Boundary calculus in conformal geometry

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background:
G-., Waldon, Conf. hypersurf. geom. $\cdots$, *CAG*, (2021), and Renormalized Volume, *CMP*, (2017);
Curry, **G-.** $\cdots$Conformal Geometry $\cdots$ GR$\cdots$, LMS Series, Cambridge,
Čap, **G-.** Hammerl: Holonomy reductions etc, *Duke Math. J.* (2014)
G-. *J. Geom. Phys.*, (2010), 182–204.

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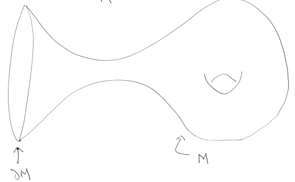
Workshop: Geometry for higher spin gravity, Conformal structure, PDEs, and Q-manifolds

L1 Plan: We begin with a discussion of the key problems, viz. the construction of natural invariants and invariant operators for conformally embedded hypersurfaces. These have applications in Scattering, String theory (especially surrounding the AdS/CFT programme, PDE boundary problems, and representation theory. Then we discuss: Tractor calculus. Hypersurface tractor theory. The proliferation of conformally invariant boundary operators, and connections of these to Q and T curvatures.

Curry+G., An introduction to conformal geometry$\cdots$ GR, LMS (2018)
G.+Peterson, Conformal boundary operators,$\cdots$, Pacific J.M., (2021)

Motivation

Basic problem:



If M is an "infinite space" (complete non-compact pseudo-Riemannian manifold) can we add and make sense of ∂M as a "boundary at infinity" and then exploit?

One approach to this idea leads to a tool to study/define:

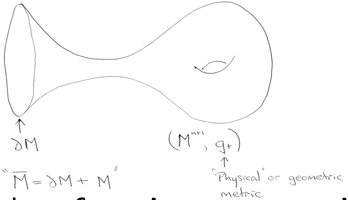
- Q-curvature;
- higher Willmore energies and invariants;
- Renormalised volume and volume anomalies, and related;
- Poincaré-Einstein manifolds;
- **Scattering and PDE boundary problems at (conformal) infinity** – including higher spin fields.
- AdS/CFT programme

Origins: Fefferman-Graham, $\cdots$, (t'Hooft), Maldacena, $\cdots$

A main tool – Conformally compact manifolds

Henceforth in these talks, **conformal compactification** of pseudo-Riemannian manifold (M^{n+1}, g_+) is a manifold $\overline{M}$ with boundary ∂M s.t.:

- $\exists\ \overline{g}$ on $\overline{M}$, with
- $g_+ = r^{-2}\overline{g}$, where r a defining function for ∂M.

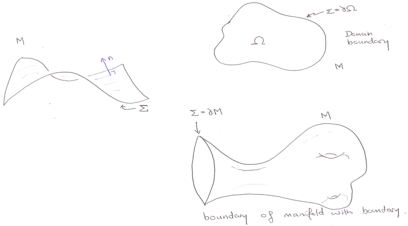


$\Rightarrow$ canonical **conformal structure on boundary**: $(\partial M, [\overline{g}|_{\partial M}])$ (where dr not null).

- Called here **Poincaré-Einstein** metric if also g_+ Einstein. (Usually this term used for negative Einstein – especially in Riemannian signature.) $\boxed{d = n+1}$

Some related problems – this lecture's focus

Hypersurface Σ here means a codimension-1 embedded submanifold.



– arise as the boundary of domains and manifolds with boundary in pseudo-Riemannian d-manifolds (M^d, g), so are important for **PDE boundary problems** and corresponding problems in **physics** and **representation theory**. Problems:

- What if Σ is the bdy **at** ∞ of a conformally compact metric?
- What if instead $\Sigma \hookrightarrow (M, \mathbf{c})$ where $\mathbf{c} = [g]$ is just a conformal class of metrics (cf. conformal boundary problems . . .)?
- In these settings: How do we make invariants and invariant operators that are natural for the embedding $\Sigma \hookrightarrow M$?

Conformal Geometry

A conformal d-manifold ($d \geq 3$) is the structure $(M^d, \mathbf{c})$ where

- $\mathbf{c}$ is a conformal equivalence class of signature (p, q) metrics, i.e. $\boxed{g, \widehat{g} \in \mathbf{c} \overset{\text{def.}}{\Longleftrightarrow} \widehat{g} = \Omega^2 g}$ and $C^\infty(M) \ni \Omega > 0$.

Because there is no distingushed metric on $(M, \mathbf{c})$ an important role is played by the **density bundles**. Note $(\Lambda^d TM)^2$ is an oriented real line bundle $\mathcal{K}$. We write $\mathcal{E}[w]$ for the roots

$$\mathcal{E}[w] = \mathcal{K}^{\frac{w}{2d}}, \qquad \text{so} \qquad \mathcal{K} = \mathcal{E}[2d],$$

$\mathcal{E}[0] := \mathcal{E}$ (the trivial bundle with fibre $\mathbb{R}$), and $\mathcal{E}_+[w]$ for the positive elements. $\mathcal{B}[w] := \mathcal{B} \otimes \mathcal{E}[w]$. With this notation there is tautologically a **conformal metric**

$$\boldsymbol{g} \in S^2 T^* M[2], \qquad \text{so that} \qquad g^\sigma := \sigma^{-2} \boldsymbol{g} \in \mathbf{c}, \quad \sigma \in \Gamma(\mathcal{E}_+[1]),$$

and

$$\otimes^{n+1} \boldsymbol{g} : \left(\Lambda^{n+1} TM\right)^2 \xrightarrow{\simeq} \mathcal{E}[2n+2].$$

General Problem: Understand *hypersurface* type submanifold embeddings

$$\iota : \Sigma^n \longrightarrow (M^{d=n+1}, \mathbf{c})$$

where $\mathbf{c}$ is the conformal structure $[g]$, and Σ is a smooth n-manifold.

Problem: How do we construct the conformal invariants of Σ?

Problem: How do we construct and understand conformally invariant (natural) differential operators along Σ?

• we restrict to Σ with the property that the any conormal field along Σ is nowhere null (i.e. Σ is nondegenerate).
Then:

• restriction of any $g \in \mathbf{c}$ gives metric $\bar{g}$ on $\Sigma \rightsquigarrow \mathbf{c}$ induces $\bar{\mathbf{c}}$ on Σ.
• It is natural to work with a weight 1 co-normal n_a along Σ satisfying $\boldsymbol{g}^{ab} n_a n_b = \pm 1$.

On a conformal manifold $(M, \mathbf{c})$ there is a conformally invariant **tractor bundle** $\mathcal{T}$ and **connection** $\nabla^{\mathcal{T}}$.

$$\mathcal{T} = \mathcal{E}[1] \mathbin{\lhd\!\!\!+} T^*M[1] \mathbin{\lhd\!\!\!+} \mathcal{E}[-1], \qquad X_A : \mathcal{E}[-1] \hookrightarrow \mathcal{T} =: \mathcal{T}_A$$

Given $g \in \mathbf{c}$

$$\mathcal{T} \stackrel{g}{=} \mathcal{E}[1] \oplus T^*M[1] \oplus \mathcal{E}[-1],$$

$$\nabla^{\mathcal{T}}_a(\sigma, \mu_b, \rho) = (\nabla_a \sigma - \mu_a,\ \nabla_a \mu_b + P_{ab}\sigma + \boldsymbol{g}_{ab}\rho,\ \nabla_a \rho - P_{ab}\mu^b),$$

(P_{ab} :=Schouten tensor) and $\nabla^{\mathcal{T}}$ preserves a tractor metric h

$$\mathcal{T} \ni V = (\sigma, \mu_b, \rho) \mapsto 2\sigma\rho + \mu_b\mu^b = h(V, V).$$

There is also a second order Thomas operator:

$$\Gamma(\mathcal{E}[w]) \in f \mapsto D_A f \stackrel{g}{=} \begin{pmatrix} (d+2w-2)wf \\ (d+2w-2)\nabla_a f \\ -(\Delta f + wJf) \end{pmatrix}$$

where J is a number times $\mathrm{Sc}(g)$.

For $g \in \mathbf{c}$, the **second fundamental form** L_{ab} is the restriction of $\nabla_a n_b$ to $T\Sigma \times T\Sigma \subset (TM \times TM)|_\Sigma$, where $\nabla = \nabla^g$; i.e.

$$L_{ab} := \nabla_a n_b \mp n_a n^c \nabla_c n_b \quad \text{along} \quad \Sigma.$$

This is not conformally invariant. But under a conformal rescaling, $g \mapsto \widehat{g} = e^{2\omega} g$, L_{ab} transforms according to

$$L^{\widehat{g}}_{ab} = L^g_{ab} + \overline{\boldsymbol{g}}_{ab} \Upsilon_c n^c, \qquad \text{where} \qquad \Upsilon = d\omega$$

Thus:

Proposition

The trace-free part of the second fundamental form

$$\mathring{L}_{ab} = L_{ab} - H \overline{\boldsymbol{g}}_{ab}, \quad \textit{where}, \quad H := \frac{1}{d-1} \overline{\boldsymbol{g}}^{cd} L_{cd}$$

is conformally invariant.

Here $d = n + 1$ is the dimension of the ambient manifold M.

The normal tractor

Evidently, under a conformal rescaling $g \mapsto \widehat{g} = e^{2\omega} g$, the **mean curvature** H^g transforms to $H^{\widehat{g}} = H^g + n^a \Upsilon_a$. Thus we obtain a conformally invariant section N of $\mathcal{T}|_\Sigma$

$$N_A \stackrel{g}{=} \begin{pmatrix} 0 \\ n_a \\ -H^g \end{pmatrix}, \qquad \leftarrow !!$$

and $h(N, N) = \pm 1$ along Σ. This is the **normal tractor** of Bailey-Eastwood-G. Differentiating N tangentially along Σ using $\nabla^{\mathcal{T}}$, we obtain the following result.

Proposition (Conformal Shape operator)

$$\mathbb{L}_{aB} := \underline{\nabla}_a N_B \stackrel{g}{=} \begin{pmatrix} 0 \\ \mathring{L}_{ab} \\ -\frac{1}{d-2} \nabla^b \mathring{L}_{ab} \end{pmatrix}$$

where $\underline{\nabla}$ is the pullback to Σ of the ambient tractor connection. Thus Σ is **totally umbilic** *iff N is parallel along Σ.*

The classical **Gauss formula**

$$\underline{\nabla}_a v^b = \overline{\nabla}_a v^b \mp n^b L_{ac} v^c \qquad v \in \Gamma(T\Sigma) \subset \Gamma(TM),$$

is the basis of pseudo-Riemannian hypersurface calculus.

