Conformal differential geometry: the ambient metric

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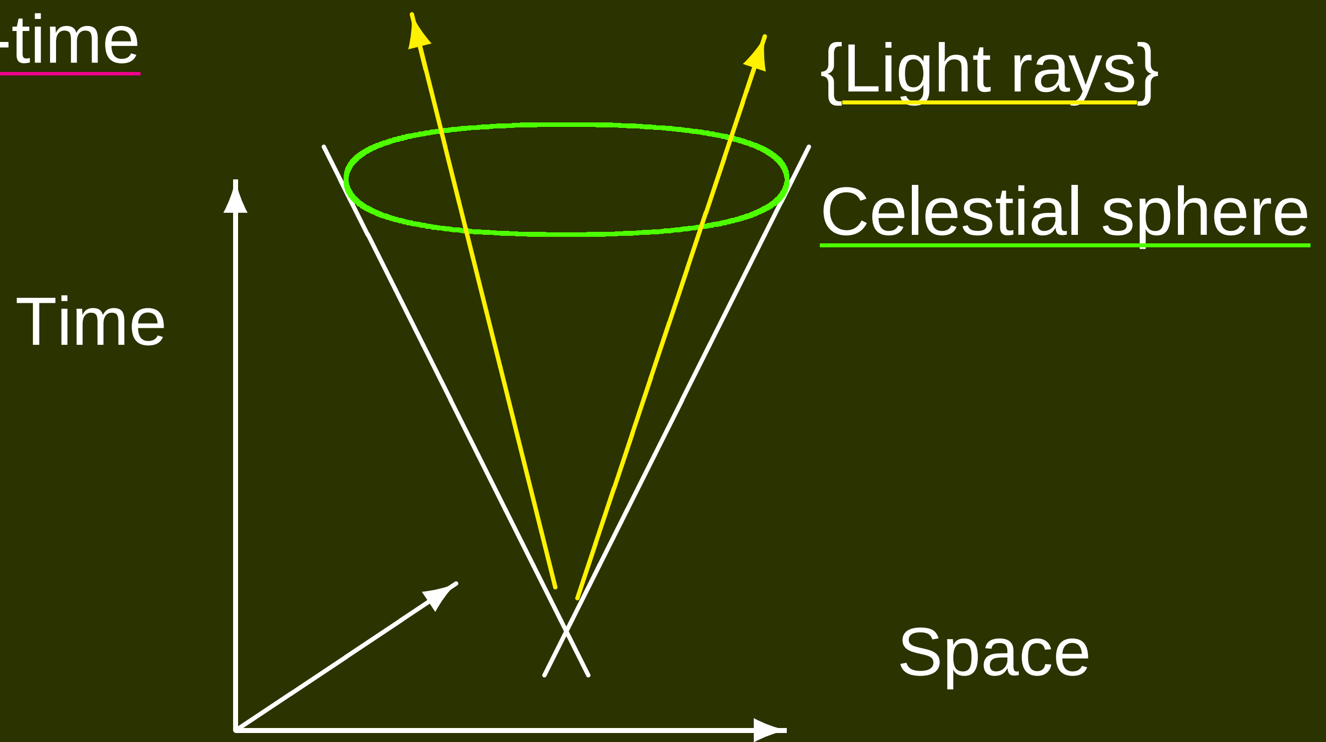
Minkowski space

