X_1, X_2, \underset{\widehat{j}}{\ldots}, X_{p+1})$$

$$+ \sum_{1 \leq j < k \leq p+1} (-1)^{S(X_j) + S(X_k) + |X_j| \cdot |X_k|} \underset{(p)}{\varphi}([X_j, X_k\}, X_1, X_2, \underset{\widehat{j,k}}{\ldots}, X_{p+1}) \,,$$

$$S(X_i) := |X_i| \cdot \sum_{j=1}^{i-1} |X_j| + i - 1 \,.$$

The $\mathbb{Z}/2\mathbb{Z}$-graded **V-valued cohomology groups of $\mathfrak{g}$** are

$$H^p(\mathfrak{g}, V) := H_0^p(\mathfrak{g}, V) \oplus H_1^p(\mathfrak{g}, V) \,, \quad H_n^p(\mathfrak{g}, V) := \frac{\ker \delta_{\mathfrak{g}}^{(p)} \restriction_{C_n^p(\mathfrak{g}, V)}}{\mathrm{im}\, \delta_{\mathfrak{g}}^{(p-1)} \restriction_{C_n^{p-1}(\mathfrak{g}, V)}} \,.$$

Def$^{\underline{n}}$: Let $(\widehat{\mathfrak{g}}, [\cdot,\cdot\}_{\widehat{\mathfrak{g}}})$ and $(\widehat{\mathfrak{a}}, [\cdot,\cdot\}_{\widehat{\mathfrak{a}}})$ be LSAs over field $\mathbb{K}$. A **supercentral extension of $\widehat{\mathfrak{g}}$ by $\widehat{\mathfrak{a}}$** is an LSA $(\widetilde{\mathfrak{g}}, [\cdot,\cdot\}_{\widetilde{\mathfrak{g}}})$ over $\mathbb{K}$ that determines a short exact sequence of LSAs

$$\mathbf{0} \longrightarrow \mathfrak{a} \xrightarrow{\jmath_{\widehat{\mathfrak{a}}}} \widetilde{\mathfrak{g}} \xrightarrow{\pi_{\widehat{\mathfrak{g}}}} \mathfrak{g} \longrightarrow \mathbf{0}\,,$$

written in terms of an LSA mono $\jmath_{\widehat{\mathfrak{a}}}$ and of an LSA epi $\pi_{\widehat{\mathfrak{g}}}$, and s.t. $\jmath_{\widehat{\mathfrak{a}}}(\widehat{\mathfrak{a}}) \subset \mathfrak{z}(\widetilde{\mathfrak{g}})$ (the supercentre of $\widetilde{\mathfrak{g}}$). Whenever $\pi_{\mathfrak{g}}$ admits an LSA **section**, *i.e.*, there exists

$$\sigma \in \mathrm{Hom}_{\mathrm{sLie}}(\widehat{\mathfrak{g}}, \widetilde{\mathfrak{g}})\,, \qquad \pi_{\widehat{\mathfrak{g}}} \circ \sigma = \mathrm{id}_{\widehat{\mathfrak{g}}}\,,$$

the supercentral extension is said to **split**.

An **equivalence of supercentral extensions** $\widetilde{\mathfrak{g}}_\alpha, \alpha \in \{1,2\}$ of $\widehat{\mathfrak{g}}$ by $\widehat{\mathfrak{a}}$ is represented by a commutative diagram of LSAs

$$\begin{array}{ccccccc}
 & & & \widetilde{\mathfrak{g}}_1 & & & \\
 & & \nearrow & \downarrow \cong & \searrow & & \\
\mathbf{0} \longrightarrow & \widehat{\mathfrak{a}} & & & & \widehat{\mathfrak{g}} & \longrightarrow \mathbf{0}\,. \\
 & & \searrow & & \nearrow & & \\
 & & & \widetilde{\mathfrak{g}}_2 & & &
\end{array}$$

The relevant one-way ticket:

Given an LSA $(\mathfrak{g}, [\cdot,\cdot\}_\mathfrak{g})$ and its supercommutative module $\mathfrak{a}$, as well as a representative $\Theta \in Z^2_0(\mathfrak{g}, \mathfrak{a})$ of a class in $H^2_0(\mathfrak{g}, \mathfrak{a})$, we define

$$\widetilde{\mathfrak{g}} := \mathfrak{a} \oplus \mathfrak{g}$$

and put on it the Lie superbracket

$$\begin{aligned} [\cdot,\cdot\}_\Theta \quad &: \quad \widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}} \longrightarrow \widetilde{\mathfrak{g}} \\ &: \quad \big((A_1, X_1), (A_2, X_2)\big) \longmapsto \big(\Theta(X_1, X_2), [X_1, X_2\}_\mathfrak{g}\big) . \end{aligned}$$

4. A **Lie 2-algebra** is a quintuple $(V_0, V_1, \delta, [-,-], \mathrm{Jac})$ composed of

- vector spaces V_0 and V_1;
- a linear map $\delta : V_1 \longrightarrow V_0$ (the **differential**);
- a *skew* bilinear map $[-,-] : (V_0 \oplus V_1)^{\times 2} \longrightarrow V_0 \oplus V_1$;
- a *skew* trilinear map $\mathrm{Jac} : V_0^{\times 3} \longrightarrow V_1$ (the **jacobiator**),

with the following properties, written for $R, S \in V_1$, $x, y, z \in V_0$,

- $[V_i, V_j] \subset V_{i+_2 j}$;
- $[R, S] = 0$;
- $\delta[x, R] = [x, \delta R]$;
- $[\delta R, S] = [R, \delta S]$;
- $[[x, y], z] + [[z, x], y] + [[y, z], x] = \delta \mathrm{Jac}(x, y, z)$
- $\mathrm{Jac}(\delta R, x, y) = -[[x, y], R] + [[x, R], y] + [x, [y, R]]$;
- 'septagonal' coherence for Jac and $[-,-]$.

Th^m [Baez & Crans '10] There exists a one-to-one correspondence between isomorphism classes of *skeletal* Lie 1-algebras (with $\delta = 0$) and equivalence classes of quadruples $(\mathfrak{g}, V, \rho, \chi)$ composed of a Lie algebra $\mathfrak{g}$, a vector space V, a representation $\rho : \mathfrak{g} \longrightarrow \mathrm{End}(V)$, and a V-valued 3-cocycle χ on $\mathfrak{g}$.

In particular, to a quadruple $(\mathfrak{g}, V, \rho, \chi)$ as above, we associate a Lie 2-algebra with $(V_0, V_1, \mathrm{Jac}) = (\mathfrak{g}, V, \chi)$ and $[-,-]$ defined as

$$[X, Y] = [X, Y]_{\mathfrak{g}}\,, \qquad [X, v] = \rho_X(v)\,, \qquad [v, w] = 0$$

for arbitrary $X, Y \in \mathfrak{g}$ and $v, w \in V$.

The correspondence was subsequently generalised to higher *slim* L_∞ algebras and Lie-algebra modules with higher cocycles, and supersised by Baez & Huerta in [Baez & Huerta '11].