We want the conformal analogue. First we need this:

Proposition (Branson-G., Grant)

There is a natural conformally invariant (isometric) isomorphism

$$\mathcal{T}|_\Sigma \supset N^\perp \xrightarrow{\simeq} \overline{\mathcal{T}} = \textit{std tractor bdle of } (\Sigma, \bar{\boldsymbol{c}}).$$

Proof.

Calculating in a scale g on M the tractor bundle $\mathcal{T}$, and hence also $N^\perp$, decomposes into a triple. Then the mapping of the isomorphism is

$$[N^\perp]_g \ni \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} \mapsto \begin{pmatrix} \sigma \\ \mu_b \mp H n_b \sigma \\ \rho \pm \frac{1}{2} H^2 \sigma \end{pmatrix} \in [\overline{\mathcal{T}}]_{\bar{g}}.$$

The tractor Gauss equation

The above reveals two connections on $\overline{\mathcal{T}} \cong N^\perp$ that we can compare. Namely the **intrinsic tractor connection** $\overline{\nabla}^{\overline{\mathcal{T}}}$ determined by $(\Sigma, \overline{\boldsymbol{c}})$, and the **projected ambient tractor connection** $\tilde{\nabla}$. The latter is defined by

$$\tilde{\nabla}_a U^B := \Pi^B_C(\Pi^c_a \nabla_c U^C) \qquad U \in \Gamma(N^\perp) \text{ extended arb. off } \Sigma$$

where Π^B_C and Π^c_a are the orthog. projections due to N and n. Including the tractor derivative of Π^B_C gives:

Proposition (Tractor Gauss formula – Stafford, Vyatkin)

$$\underline{\nabla}_a V^B = \overline{\nabla}_a V^B \mp S_a{}^B{}_C V^C \mp \mathrm{N}^B \mathbb{L}_{aC} V^C,$$

where $\mathrm{S}_{aBC} = \overline{\mathbb{X}}_{BC}{}^c \mathcal{F}_{ac}$, *($\overline{\mathbb{X}}_{BC}{}^c$ an invariant bundle injector), and*

$$\mathcal{F}_{ab} = \frac{1}{d-3}\left(W_{acbd} n^c n^d + \mathring{L}^2_{ab} - \frac{|\mathring{L}|^2}{2(d-2)} \overline{\boldsymbol{g}}_{ab} \right).$$

Recall $\mathbb{L}_{aC} = \underline{\nabla}_a N_C$. This shows that $\mathcal{F}_{ab}$ is a conformal invariant of hypersurfaces. It is the so-called **Fialkow tensor**.

Conformal Hypersurface invariants

We now have the basic tools to proliferate hypersurface invariants: On

$$\iota : \Sigma^n \longrightarrow (M^{d=n+1}, \mathbf{c})$$

We have N_C. Thus can form

$$\overline{D}_B N_C = L_{BC} \qquad \boxed{\text{(tractor second ff - up to a const.)}}$$

where $\overline{D}$ is the ambient-$\nabla^{\mathcal{T}}$-coupled-$(\Sigma, \mathbf{c})$-Thomas-D.
Then

$$\overline{D}_A \cdots \overline{D}_B N_C, \qquad \text{and} \qquad (\overline{D}_A \cdots \overline{D}_B N_C)(\overline{D}^A \cdots \overline{D}^B N^C), \quad \text{etcetera}$$

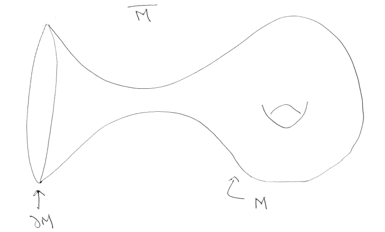
are hypersurface conformal invariants -cf. Riemannian theory. Can get all the "easy invariants" - but not the most interesting!!

$$\boxed{\text{More interesting ones require deeper ideas – see L2, L3}}$$

Let's turn to an important application of the calculus . . .

Example problem: Conformal Dirichlet-to-Neumann

Consider a **conformal manifold with boundary** $(\overline{M}^d, \mathbf{c})$ (e.g. PE):



A conformally invariant **Dirichlet-to-Robin** operator (on ∂M) can be constructed in two steps:

1. From a density f (weight $1-\frac{d}{2}$) on the boundary ∂M solve the Dirichlet problem $(\Delta - \frac{d-2}{4(d-1)}\,\mathrm{Sc})\tilde{f} = 0$ with $\tilde{f}|_{\partial M} = f$.

2. Then DtoR : $f \to \delta_1 \tilde{f}|_{\partial M}$ where

$$\delta_1 \stackrel{g}{:=} n^a \nabla_a - wH^g, \quad w = 1 - \frac{d}{2}$$

is the **Cherrier-Robin operator**. Here H^g is the mean curvature of ∂M and δ_1 is **conformally invariant**.

Problem: Higher order analogues? **Interior yes**. Analogues of δ_1??

Boundary operators: a naïve construction

In **Riemannian geometry** the basic **Neumann operator** is $n^a\nabla_a$. Higher transverse order transverse boundary operators similarly given: $n^a n^b \nabla_a \nabla_b$ etc.

The tools above suggest an immediate analogue – via **normal tractor** $\to N^A + D_A \leftarrow$**Thomas-D**. E.g. $\delta_1 \stackrel{g}{:=} n^a\nabla_a - wH^g$ - the conformal Cherrier-Robin operator. This is recovered by

$$\boxed{(d+2w-2)\delta_1 = N^A D_A.}$$

More generally (with $\mathcal{T}^\Phi[w]$ any tractor bundle/sections thereof) :

Lemma

$$\delta_K := N^{A_1} N^{A_2} \cdots N^{A_{K-1}} \delta_1 D_{A_1} D_{A_2} \cdots D_{A_{K-1}} \tag{1}$$

constructs a family of natural conformally invariant hypersurface operators $\delta_K : \mathcal{T}^\Phi[w] \to \mathcal{T}^\Phi[w-K]|_\Sigma$ *along* Σ.

Hidden problems, hidden treasures

It would **appear** from the formula

$$\delta_K := N^{A_1} N^{A_2} \cdots N^{A_{K-1}} \delta_1 D_{A_1} D_{A_2} \cdots D_{A_{K-1}} \quad \text{along} \quad \Sigma$$

that the operator has "high" transverse order and is always at least of transverse order 1. But e.g.: (where $n = \dim(\Sigma)$ etc)

$$\delta_2 f = -(\bar{\Delta} - \frac{n-2}{4(n-1)} \bar{\mathrm{Sc}}) f + \frac{n-2}{4(n-1)} \mathring{L}^{ab} \mathring{L}_{ab} f \ , \quad \text{for } f \in \mathcal{E}\left[1 - \frac{n}{2}\right] .$$

This is the **intrinsic to Σ Yamabe operator** of $(\Sigma, \mathbf{c}_\Sigma)$ (plus the conformal invariant $\mathring{L}^{ab} \mathring{L}_{ab}$). So:

$\boxed{\textbf{at this weight } \delta_2 \text{ has transverse order 0.}}$

At the **interior Yamabe weight** $1 - \frac{n}{2}$ we have instead

$$\delta_2 = -(\Delta - \frac{d-2}{4(d-1)} \mathrm{Sc}) \quad \text{along} \quad \Sigma.$$

– i.e. the **interior Yamabe operator** – so transverse order 2.

Bad weights

The above could be viewed as treasures, **BUT**, for δ_3

$$\boxed{\delta_3 = 0 \quad \text{at weight} \quad w = 2 - \tfrac{d}{2}\,,}$$

i.e. at **interior Paneitz weight**.

Leading order behaviour: a straightforward induction proves:

Proposition

Let $w \in \mathbb{R}$ and $K \in \mathbb{Z}_{>0}$ be given, and suppose that δ_K acts on $\mathcal{T}^\Phi[w]$. Then along Σ,

$$\delta_K = \Big[\prod_{i=1}^{K-1}(d + 2w - K - i)\Big](\nabla_n)^K + ltots\,.$$

So the set of **"Bad weights"** (where max. transverse order not reached) are as follows: $E(\delta_1) = \emptyset$, and for any $K \in \mathbb{Z}_{\geq 2}$,

$$E(\delta_K) = \left\{\frac{2K-1-d}{2}, \frac{2K-2-d}{2}, \cdots, \frac{K+1-d}{2}\right\}\,. \tag{2}$$

Digging deeper – conformally flat case

On **conformally flat manifolds** we **can improve** the operators and eliminate every second bad weight:

Theorem (δ^0_k – for conformally flat M)

Let $K \in \mathbb{Z}_{>0}$ and Σ in a conformally flat manifold. There is a family of natural conformally invariant differential operators along Σ, $\boxed{\delta^0_K : \mathcal{T}^{\Phi}[w] \to \mathcal{T}^{\Phi}[w-K]\ ,}$ *determined by*

$$\Big[\prod_{j=1}^{\lfloor \frac{K-1}{2} \rfloor} (d + 2w - 2K + 2j)\Big]\delta^0_K = \delta_K\ , \tag{3}$$

and polynomial continuation in w. The universal symbolic formula for δ^0_K is polynomial in w and n.

So: $E(\delta^0_K) = \left\{\frac{2K-1-d}{2}, \frac{2K-1-d}{2} - 1, \cdots, \frac{2K-1-d}{2} - \left\lfloor \frac{K-2}{2} \right\rfloor\right\}$.

In particular $\boxed{d \text{ even} \Rightarrow 0 \neq E(\delta^0_K) = \text{the bad weights.}}$

How the Theorem for δ_k^0 (c. flat M) works

The proof of the above uses e.g. that for $f \in \mathcal{T}^{\Phi}[2-\frac{d}{2}]$

$$D_A \circ D_B\, f = (0,0,\cdots,0,P_4 f) \quad \text{where} \quad P_4 = \text{Paneitz op.}$$

So at other weights $w \in \mathbb{R}$, $f \in \mathcal{T}^{\Phi}[w]$ we can deduce (using polynomial in w nature of the D operators)

$$D_A \circ D_B\, f = ((d+2w-4)*, (d+2w-4)*, \cdots, (d+2w-4)*, \Delta^2 f + lots)$$

While for $(M, \mathbf{c})$ **conformally flat** and $f \in \mathcal{T}^{\Phi}[3-\frac{d}{2}]$ we have

$$D_A \circ D_B \circ D_C f = (0,0,\cdots,0,-P_6 f),$$

and so for $w \in \mathbb{R}$, $f \in \mathcal{E}[w]$

$$D_A \circ D_B \circ D_C f = ((d+2w-6)*, (d+2w-6)*, \cdots, (d+2w-6)*, \star).$$

We then show that **these factors survive in the formulae for** δ_K.

Curved case ? : $(M, \mathbf{c})$ **conformally curved** then e.g. $f \in \mathcal{T}^{\Phi}[3-\frac{d}{2}]$ gives

$$D_A \circ D_B \circ D_C f = (0,0,\mathrm{mess},\cdots,\mathrm{mess},-\Delta^3 f + \mathrm{mess} f)$$

Recovering the curved case

In fact the "mess" terms have been understood from earlier work. For example for $f \in \mathcal{E}[3-\frac{d}{2}]$

$$P_{ABC} f := D_A \circ D_B \circ D_C f - \frac{2}{d-4} X_A W_B{}^F{}_C{}^E D_F D_E f = (0, \cdots, 0, -P_6 f),$$

and so we may replace $D_A \circ D_B \circ D_C$ with P_{ABC} in a construction of new δ-operators and we retain polynomiality wrt w.

Using similar results from work in G.+Peterson CMP 2003, Pacific 2006 there is an algorithm for similarly modifying any power D^k
This leads to what is a, possibly optimal, construction higher order conformally invariant Robin operators:

$$\delta_1, \delta_2, \cdots, \delta_K$$

that work at "*most*" weights and on an arbitrary conformal manifold with boundary.
Branson +G. . . . G.+Peterson: Pacific J.M. 2021

higher conformal Dirichlet-to-Neumann operators

$P_{2k} = \Delta^k + lots$ the order $2k$ GJMS conformal Laplacian operator:

Theorem (On a conf mfld with bdy $(\overline{M}, \partial M, \mathbf{c})$)

Let $B = (\delta_0, \delta_1, \cdots, \delta_{k-1})$ and suppose that the conformal generalised Dirichlet problem (P_{2k}, B) has trivial kernel. Then there is a well-defined conformally invariant Dirichlet-to-Neumann operator

$$P^k_{2m} : \overline{\mathcal{E}}\left[m - \frac{d}{2}\right] \to \overline{\mathcal{E}}\left[-m - \frac{d}{2}\right]$$

given by

$$\overline{\mathcal{E}}\left[m - \frac{d}{2}\right] \ni f \mapsto \delta_{2k-1-\ell} u \ .$$

Here $m := k - 1/2 - \ell$, and u solves the conformal generalised Dirichlet problem

$$P_{2k} u = 0, \quad \delta_\ell u = f, \quad \delta_j u = 0 \text{ for } j \neq \ell \text{ and } 0 \leq j \leq k-1 \ .$$

The operator P^k_{2m} has leading term $(-\overline{\Delta})^m$.

T-curvatures

Key idea: Recall for a Riemannian differential operator formula $\text{Operator}(w) : \mathcal{E}[w] \to \mathcal{E}[w-u]$ depending polynomally on weight:

$$\text{Operator}(w) = \text{Operator}'(w) \circ \nabla + wQ^g(w)$$

Then Branson's argument (e.g.) implies the conformal transformation (for $Q^g := Q^g(0)$)

$$e^{u\Upsilon} Q^{\widehat{g}} = Q^g + \text{Operator}(0)\Upsilon, \quad \widehat{g} = e^{2\Upsilon} g.$$

Definition: For $\text{Operator}(w)$ along Σ, say that Q^g is a T-**curvature** (and denote T^g) if Operator(0) has **maximal transverse order**, (as allowed by the weight/order).

$$\boxed{\text{In particular we can take Operator}(w) \text{ to be our } \delta_K}$$

T curvatures turn up as boundary/transgression terms in conformal anomaly calcs. Chang-Qing: JFA 1997 G+Waldron . . .

One T-curvature result (d even)

The weight 0 can be removed from the trouble weight list at each order if Σ is **odd dimensional**. So:

Theorem

Let a hypersurface Σ of a Riemannian manifold (M, g) be given, and suppose that the dimension d, of M, is even. Then there are canonical T-curvature pairs

$$(\delta_K, T^g_K)$$

of orders $K = 1, 2, 3, \ldots,$ respectively. In each case, $T^g_K := Q_g(\delta_K)$.

For each T-curvature here: $e^{K\Upsilon} T^{\widehat{g}}_K = T^g_K + \delta_K \Upsilon$, if $\widehat{g} = e^{2\Upsilon} g$ – so **generalise the mean curvature**.

THE END

of lecture one

Application – Scale normalisation

An easy inductive argument shows that that we can arrange:

Proposition

On Σ^{odd}: given any metric g_Σ in the conformal class, there is a metric $g \in \mathbf{c}$ inducing g_Σ s.t.

$$T_1^g = T_2^g = \cdots = T_m^g = 0 \quad \text{any } m \in \mathbb{Z}_{\geq 1},$$

along Σ. This g is determined uniquely by g_Σ (given $(M, \mathbf{c}, \Sigma)$) up to the given order.

In fact one can proceed to $m = \infty$. This means a choice of scale g_Σ on Σ canonically (formally) determines the ambient scale.

Examples

For any $w \in \mathbb{R}$, the operator $\delta_2 : \mathcal{T}^\Phi[w] \to \mathcal{T}^\Phi[w-2]$ is given by $\delta_2 := N^A \delta_1 D_A$. For all $f \in \mathcal{T}^\Phi[w]$,

$$\begin{aligned}\delta_2 f = &\\ & -(\Delta + w\mathrm{J})f + (n+2w-2)n^a n^b \nabla_a \nabla_b f \\ & -2(w-1)(n+2w-2)Hn^a\nabla_a f + (w-1)w(n+2w-2)H^2 f \\ & +w(n+2w-2)n^a n^b \mathrm{P}_{ab} f \ .\end{aligned}$$

Thus:

$$\boxed{T_g(\delta_2) = \mathrm{J} + (n-2)H^2 - (n-2)n^a n^b \mathrm{P}_{ab}}.$$

$E(\delta_2) = \{(3-n)/2\}$, so for all $n \geq 4$, $T_g(\delta_2)$ is a hypersurface T-curvature.

We see for $w = 1 - \frac{n}{2}$ (interior Yamabe weight)

$$\delta_2 = -(\Delta + w\mathrm{J})f = -(\mathrm{Yamabe})f \ .$$

Third order

The operator $\delta_{1,2} : \mathcal{E}[w] \to \mathcal{E}[w-3]$ is given simply by

$$(n+2w-4)\delta_{1,2} = N^A N^B \delta_1 D_A D_B.$$

Expanding. For $f \in \mathcal{E}[w]$

$$\delta_{1,2} f = (n+2w-5)\delta_1 \Box f - (n+2w-2)\big(\Box_\Sigma \delta_1 f + \text{ lower order}\big).$$

Where $\Box := \Delta + w\mathrm{J}$ and $\Box_\Sigma$ is the intrinsic to Σ equivalent. When $w = 1 - \frac{n}{2}$ this factors: $\delta_{1,2} f = -3\delta_1 \Box f$. When $w = 2 - \frac{\bar{n}}{2}$ then $\delta_{1,2} f = -3\Box_\Sigma \delta_1 f + \star f$ where $\star$ is a manifestly invariant lower order operator.

The T-curvature is:

$$\begin{aligned} T_g(\delta_{1,2}) = {} & 3n^a \nabla_a \mathrm{J} - (n-2) n^a n^b n^c \nabla_a \mathrm{P}_{bc} + 6H\mathrm{J} \\ & -6(n-2) H n^a n^b \mathrm{P}_{ab} + 2(n-2) H^3 \ . \end{aligned}$$

Lecture Two

Plan

We introduce first steps toward the holographic approach to hypersurfaces and boundary calculus. The following topics will be treated.

A conceptual approach to compactification. The scale tractor and a tractor interpretation of conformally compact manifolds. The scattering Laplacian, and the sl(2) of the Laplace-Robin operator I.D. Formal asymptotics.

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G. Nurowski, Obstructions to conformally Einstein metrics, *J. Geom. Phys.*, (2006)

G-. Almost Einstein and Poincaré-Einstein manifolds etc, *J. Geom. Phys.*, (2010), 182–204.

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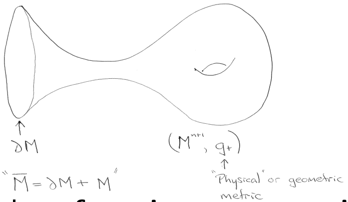
G. + Waldron, Boundary calculus for conformally compact manifolds, *Indiana* (2014)

Conformal compactification

Recall from L1: A **conformal compactification** of pseudo-Riemannian manifold (M^{n+1}, g_+) is a manifold $\overline{M}$ with boundary ∂M s.t.:

- $\exists\ \overline{g}$ on $\overline{M}$, with
- $g_+ = r^{-2}\overline{g}$, where r a defining function for ∂M.



$\Rightarrow$ canonical **conformal structure on boundary**: $(\partial M, [\overline{g}|_{\partial M}])$ (where dr not null).

- Called here **Poincaré-Einstein** metric if also g_+ Einstein.

Let's redicover this **conceptually/geometrically** and so learn **new tools** to treat it. $\boxed{d = n+1}$

Conformal compactification of $\mathbb{H}^{n+1}$ – the Poincaré ball

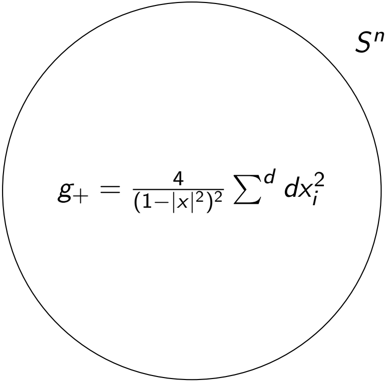
Escher's circle limit

$$\overline{\mathbb{H}^2} = \mathbb{H}^2 + \partial\mathbb{H}^2$$

The embedding gives the compactification

$\mathbb{H}^d$ embedded conformally in Euclidean $\mathbb{E}^d$ – Poincaré-Ball

$S^n = \partial\mathbb{H}^{n+1}$



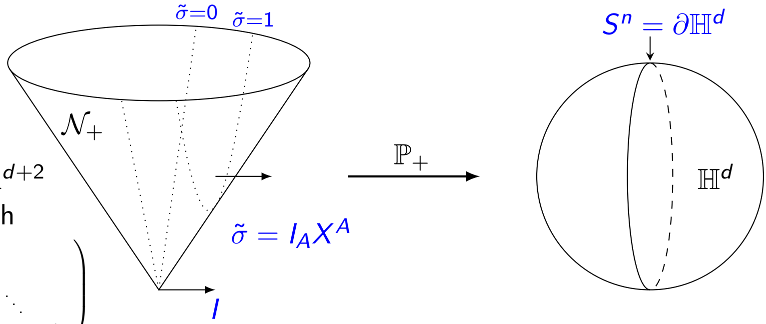
$$g_+ = \frac{4}{(1-|x|^2)^2}\sum^d dx_i^2$$

$$\overline{\mathbb{H}^d} = \mathbb{H}^d + \partial\mathbb{H}^d$$

Poincaré compactification via $\mathbb{P}_+(\textit{nullcone})$

Conformal compactification of $\mathbb{H}^d$ by **symmetry breaking**:

i.e. $\mathbb{R}^{d+2}$ with
$$\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

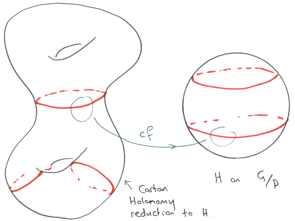


$S^d = \mathbb{P}_+(\mathcal{N}_+ \subset \mathbb{R}^{d+2} \setminus \{0\})$ is model of flat conformal geometry.
$G := SO_o(d+1,1)$ acts transitively. $I \in \mathbb{R}^{d+2}$, spacelike $h(I,I) = 1$
Symmetry reduction by I: $\Rightarrow$ $\boxed{H = SO_o(d,1) \text{ orbits.}}$ Right hemi. is conf. compactification $\overline{M}_c$ of $\mathbb{H}^d$; $\sigma = 0$ conformal ∞ with conformal str.

Theorem (**Curved orbit decomposition** - Čap,G., Hammerl)

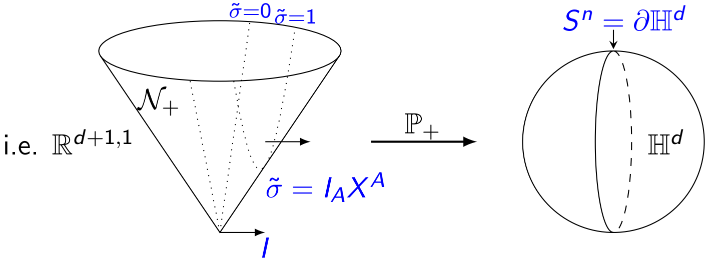
Suppose $(\mathcal{G},\omega) \to M$ is a Cartan geometry (modelled on $G \to G/P$) endowed with a parallel tractor field h giving a Cartan holonomy reduction with **holonomy group** *H. Then:*
*(1) M is canonically stratified $M = \bigcup_{i \in H \backslash G/P} M_i$ in a way locally diffeomorphic to the the H-***orbit decomposition** *of G/P; and*
(2) there $\exists$ a **Cartan geometry on** *M_i of the same type as the model.*

Thus there is a **general way to define a curved analogue of an orbit decomposition of a homogeneous space**.

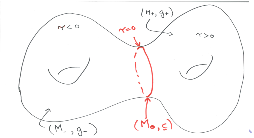


Curving the conformal compactification of $\mathbb{H}^d$

Recall the $H = SO_o(d,1)$ orbits on conformal sphere G/P, where $G = SO_o(d+1,1)$, H fixes $I \in \mathbb{R}^{d+2}$ spacelike:



Curved: A conformal manifold has a canonical Cartan bundle $\mathcal{G}$ modeled on (G,P). If this supports a **parallel spacelike tractor** I_A then the **curved orbit theorem** (**plus** some interpretation) states either M Einstein or M stratifies into disjoint union $M = M_- \cup M_0 \cup M_+$ and M_0 is a separating hypersurface. Moreover $M \setminus M_\mp$ is a **conf. compactification** of the **Einstein** $M_\pm$.



Parallel standard tractors

Note that from the formula

$$\nabla^{\mathcal{T}}_a(\sigma, \mu_b, \rho) = (\nabla_a\sigma - \mu_a,\ \nabla\mu_b + P_{ab}\sigma + \boldsymbol{g}_{ab}\rho,\ \nabla_a\rho - P_{ab}\mu^b),$$

if $I_A \stackrel{g}{=} (\sigma, \mu_a, \rho)$ is a parallel tractor then $\mu_a = \nabla_a\sigma$, and $\rho = -(\Delta\sigma + wJ\sigma)$. This gives the first statement of:

Proposition

I parallel implies $I_A = \frac{1}{d}D_A\sigma$. So $I \neq 0 \Rightarrow \sigma$ is nonvanishing on an open dense set $M_{\sigma\neq 0}$. On $M_{\sigma\neq 0}$, $g^o = \sigma^{-2}\mathbf{g}$ is Einstein. Conversely if $g^o = \sigma^{-2}\mathbf{g}$ is Einstein then $I := \frac{1}{d}D\sigma$ is parallel.

Proof.

On $M_{\sigma\neq 0}$ we have locally $\pm\sigma \in \Gamma(\mathcal{E}_+[1])$ so $\mu_a = \nabla_a\sigma = 0$ for $\nabla = \nabla^{g^\sigma}$. Thus

$$P_{ab} + \frac{\rho}{\sigma}\boldsymbol{g}_{ab} = 0.$$

The converse is easy. $\square$

So we say $(M, \mathbf{c})$ with parallel $I \neq 0$ is **almost Einstein**.

Almost pseudo-Riemannian geometry

We now drop the PE condition to understand all conf. compact

For convenience we say that a structure

$$(M^d, \mathbf{c}, \sigma) \quad \text{where} \quad \sigma \in \Gamma(\mathcal{E}[1])$$

is **almost pseudo-Riemannian** if the tractor

$$\boxed{I_A := \tfrac{1}{d} D_A \sigma \qquad \textbf{is nowhere zero} \overset{def.}{\leftrightarrow} I \text{ is a } \textbf{scale tractor}}$$

Note then that σ is non-zero on an open dense set, since $D_A\sigma$ encodes part of the 2-jet of σ. So on an almost pseudo-Riemannian manifold there is the pseudo-Riemannian metric $g^o = \sigma^{-2}\boldsymbol{g}$ on the same open dense set.

Lemma

A conf. compact mfld is an almost Riemannian manifold $(\overline{M}, \mathbf{c}, \sigma)$ with boundary ($\overline{M} = M_+ \cup \partial M_+$) such that σ **defines*** *∂M_+*

$*$ i.e. $\partial M = \sigma^{-1}(0) =: \mathcal{Z}(\sigma)$ and $\nabla\sigma$ nowhere 0 on ∂M.

Now from the formula for I and the tractor metric we have

$$I^A I_A =: I^2 \stackrel{g}{=} \boldsymbol{g}^{ab}(\nabla_a\sigma)(\nabla_b\sigma) - \frac{2}{d}\sigma(\mathrm{J}+\Delta)\sigma \qquad (4)$$

where g is any metric from **c** and ∇ its Levi-Civita connection. This is well-defined everywhere on an almost pseudo-Riemannian manifold. Where σ is non-zero, it computes

$$I^2 = -\frac{2}{d}\mathrm{J}^{g^o} = -\frac{\mathrm{Sc}^{g^o}}{d(d-1)} \quad \text{where} \quad g^o = \sigma^{-2}\boldsymbol{g}.$$

Thus I^2 gives a **generalisation of the scalar curvature** (up to a constant factor $-1/d(d-1)$); it is canonical and smoothly extends the scalar curvature to include the zero set of σ.

ASC manifold (where ASC means **almost scalar constant**): means an almost pseudo-Riemannian manifold with $I^2 = \mathit{constant}$. Since the tractor connection preserves h, then I parallel implies $I^2 = \mathit{constant}$. So an almost Einstein manifold is ASC, just as Einstein manifolds have constant scalar curvature.

Non-zero generalised scalar curvature.

Much of the almost Einstein curved orbit picture remains in the almost pseudo-Riemannian setting when I^2 is non-vanishing:

Theorem

Let $(M, \mathbf{c}, I)$ be an almost pseudo-Riemannian manifold with I^2 **nowhere zero**. *Then $\mathcal{Z}(\sigma)$, if not empty, is a smooth embedded separating hypersurface. This has a spacelike (resp. timelike) normal if g^o has negative scalar (resp. positive) scalar curvature. If $\mathbf{c}$ has Riemannian signature and $I^2 < 0$ then $\mathcal{Z}(\sigma)$ is empty.*

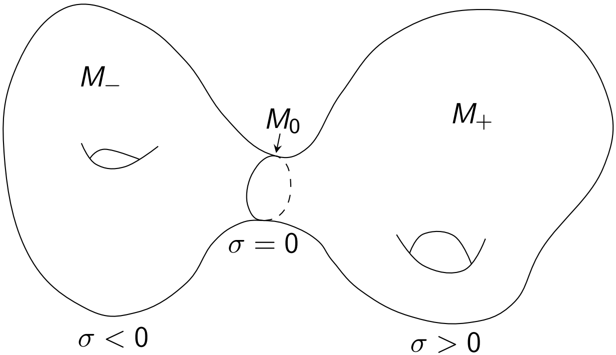
Key aspect of Proof.

From $I^2 \stackrel{g}{=} \boldsymbol{g}^{ab}(\nabla_a\sigma)(\nabla_b\sigma) - \frac{2}{d}\sigma(\mathrm{J} + \Delta)\sigma$: Along $\mathcal{Z}(\sigma)$ we have

$$I^2 = \boldsymbol{g}^{ab}(\nabla_a\sigma)(\nabla_b\sigma).$$

in particular $\nabla\sigma$ is nowhere zero on $\mathcal{Z}(\sigma)$, and so σ is a **defining density**. Thus $\mathcal{Z}(\sigma)$ is a smoothly embedded hypersurface by the implicit function theorem.

The picture if $I_A = \frac{1}{d} D_A \sigma$ s.t $I^2 \neq 0$:



$(M, \boldsymbol{c})$ equipped with a **scale tractor** $I = \frac{1}{d} D\sigma$, with I^2 nowhere zero has I nowhere zero and so is almost pseudo-Riemanian. Where $\sigma = X^A I_A$ is nonzero (almost everywhere) there is the pseudo-Riemannian metric $g^o = \sigma^{-2}\boldsymbol{g}$, and σ is a defining density for the separating hypersurface $M_0 = Z(\sigma)$. Hence $M \setminus M_\pm$ is **conformally compact** with conf. infinity $(M_0, \boldsymbol{c}|_{M_0})$. Conversely all conformally compact manifolds arise this way*.

The moral so far, and three (other) powerful facts

Moral: *Replace* (M, g) *with* $(M, \boldsymbol{c}, I)$ *where* I *is the* **scale tractor**. *This* **generalises our notion of geometry** *in a way that builds in the compactification data.*

E.g.$(*)$ $(\overline{M}, g_o)$ *a* **conformal compactification**, *with the scalar curvature bounded away from zero, means just* $(\overline{M}, \boldsymbol{c}, I)$ *where* $\overline{M} = M + \partial M$, $\partial M = \mathcal{Z}(\sigma)$ *and* I^2 *non-vanishing. (On* M, $g_o = \sigma^{-2}\boldsymbol{g}$*.)*

Next we note three remarkable facts about the scale tractor

1. With mild restrictions it recovers the normal tractor on the boundary of conformally compact manifolds.
2. On the interior it combines with D_A to give the "scattering Laplacian".
3. It yields an $sl(2)$ structure for boundary calculus.

AH or asymp de Sitter then $I|_{\partial M} = N$

Think of **conformally compact geometries** $(M, \mathbf{c}, I)$. Recall the **scale tractor** I is given $I = (\sigma, \nabla\sigma, -\frac{1}{d}(\Delta\sigma + J\sigma))$.

We will consider in particular $(M, \mathbf{c}, I)$ which near the conformal infinity are **asymptotically of constant nonzero scalar curvature**. By imposing a constant dilation we may assume that I^2 approaches ± 1, i.e. **asymptotically hyperbolic/AdS** resp. **asymptotically de Sitter**.

The σ, equivalently scale tractor I, strongly links the geometry of $\Sigma = \mathcal{Z}(\sigma)$ to the ambient by a beautiful agreement of I and the normal tractor: $\Sigma = \partial M$ if conf. compact

Proposition

Let $(M^d, \mathbf{c}, I)$ be an almost pseudo-Riemannian structure with scale singularity set $\Sigma \neq \emptyset$ and $I^2 = \pm 1 + \sigma^2 f$ for some smooth (weight -2) density f. Then Σ is a smoothly embedded hypersurface and, with N denoting the normal tractor for Σ, we have $\boxed{N = I|_\Sigma}$.

Proof.

For simplicity assume the case $I^2 = \pm 1$ (so $f = 0$ and the structure is ASC). As usual let us write $\sigma := h(X, I)$. Along $\mathcal{Z}(\sigma)$

$$I_A = \frac{1}{d} D_A \sigma \stackrel{g}{=} \begin{pmatrix} 0 \\ \nabla_a \sigma \\ -\frac{1}{d}\Delta\sigma \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{g}^{ab}(\nabla_a \sigma)\nabla_b \sigma = \pm 1$$

so $n_a := \nabla_a \sigma$ is the unit conormal and a computation gives $-\frac{1}{d}\Delta\sigma = -\frac{1}{d-1}\boldsymbol{g}^{ab} L^g_{ab} = -H^g$. □

Corollary

Let $(M^d, \mathbf{c}, I)$ be an almost pseudo-Riemannian structure with scale singularity set $\Sigma \neq \emptyset$, and that is asymptotically Einstein in the sense that $I^2|_\Sigma = \pm 1$, and $\nabla_a I_B = \sigma f_{aB}$ for some smooth (weight -1) tractor valued 1-form f_{aB}. Then Σ is a **totally umbilic hypersurface**.

Agreement of tractor connections

If we assume the stronger asymptotics: $I^2|_\Sigma = \pm 1$, and $\nabla_a I_B = \sigma^2 f_{aB}$ Then along Σ, I_B is parallel to the given order, and so the tractor curvature satisfies

$$\kappa_{ab}{}^C{}_D I^D = \kappa_{ab}{}^C{}_D N^D = 0 \quad \text{along} \quad \Sigma.$$

This implies

$$\boxed{W_{ab}{}^c{}_d n^d = 0}, \quad \text{along} \quad \Sigma = \mathcal{Z}(\sigma)$$

$\therefore$ Fialkow $\mathcal{F}_{ab} = \frac{1}{d-3}(W_{acbd} n^c n^d + \mathring{L}^2_{ab} - \frac{|\mathring{L}|^2}{2(d-2)} \overline{\boldsymbol{g}}_{ab}) = 0$, & L1 $\Rightarrow$

Theorem

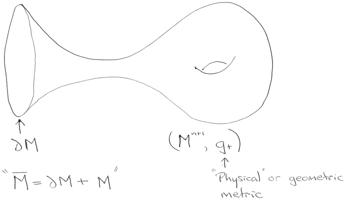
Let $(M^{d \geq 4}, \mathbf{c}, I)$ be an almost pseudo-Riemannian structure with scale singularity set $\Sigma \neq \emptyset$, and that is asymptotically Einstein in the sense that $I^2|_\Sigma = \pm 1$, and $\nabla_a I_B = \sigma^2 f_{aB}$. Then the tractor connection of $(M, \mathbf{c})$ preserves the intrinsic tractor bundle of Σ, where the latter is viewed as a subbundle of the ambient tractors: $\mathcal{T}_\Sigma \subset \mathcal{T}$. Furthermore the restriction of the parallel transport of $\nabla^{\mathcal{T}}$ coincides with the intrinsic tractor parallel transport of $\nabla^{\mathcal{T}_\Sigma = \overline{\mathcal{T}}}$.

Summary to this point

An almost pseudo-Riemannian manifold with **non-zero generalised scalar curvature** has $\Sigma = \mathcal{Z}(\sigma)$ smoothly embedded.

Questions: E.g. $g = \sigma^{-2}\boldsymbol{g}$ – is **asymptotically Einstein** then:

1. **Asymptotics of** g near $\Sigma = \partial M$?:
   $I^2 = \pm 1 + \sigma f$ so g is asymptotically of constant scalar curvature and is resp. asymp. de Sitter/asyp. hyperbolic.
   $$R^g_{abcd} = \pm(g_{ac}g_{bd} - g_{ad}g_{bc}) + O(\sigma^{-3})$$
2. **Extrinsic geometry** of $(\partial M, \boldsymbol{c}|_{\partial M})$?:
   $$\mathring{L}_{ab} = 0, \quad \mathcal{F}_{ab} = 0, \cdots \text{(see next lect. \& arXiv:2107.10381)}$$
   Conformal geometry of $(M, \boldsymbol{c})$ near ∂M, e.g. $W_{ab}{}^c{}_d n^d = 0$.
3. **Intrinsic geometry** of $(\partial M, \boldsymbol{c}|_{\partial M})$?:
   For d odd, n even and $\nabla I = 0$ to high order (approx. σ^{d-1}) then
   $$0 = \overline{B}_{ab} = \overline{\Delta}^{n/2-2}\overline{\nabla\nabla W}_{acbd} + \text{lower order}$$
   the Fefferman-Graham obstruction tensor of $(\partial M, \boldsymbol{c}|_{\partial M})$



Suppose on the interior one wants to solve

$$\left(\Delta^g + s(n-s)\frac{J^g}{d}\right)f = 0$$

where Δ^g is, as usual, the **wave operator** or metric **Laplacian** $g^{ab}\nabla_a\nabla_b$ for the conformally compact metric

$$g = g_+ = \sigma^{-2}\boldsymbol{g}$$

that is singular at the boundary ∂M. What are the right "Dirichlet" and "Neumann" boundary conditions? Mapping between these is one idea in scattering. Then s is the **spectral parameter**.

Differential operators by prolonged coupling

On an almost pseudo-Riemannian manifold $(M, \mathbf{c}, I)$ there is a canonical differential operator by **coupling** I^A **to** D_A , namely

$$I \cdot D := I^A D_A .$$

This acts on any weighted tractor bundle, preserving its tensor type but lowering the weight:

$$I \cdot D : \mathcal{E}^\Phi[w] \to \mathcal{E}^\Phi[w-1].$$

It will be useful to define define the *weight operator* $\mathbf{w}$: if $\beta \in \Gamma(\mathcal{B}[w_0])$ we have

$$\mathbf{w}\,\beta = w_0 \beta .$$

Then on $\mathcal{E}^\Phi[w]$ we have

$$I \cdot D \stackrel{g}{=} \begin{pmatrix} -\frac{1}{d}(\Delta\sigma + \mathrm{J}\sigma) & \nabla^a\sigma & \sigma \end{pmatrix} \begin{pmatrix} \mathbf{w}(d+2\mathbf{w}-2) \\ \nabla_a(d+2\mathbf{w}-2) \\ -(\Delta + \mathrm{J}\mathbf{w}) \end{pmatrix} .$$

$$= -\sigma\Delta + (d+2w-2)[(\nabla^a\sigma)\nabla_a - \frac{w}{d}(\Delta\sigma)] - \frac{2w}{d}(d+w-1)\sigma \mathrm{J}$$

The canonical degenerate Laplacian

Now on $M \setminus \mathcal{Z}(\sigma)$ in the metric $g_{\pm} = \sigma^{-2}\boldsymbol{g}$, with densities trivialised accordingly, we have

$$I \cdot D \stackrel{g_{\pm}}{=} -\Big(\Delta^{g_{\pm}} + \frac{2w(d+w-1)}{d} \mathsf{J}^{g_{\pm}}\Big).$$

In particular if $g_{\pm}$ satisfies $\mathsf{J}^{g_{\pm}} = \mp \frac{d}{2}$ (*i.e.* $\mathrm{Sc}^{g_{\pm}} = \mp d(d-1)$ or equivalently $I^2 = \pm 1$) then, relabeling $d + w - 1 =: s$ and $d - 1 =: n$, we have $\boxed{I \cdot D \stackrel{g_{\pm}}{=} -\big(\Delta^{g_{\pm}} \pm s(n-s)\big)}$.

so solutions are **eigenvectors of the Laplacian** (and s is called the **spectral parameter**) as in **scattering theory**.

But on $\Sigma = \mathcal{Z}(\sigma) \neq \emptyset$, the conformal infinity, $I \cdot D$ degenerates and there the operator is first order. In particular if the structure is asymptotically ASC, in the sense that $I^2 = \pm 1 + \sigma^2 f$, for some smooth f, then along Σ

$\boxed{I \cdot D = N^A D_A = (d + 2w - 2)\delta_1}$, $\delta_1 \stackrel{g}{=} n^a \nabla_a^g - wH^g =$ conformal Robin

Thus $I \cdot D$ is a **degenerate Laplacian**, natural to $(M, \boldsymbol{c}, I)$.

The $\mathfrak{sl}(2)$-algebra

$(M, \mathbf{c})$ be a conformal structure of dimension $d \geq 3$, $\sigma \in \Gamma(\mathcal{E}[1])$ and $I_A = \frac{1}{d} D_A \sigma$ (as usual). Then a direct computation gives

Lemma

Acting on any section of a weighted tractor bundle we have

$$[I \cdot D, \sigma] = I^2(d + 2\mathbf{w}),$$

where $\mathbf{w}$ *is the weight operator.*

Thus with **only the restriction that generalised scalar curvature is non-vanishing** we have:

Proposition (G.-Waldron)

Suppose that (M, c, σ) *is such that* I^2 *is nowhere vanishing. Setting* $x := \sigma$, $y := -\frac{1}{I^2} I \cdot D$, *and* $h := d + 2\mathbf{w}$ *we obtain the commutation relations*

$$[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h,$$

of standard $\mathfrak{sl}(2)$*-algebra generators.*

Application: Conformal Laplacian powers

Theorem (G+Waldron)

Let $\mathcal{E}^\Phi$ be any tractor bundle and $k \in \mathbb{Z}_{\geq 1}$. Then, for each $k \in \mathbb{Z}_{\geq 1}$, along $\Sigma = \mathcal{Z}(\sigma)$

$$P_k : \mathcal{E}^\Phi[\frac{k-n}{2}] \to \mathcal{E}^\Phi[\frac{-k-n}{2}] \quad \text{given by} \quad P_k := \left(-\frac{1}{I^2} I\!\!\!/ D\right)^k \tag{5}$$

is a tangential differential operator, and so determines a canonical differential operator $P_k : \mathcal{E}^\Phi[\frac{k-n}{2}]|_\Sigma \to \mathcal{E}^\Phi[\frac{-k-n}{2}]|_\Sigma$. For k even this takes the form

$$P_k = \overline{\Delta}^k + \text{lower order terms.} \tag{6}$$

Proof.

From the $\mathfrak{s}l(2)$-identities we have $[x, y^k] = y^{k-1}k(h-k+1)$. Thus on $\mathcal{E}^\Phi[\frac{k-n}{2}]$

$$P_k(f + \sigma h) = y^k(f + xh) = P_k f + \sigma \widetilde{P}_k h.$$

So P_k is **tangential**. Expanding the $I{\cdot}D$s yields (6). □

Natural boundary problems

Suppose on a conformally compact manifold (M_+, g_+) (with $M_+ \cup \partial M_+ = \overline{M}$) we wish to study solutions to

$$Pf := \left(\Delta^{g_+} + \frac{2w(d+w-1)}{d} \mathsf{J}^{g_+}\right) f = 0.$$

E.g. as in the usual Poincaré-Einstein scattering program.

boundary conditions ?? Since the boundary ∂M_+ is at infinity, with g_+ singular along ∂M_+, this is non-trivial.

From above, if we view f as the trivialisation of a density of weight w then

$$\boxed{Pf \stackrel{g_+}{=} I \cdot Df}$$

and $I \cdot D$ is well defined on all of $\overline{M}$ (and its smooth extension to M beyond ∂M_+). Thus it is natural to study the $I \cdot D$ problem. We do this **formally**.

First we treat an obvious Dirichlet-like problem where we view $f|_\Sigma$ as the initial data.

Asymptotic solutions of the first kind

Problem

Given $f|_\Sigma$, and an arbitrary extension f_0 of this to $\mathcal{E}^\Phi[w_0]$ over M, find $f_i \in \mathcal{E}^\Phi[w_0 - i]$ (over M), $i = 1, 2, \cdots$, so that

$$f^{(\ell)} := f_0 + \sigma f_1 + \sigma^2 f_2 + \cdots + O(\sigma^{\ell+1})$$

solves $I{\cdot}Df = O(\sigma^\ell)$, off Σ, for $\ell \in \mathbb{N} \cup \infty$ as high as possible.

$I{\cdot}Df = 0 \Leftrightarrow -\frac{1}{I^2} I{\cdot}Df = 0$ so we recast this via $\mathfrak{sl}(2) = \langle x, y, h \rangle$.
Set $h_0 = d + 2w_0$. By the identity $[x^k, y] = x^{k-1} k(h + k - 1)$:

$$yf^{(\ell+1)} = yf^{(\ell)} - x^\ell(\ell+1)(h+\ell)f_{\ell+1} + O(x^{\ell+1}).$$

Now $hf_{\ell+1} = \big(h_0 - 2(\ell+1)\big) f_{\ell+1}$, thus

$$yf^{(\ell+1)} = yf^{(\ell)} - x^\ell(\ell+1)(h_0 - \ell - 2)f_{\ell+1} + O(x^{\ell+1}). \qquad (7)$$

By assumption $yf^{(\ell)} = O(x^\ell)$, thus if $\boxed{\ell \neq h_0 - 2}$ we can solve $yf^{(\ell+1)} = O(x^{\ell+1})$ and this **uniquely determines** $f_{\ell+1}|_\Sigma$.

The obstruction on conformally compact manifolds

So we can solve to all orders provided we do not hit $\ell = h_0 - 2$ i.e. provided $w_0 \notin \{\frac{k-n}{2} \ : \ k \in \mathbb{Z}_{\geq 1}\}$. Otherwise (7) shows that

$$\ell = h_0 - 2 \quad \Rightarrow \quad yf^{(\ell)} = y\big(f^{(\ell)} + x^{\ell+1} f_{\ell+1}\big), \qquad \text{modulo} \ \ O(x^{\ell+1}),$$

regardless of $f_{\ell+1}$. It follows that the map $f_0 \mapsto x^{-\ell} y f^{(\ell)}$ is tangential and $x^{-\ell} y f^{(\ell)}|_\Sigma$ is the obstruction to solving $yf^{(\ell+1)} = O(x^{\ell+1})$. Then by a simple induction this is seen to be a non-zero multiple of $y^{\ell+1} f_0|_\Sigma$:

Proposition (G+Waldron)

If $\ell = h_0 - 2$ then the smooth extension is (in general) obstructed by $P_{\ell+1} f_0|_\Sigma$, where $P_{\ell+1} = (-\frac{1}{I^2} I \cdot D)^{\ell+1}$ is the tangential operator on densities of weight w_0 discussed above.

If $\ell = h_0 - 2$ then the extension can be continued with **log terms**. If $\overline{M}$ is **almost Einstein** to sufficiently high order then:

- the **odd order** $P_{\ell+1}$ **vanish identically**; and
- the **even order** $P_{\ell+1}$ are the **GJMS operators** on $(\partial M_+, \bar{\mathbf{c}})$.

(Formal) solutions of the second kind

Now we consider the more general type of solution:

Problem

Given $\overline{f}_0|_\Sigma \in \Gamma\mathcal{E}^\Phi[w_0-\alpha]|_\Sigma$ and an arbitrary extension $\overline{f}_0$ of this to $\Gamma\mathcal{E}^\Phi[w_0-\alpha]$ over $\overline{M}$, find $\overline{f}_i \in \mathcal{E}^\Phi[w_0-\alpha-i]$ (over $\overline{M}$), $i=1,2,\cdots$, so that

$$\overline{f} := \sigma^\alpha\big(\overline{f}_0 + \sigma\,\overline{f}_1 + \sigma^2\,\overline{f}_2 + \cdots + O(\sigma^{\ell+1})\big) \tag{8}$$

solves $I\cdot D\overline{f} = O(\sigma^{\ell+\alpha})$, off ∂M_+, for $\ell \in \mathbb{N}\cup\infty$ as high as possible.

Now α, if not integral, this problem takes us outside the realm of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ and its modules. But it is straightforward to show that for any $\alpha \in \mathbb{R}$:

$$[x^\alpha, y] = x^{\alpha-1}\alpha(h+\alpha-1). \tag{9}$$

It follows immediately from (9) that $I \cdot D\overline{f} = 0$ has:

- no solution if $\alpha \notin \{0, h_0 - 1\}$, where $h\overline{f} = h_0\overline{f}$; and
- if $\alpha = h_0 - 1$ and $\overline{f} = \sigma^\alpha f$ then

$$I \cdot D\overline{f} = \sigma^\alpha I \cdot Df \qquad \boxed{\text{So } \overline{f} \text{ is a solution iff } f \text{ is!}}$$

So in this way **second solutions** arise from **first** and vv.

For $w_0 \notin \{\frac{k-n}{2} \ : \ k \in \mathbb{Z}_{\geq 1}\}$, and writing $F = f$, $G = \sigma^{-\alpha}\overline{f}$ we can combine these to a general solution

$$F + \sigma^{h_0 - 1} G = F + \sigma^{n+2w_0} G$$

or, trivialising the densities on M_+ using the generalised scale σ:

$$f = \sigma^{n-s} F + \sigma^s G = \sigma^{-w_0}(F + \sigma^{h_0-1} G)$$

where $s := w_0 + n$. Which is the form of solution used in the **scattering theory** (of Mazzeo-Melrose, Graham-Zworski, $\cdots$). (For global solns f the **scattering matrix** is the map $F|_\Sigma \mapsto G|_\Sigma$ – cf. Dirichlet-to-Neumann.) $\mathfrak{sl}(2)$ above $\to$ asymptotics of F & G.

THE END

of lecture two

Lecture Three = talk

Higher conformal fundamental forms and the asymptotically PE condition

Part I. Singular Yamabe problem and higher Willmore invariants

* G. + Waldron, Conformal hypersurface geometry via a boundary Loewner-Nirenberg-Yamabe problem. *Comm. Anal. Geom.* (2021),
* G. + Waldron, Andrew Renormalized volume. *Comm. Math. Phys.* (2017)
* Arias, G. , Waldron Conformal geometry of embedded manifolds with boundary from universal holographic formuae. *Adv. Math.* (2021),

Part II: Higher conformal fundamental forms

Blitz, G. , Waldron, Conformal Fundamental Forms and the Asymptotically Poincaré–Einstein Condition, arXiv:2107.10381

Part I. Singular Yamabe and higher Willmore invariants

The Poincaré-Einstein construction of Fefferman-Graham is a tool for studying a conformal manifold $(\Sigma, \bar{\mathbf{c}})$ **holographically**. That is for obtaining the invariants and invariant operators of $(\Sigma, \bar{\mathbf{c}})$ in terms (pseudo-)Riemannian objects on the manifold M_+ of 1 greater dimension that has $\Sigma = \partial M_+$.

But requiring g_+ to be Einstein (even asymp. near ∂M_+) is **highly restrictive**. It means that the conformal manifold with boundary $(\overline{M}, \mathbf{c})$ has $\Sigma = \partial M_+$ totally umbilic, Fialkow vanishes, etcetera.

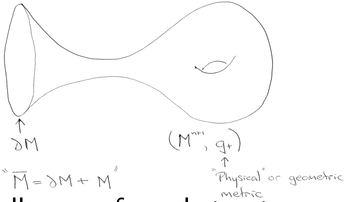
Here we seek to set up the analogous program for $(\overline{M}, \mathbf{c})$ **a general manifold with boundary**.

Thus, given $(\overline{M} = M_+ \cup \partial M_+, \mathbf{c})$ we **need a way to determine a distinguished metric** $g_+ \in \mathbf{c}|_{M_+}$ on M_+ so that (M_+, g_+) is conformally compact.

Generalising Poincaré ⇝ A singular Yamabe problem

Recall a **conformal compactification** of a complete Riemannian manifold (M^{n+1}, g_+) is a manifold $\overline{M}$ with boundary ∂M s.t.:

- $\exists\ \overline{g}$ on $\overline{M}$, with $g_+ = r^{-2}\overline{g}$, where
- r a **defining function** for ∂M: $\partial M = \mathcal{Z}(r)$ & $dr_p \neq 0\ \forall p \in \partial M$.



$\Rightarrow$ canonically a conformal structure on boundary: $(\partial M, [\overline{g}|_{\partial M}])$.

Question/variant: Given $\overline{g}$ (or really $\mathbf{c} = [\overline{g}]$) can we find a defining function $r \in C^\infty(\overline{M})$ for $\Sigma = \partial M$ s.t.

$\mathrm{Sc}(r^{-2}\overline{g}) = -n(n+1)$?   NB: This satisfied for Poincaré-Einstein

cf. Loewner-Nirenberg, Aviles and McOwen – related interior problems.

The obstruction density of ACF

Can we solve $\mathrm{Sc}(r^{-2}\overline{g}) = -n(n+1)$? formally (i.e. power series) along the boundary? **Answer: No** - in general can get:

Theorem (Andersson, Chruściel, & Friedrich)

$$\mathrm{Sc}(r^{-2}\overline{g}) = -n(n+1) + r^{n+1}\mathcal{B}_n.$$

Furthermore (they show)

$$\mathcal{B}_2 = \delta \cdot \delta \cdot \mathring{L} + \textit{lower order}$$

is a conformal invariant of $\Sigma^2 = \partial M$.

Theorem.[G. + Waldron] For $n \geq 2$ $\mathcal{B}_n$ is a conformal invariant of $\Sigma = \partial M$, and $\mathcal{B}_2 =$ **Willmore Invariant** $= \overline{\Delta} H +$ lower order!

•For n even the invariant $\mathcal{B}_n$ is **higher order analogue** of $\mathcal{B}_2 = \mathcal{B}$.

NB. The existence of such a higher analogue was not previously obvious as the weight and leading order of $\mathcal{B}_n$ means standard tractor/ambient metric approaches fail.

Recasting the problem and holography

Recall the constant scalar curvature condition in terms of scale. A conformal manifold has a canonical conformal metric $\boldsymbol{g} \in S^2T^*M[2]$. A metric $g_+ \in \mathbf{c}$ is equivalent to a **scale**:

$$\boxed{g_+ = \sigma^{-2}\boldsymbol{g}} \quad \Leftrightarrow \quad \boxed{\sigma \in \Gamma(\mathcal{E}_+[1]).}$$

Via the Thomas-D operator $\dot{D} = \frac{1}{d}D$ the scale is equivalent to the

$$\boxed{\textbf{scale tractor} \quad I_A := \dot{D}_A\sigma,} \qquad \text{and}$$

Lemma

$\mathrm{Sc}(g_+) = -n(n+1) \Leftrightarrow I^2 := h(I, I) = 1$

So we come to a "conformal Eikonal equation" $(\dot{D}_A\sigma)(\dot{D}^A\sigma) = 1$, where σ a **defining density** for Σ. **NB:**

- If $\boxed{\text{we could solve uniquely then } \Sigma \hookrightarrow (M, \mathbf{c}) \textbf{ determines } g \in \mathbf{c}}$.

Then invariants of conf. compact (M, g_+) would be invariants of Σ.

The conformal Eikonal equation

Thus to solve the singular Yamabe problem formally we come to the following **non-linear** problem:

Problem: For a conformal manifold $(M, \mathbf{c})$ and an embedding $\iota : \Sigma \to M$ solve

$$I_A I^A = (\dot{D}_A \sigma)(\dot{D}^A \sigma) = 1 + O(\sigma^\ell)$$

for ℓ as high as possible, and σ a Σ defining density.

A key observation is that the **linearisation** of $I^A I_A = 1$ is $I^A D_A \dot{\sigma} = 0$ – the $I \cdot D$ problem on $\mathcal{E}[1]$. Thus $\exists$ hope that the $\mathfrak{sl}(2)$ generated by $x := \sigma$, $y := -\frac{1}{I^2} I^A D_A$ will again be useful.

Recall from the standard $\mathfrak{sl}(2)$ identities we have

$$[I \cdot D, \sigma^{k+1}] = I^2 \sigma^k (k+1)(d + k + 2\boldsymbol{w}),$$

and this allows an inductive solution (using also **other tractor identities**) that mimics the linear case!

Lemma

Suppose that $\underline{\sigma \in \Gamma(\mathcal{E}[1]) \text{ defines } \Sigma = \partial M_+ \text{ in } (\overline{M}, \mathbf{c})}$ *and*

$$I_\sigma^2 = 1 + \sigma^k A_k \quad where \quad A_k \in \Gamma(\mathcal{E}[-k])$$

is smooth on M, *and* $k \geq 1$, *then*

- *if* $k \neq (n+1)$ *then* $\exists\ f_k \in \Gamma(\mathcal{E}[-k])$ *s.t.* $\sigma' := \sigma + \sigma^{k+1} f_k$ *satisfies* $I_{\sigma'}^2 = 1 + \sigma^{k+1} A_{k+1}$, *where* A_{k+1} *smooth*;
- *if* $k = (n+1)$ *then:* $I_{\sigma'}^2 = I_\sigma^2 + O(\sigma^{n+2})$.

Proof.

Squaring with the tractor metric, using the $\mathfrak{sl}(2)$, etc

$$\begin{aligned}
(\dot{D}\sigma')^2 &= (\dot{D}\sigma + \dot{D}(\sigma^{k+1} f_k))^2 \\
&= I_\sigma^2 + \frac{2}{n+1} I_\sigma \cdot D(\sigma^{k+1} f_k) + (\dot{D}(\sigma^{k+1} f_k))^2 \\
&= 1 + \sigma^k A_k + \frac{2\sigma^k}{n+1}(k+1)(n+1-k) f_k + O(\sigma^{k+1}).
\end{aligned}$$

$\square$

The distinguished defining density and obstruction density

Theorem (G.-, Waldron arXiv:1506.02723 = CAG '21)

For Σ^n embedded in $(M^{n+1}, \mathbf{c})$ there is a distinguished defining density $\bar{\sigma}$, **unique** *modulo $+O(\sigma^{n+2})$, s.t.*

$$I^2_{\bar{\sigma}} = 1 + \bar{\sigma}^{n+1}\mathcal{B}_{\bar{\sigma}}.$$

Moreover:

$$\mathcal{B} := \mathcal{B}_{\bar{\sigma}}|_\Sigma \in \Gamma(\mathcal{E}_\Sigma[-n-1])$$

is determined by $(M, \mathbf{c}, \Sigma)$ and is a **natural conformal invariant**.

For n even $\mathcal{B} = 0$ **generalises the Willmore equation** *in that:*

$$\mathcal{B} = \bar{\Delta}^{\frac{n}{2}} H + \textit{lower order terms};$$

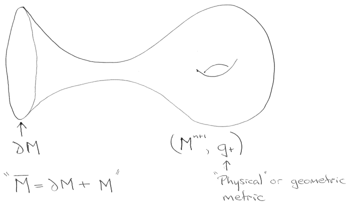
while for n odd $\mathcal{B}$ has no linear leading term.

Corollary (ACF + above implies)

On a closed (M, g) if there is a sign changing smooth solution of sing. Yamabe: $|du|^2 - \frac{2}{n+1} u \left(\Delta^g + \frac{\mathrm{Sc}^g}{2n}\right) u = 1$ then $\Sigma := \mathcal{Z}(u)$ is a higher Willmore hypersurface – i.e. it satisfies $\mathcal{B} = 0$. –
$\mathrm{Sc}^{u^{-2}g} = -n(n+1)$ *eqn*

It turns out that the **obstruction density** $\mathcal{B}$ is the variation, with respect to variation of embedding, of an "energy" or "action". To see where this comes from we needf to study further subtle invariants.

Conformally compact Riemannian manifolds have **infinite volume**



But we can **regularise** by cutting away the part within ϵ of ∂M – (according to some coordinate) and leaving $\overline{M}_\epsilon$

$\mathcal{B}$ is variational

For suitable regularisations $\overline{M}_\epsilon$ of conformally compact manifolds $\overline{M}$:

$$\mathrm{Vol}_\epsilon = \int_{\overline{M}_\epsilon} \sqrt{g_+} = \frac{v_n}{\epsilon^n} + \cdots + \frac{v_1}{\epsilon} + \mathcal{A} \log \epsilon + V_{ren} + O(\epsilon).$$

Theorem (Graham 2016: PAMS 2017, arXiv:1606.00069)

If $g_+ = \bar{\sigma}^{-2}\mathbf{g}$, where $\bar{\sigma}$ an approximate solution of the sing. Yamabe problem then $\mathcal{A}$ a conformal invariant of $\Sigma \hookrightarrow M$ and

$$\frac{\delta \mathcal{A}}{\delta \Sigma} = \frac{(n+1)(n-1)}{2}\mathcal{B}$$

So the anomaly term in the renormalised volume expansion provides an **energy** with **functional gradient the obstruction density**, in other words $\mathcal{A}$ is an energy generalising the Willmore energy.

Extrinsic Q-curvature and the anomaly

In fact – also in analogy with the treatment of Poincaré-Einstein manifolds – there is nice local quantity giving the anomaly:

Theorem (G.- Waldron, CMP 2017, arXiv:1603.07367)

$$\mathcal{A} = \frac{1}{n!(n-1)!} \int_\Sigma Q$$

where, with $\tau \in \Gamma\mathcal{E}_+[1]$ *a scale giving the boundary metric,* $Q := (-I \cdot D)^n \log \tau$.

- Q here is an **extrinsically coupled** Q**-curvature** meaning e.g.

$$Q^{\widehat{g}_\Sigma} = e^{-nf}(Q^{g_\Sigma} + P_n f) \quad \text{where} \quad \widehat{g}_\Sigma = e^{2f} g_\Sigma$$

and for n even

$$P_n = \Delta_\Sigma^{\frac{n}{2}} + \text{lower order terms}; \ \ P_n \text{ FSA, and } P_n 1 = 0,$$

is an **extrinsically coupled** GJMS type operator. Q and P_n are from G.-, Waldron arXiv:1104.2991 = Indiana U.M.J. 2014.

Idea of proof

Use a Heaviside function θ to "cut off" an integral over all $\overline{M}$

$$\mathrm{Vol}_\epsilon = \int_{\overline{M}} \frac{dV^{g_\tau}}{\sigma^{n+1}} \theta(\frac{\sigma}{\tau} - \epsilon).$$

Then the divergent terms and anomaly are given by

$$v_k \sim \frac{d^{n-k}}{d\epsilon^{n-k}} \left(\epsilon^{n+1} \frac{d}{d\epsilon} \mathrm{Vol}_\epsilon \right) \Big|_{\epsilon=0},$$

So

$$v_k \sim \int_{\overline{M}} \frac{\delta^{n-k}(\sigma)}{\tau^k} \quad \text{and} \quad \mathcal{A} \sim \int_{\overline{M}} \delta^{n-1}(\sigma) I \cdot D \log \tau$$

Then via identities, and the $sl(2)$ again

$$v_k \sim \int_\Sigma (I \cdot D)^{n-k} \frac{1}{\tau^k} \quad \text{and} \quad \mathcal{A} \sim \int_\Sigma (I \cdot D)^n \log \tau$$

Further Invariants by conformal holography.

Recall:

Theorem (G.-, Waldron arXiv:1506.02723 = CAG '21)

For Σ^n embedded in $(M^{n+1}, \boldsymbol{c})$ there is a distinguished defining density $\bar{\sigma}$, **unique** *modulo $+O(\sigma^{n+2})$, s.t.*

$$I^2_{\bar{\sigma}} = 1 + \bar{\sigma}^{n+1}\mathcal{B}_{\bar{\sigma}}.$$

Moreover:

$$\mathcal{B} := \mathcal{B}_{\bar{\sigma}}|_{\Sigma} \in \Gamma(\mathcal{E}_{\Sigma}[-n-1])$$

is a natural invariant $\cdots$
Etcetera

Corollary (above implies)

A $(\overline{M}, \boldsymbol{c})$ has a **canonical** *conformally compact structure up to $+O(\sigma^d)$.*

Part II. All hypersurface invariants via holography?

The construction can be used to obtain other hypersurface invariants: Our Theorem above shows that:

$$(M, \mathbf{c}, \Sigma) \quad \text{determines} \quad \bar{\sigma} \quad \text{modulo} + O(\sigma^{n+2}).$$

Suppose that $\mathcal{I}$ is any coupled conformal invariant of $(M, \mathbf{c}, \bar{\sigma})$ involving only the jet $j^{n+1}\bar{\sigma}$. Then along Σ

$$\boxed{\mathcal{I}\big|_{\Sigma} \quad \text{is a conformal invariant of} \quad (M, \mathbf{c}, \Sigma).}$$

This **holographic** approach fails at order $n+2$ when because of the existence of the **obstruction invariant** $\mathcal{B}$ and ambiguity. This is an analogue of the use Fefferman-Graham's Poincaré and ambient metric constructions to find conformal invariants – that fails at order $n+1$ because of **Bach** B_{ab} in dimension 4 and the **Fefferman-Graham obstruction tensor** in higher even dimensions.

The obstructions to Poincaré-Einstein (PE)

Conformally compact manifolds are often assumed to be PE, or asymptotically PE. Often for simplicity. But what does it mean?

Given a conformal manifold with boundary $(\overline{M}, \mathbf{c})$ does it admit a smooth PE metric with ∂M the conformal infinity?

Forgetting the boundary and global, in general there are local obstructions to $\nabla I = 0$ on M. E.g. this will obviously fail if the tractor curvature has max. rank: G. +Nurowksi, G&Phys (2006)

Given a conformal manifold with boundary $(\overline{M}, \mathbf{c})$ does it admit a smooth **asymptotically** PE metric with ∂M the conformal infinity?

It turns out that the trace free second fundamental form $\mathring{\mathrm{II}} := \mathring{L}$ is the first obstruction. At the next order the tf Fialkow tensor is the next obstruction:

$$\mathcal{F}_{ab} = \frac{1}{d-3}(W_{acbd} n^c n^d + \mathring{L}^2_{ab})$$

Both of these were seen above as consequences of $\nabla I = 0$ along ∂M. How do we systematically find the higher order obstructions? There's a very nice answer!!

The almost Einstein tensor E_{ab}

In a PE manifold $(\overline{M} = M \cup \partial M, g^o)$ the Schouten tensor of the metric satisfies

$$P^{g^o}_{ab} = \lambda g^o_{ab} \qquad \text{on} \quad M \tag{10}$$

But both g^o and P^{g^o} are singular at ∂M. HOWEVER given $\sigma \in \Gamma(\mathcal{E}[1])$ the quantity

$$\text{Trace-Free}(\nabla^g_a \nabla^g_b \sigma + \sigma P^g_{ab}) \qquad g \in \mathbf{c},$$

is conformally invariant.
If: 1. $Z(\sigma) = \partial M$, 2. $I^2 = \pm 1 + \sigma^d B$ (B smooth), then

$$\boxed{E_{ab} := \text{Trace-Free}(\nabla^g_a \nabla^g_b \sigma + \sigma P^g_{ab})} \qquad \text{is determined by } (\overline{M}, \mathbf{c})$$

up to $+O(\sigma^n)$. On the interior $\boxed{E_{ab} = \sigma P^{g^o}_{ab}}$ where $g^o := \boldsymbol{g}/\sigma^2$.

So E_{ab} extends $\sigma P^{g^o}_{ab}$ smoothly to the boundary – it vanishes iff g^o is a PE metric.

Summarising:

• $E_{ab} :=$ Trace-Free$(\nabla^g_a \nabla^g_b \sigma + \sigma P^g_{ab})$ extends $\sigma P^{g^o}_{ab}$ smoothly to the boundary.

• E_{ab} vanishes iff g^o is a PE metric.

• E_{ab} depends only on the conformal embedding $\partial M \hookrightarrow \overline{M}, \mathbf{c})$, up to $+O(\sigma^n)$. Thus

Lemma: "The jets of E_{ab} along ∂M are extrinsic hypersurface invariants that obstruct the existence of PE metrics in $\mathbf{c}|_M$".

E.g. zero jet:

Proposition: $E_{ab}|_{\partial M} = \mathring{\mathrm{II}}_{ab}$.

Proof: 1. σ SY means $I_A|_{\partial M} = N_A$ and $n_a := \nabla_a \sigma$ is a weight 1 unit conormal. Then 2. differentiating $I^2 = 1 + O(\sigma^{n+1})$ gives

$$N^B \nabla_b I_B = 0 \quad \& \quad n^b \nabla_b I_A = 0 \qquad \text{along } \partial M.$$

So

$$\begin{pmatrix} 0 \\ E_{ab} \\ * \end{pmatrix} = \nabla_a I_B \overset{\partial M}{=} \underline{\nabla}_a N_B \overset{\text{see L1}}{=} \begin{pmatrix} 0 \\ \mathring{\mathrm{II}}_{ab} \\ * \end{pmatrix}$$

Higher Fundamental Forms

So $\boxed{\mathring{\mathrm{II}} \text{ is an obstruction to Poincaré-Einstein (PE).}}$

Next recall the Cherrier-Robin operator from L1:

$$\delta_1 \stackrel{g}{:=} n^a \nabla_a - wH^g : \Gamma(\mathcal{T}^\Phi[w]) \to \Gamma(\mathcal{T}^\Phi[w-1]|_{\partial M})$$

where $\mathcal{T}^\Phi[w]$ means any weight w tractor bundle – or simply densities of that weight.

There's a version for rank 2 trace-free symmetric tensors of weight w. And

$$\delta_1 E_{ab} = n^c n^d W_{cabd} - \mathring{\mathrm{II}}^2_{(ab)\circ} = -(d-3)\mathring{\mathcal{F}}_{ab} \in \Gamma(S^2_o T^* \partial M)$$

where $\mathcal{F}$ is the Fialkow tensor and $\mathring{\mathrm{II}}^2$ is the obvious composition of II_0 with itself. Note that $J^1_{\partial M} E_{ab}$ is captured by the two extrinsic invariants $\mathring{\mathrm{II}}$ (which gets $J^0_{\partial M} E$) and $\delta_1 E_{ab}$. So

$\boxed{\mathring{\mathrm{III}} := \delta_1 E_{ab} \text{ is an obstruction to PE.}}$

Making higher fundamental forms

To make higher order analogues of $\mathring{\mathrm{II}}$ and $\mathring{\mathrm{III}}$ we need higher analogues of the operator δ_1.

STEP 1: We want E in a tractor quantity.

$$P_{AB} := \dot{D}_A I_B = \dot{D}_A \dot{D}_B \sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & E_{ab} & * \\ 0 & * & * \end{pmatrix}$$

does this.

STEP 2: We can then form e.g.

$$\delta_1 (I \cdot D)^{K-1} P_{AB} \quad \text{or better} \quad \delta_K P_{AB}$$

where δ_K are the conformal higher Neumann operators from L1.

STEP 3: Actually STEP 2 needs a lot of refining to extract a symmetric trace-free tensor again . . . see Blitz, G. Waldron arXiv:2107.10381

Theorem BGW: Let $d \geq 3$ and let $2 \leq K < \frac{d+3}{2}$. For any embedded hypersurface Σ in a conformal d manifols there is a well-defined *canonical Kth fundamental form* $\mathring{\underline{\overline{\mathrm{K}}}}$ is defined by

$$\mathring{\underline{\overline{\mathrm{K}}}} := \delta_{(K-2)} E.$$

• Each K^{th}-fundamental form is an extrinsic hypersurface conformal invariant that depends, along Σ, on $K-1$ transverse derivatives of the ambient conformal structure $\boldsymbol{c}$.

• Each $\mathring{\underline{\overline{\mathrm{K}}}}$ is an obstruction to the existence of an asymptotically PE $g_+ \in \boldsymbol{c}$.

Next:

• If $\mathring{\mathrm{II}}$, $\mathring{\mathrm{III}}$, $\cdots$, $\mathring{\underline{\overline{\lfloor \frac{d+3}{2} \rfloor}}}$ vanish then we can define higher fundamental forms to $K = n = d-1$ and :

Theorem BGW If $\mathring{\mathrm{II}}$, $\mathring{\mathrm{III}}$, $\cdots$ $\mathring{\underline{\overline{d-1}}}$ vanish, then

$$g_+ = \boldsymbol{g}/\sigma^2$$

is asymptotically PE meaning $E = O(\sigma^{n-1})$

THE END !!

Extrinsically coupled GJMS operators

Recall on any almost Riemannian manifold (M, c, I) we had:

Theorem

Let $\mathcal{E}^\Phi$ be any tractor bundle and $k \in \mathbb{Z}_{\geq 1}$. Then, for each $k \in \mathbb{Z}_{\geq 1}$, along $\Sigma = \mathcal{Z}(\sigma)$

$$P_k^\sigma : \mathcal{E}^\Phi[\frac{k-n}{2}] \to \mathcal{E}^\Phi[\frac{-k-n}{2}] \quad \textit{given by} \quad P_k^\sigma := \Big(-\frac{1}{I^2} I\cdot D \Big)^k$$

is a tangential differential operator, and so determines a canonical differential operator $P_k^\sigma : \mathcal{E}^\Phi[\frac{k-n}{2}]|_\Sigma \to \mathcal{E}^\Phi[\frac{-k-n}{2}]|_\Sigma$. For k even this takes the form

$$P_k = \overline{\Delta}^k + \textit{lower order terms.}$$

Because $(M, \mathbf{c}, \Sigma)$ determines $\bar{\sigma}$ modulo $+ O(\sigma^{n+2})$, we have:

Theorem

For $k \leq n = d-1$ the operators P_k are determined canonically by the data $(M, \mathbf{c}, \Sigma)$.

ADD

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