R. Penrose (1959)

The apparent shape of a relativistically moving sphere

Space-time

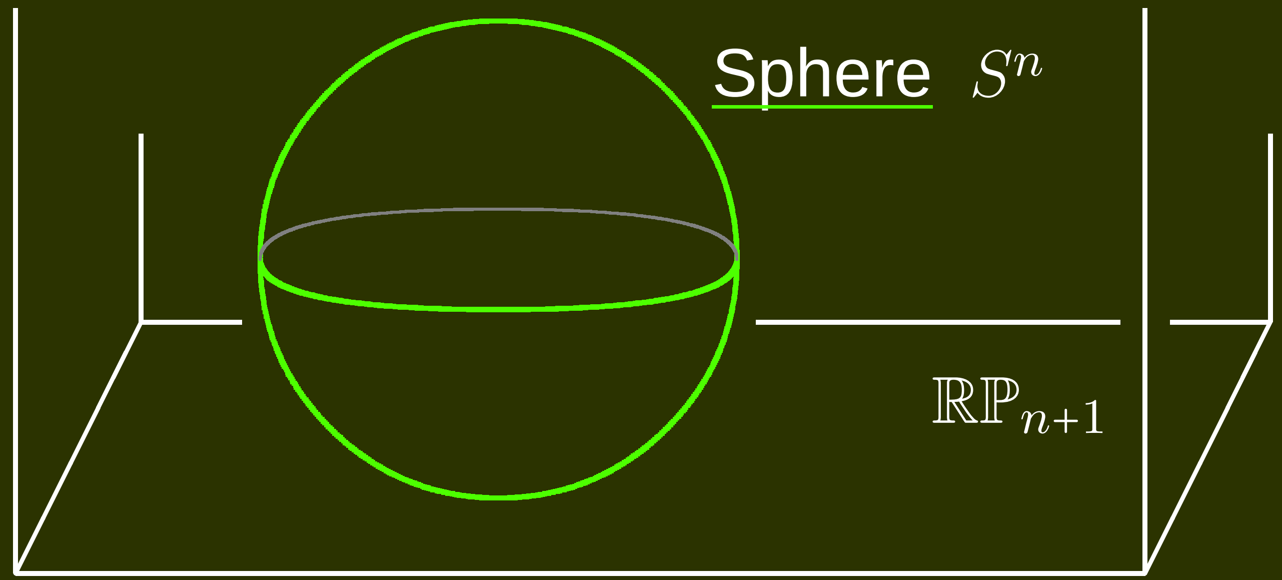


Lorentz group $\mathrm{SO}(3,1)$

acting (conformally) on the celestial sphere!

Hyperbolic space

E. Beltrami (1868)
Teoria fondamentale degli spazii di curvatura costante



Symmetry group $= \mathrm{SO}^{\uparrow}(n+1,1)$

- conformal motions of the n-sphere (J. Liouville (1850))
- Riemannian motions of hyperbolic (n+1)-space

!!

Holography

P.A.M. Dirac (1935)

The electron wave equation in de-Sitter space

$$\underbrace{\widetilde{g}_{AB}x^A x^B}_{\displaystyle \overset{\text{|||}}{r}} \equiv 2x^0 x^\infty + g_{ab}x^a x^b \quad \text{for} \quad x^A = (x^0, x^a, x^\infty) \in \mathbb{R}^{n+2}$$

$$\widetilde{\Delta} \equiv \widetilde{g}^{AB} \frac{\partial^2}{\partial x^A \partial x^B}$$ ambient wave operator

$$f(\lambda x^A) = \lambda^w f(x^A) \ \forall \lambda > 0$$ homogeneous of degree w

$$f(x^A) \text{ on } \{r = 0\} \longmapsto \widetilde{f}(x^A) \text{ near } \{r = 0\}$$

$$\widetilde{f} \rightsquigarrow \widetilde{f} + rg \qquad \widetilde{\Delta}(rg) = r\widetilde{\Delta}(g) + 2(n + 2w - 2)g$$

vanishes if $w = 1 - n/2$

$\therefore f \mapsto \widetilde{f} \mapsto (\widetilde{\Delta}\widetilde{f})|_{\{r=0\}}$ is well-defined: conformal Laplacian!!

Applications

L.P. Hughston and T.R. Hurd (1983)
A CP^5 calculus for space-time fields

twistor theory !

M.G.E. and C.R. Graham (1991)
Invariants of conformal densities

T.N. Bailey, M.G.E., and C.R. Graham (1994)
Invariant theory for conformal and CR geometry

J. Maldecena (1998)
The large N limit of superconformal field theories and supergravity

E. Witten (1998)
Anti de Sitter space and holography

M.G.E. (2005)
Higher symmetries of the Laplacian

The ambient metric

C. Fefferman (1974)
The Bergman kernel and biholomorphic mappings of pseudoconvex domains

C. Fefferman (1976)
Monge-Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains

C. Fefferman (1979)
Parabolic invariant theory in complex analysis

C. Fefferman and C.R. Graham (1985)
Conformal invariants (22 pages)

C. Fefferman and C.R. Graham (2012)
The ambient metric (111 pages)

$\Big\} \star\star\star$

A remarkable calculation

Flat Lorentzian metric (again)

$$2\rho\, dt^2 + 2t\, dt\, d\rho + t^2 g_{ab}\, dx^a dx^b \quad \text{for} \quad (t, x^a, \rho) \in \mathbb{R}^{n+2}$$

Curved version

$$2\rho\, dt^2 + 2t\, dt\, d\rho + t^2 g_{ab}(x,\rho)\, dx^a dx^b \qquad (\heartsuit\heartsuit\heartsuit)$$

- $g_{ab}(x, 0) = g_{ab}(x)$ is a given initial metric
- insist that ($\heartsuit\heartsuit\heartsuit$) be Ricci flat

'Straightforward but tedious' calculation

$$g_{ab}(x,\rho) = g_{ab}(x) + 2\mathrm{P}_{ab}(x)\rho + \mathrm{O}(\rho^2),$$

where $\mathrm{P}_{ab} \equiv \frac{1}{n-2}\left(R_{ab} - \frac{1}{2(n-1)} R g_{ab}\right)$ = Schouten tensor !!

Higher order terms

Try $g_{ab}(x,\rho) = g_{ab} + 2\mathrm{P}_{ab}\,\rho + \mathrm{P}_a{}^c\mathrm{P}_{bc}\,\rho^2$

The Lorentzian metric

$$2\rho\, dt^2 + 2t\, dt\, d\rho + t^2 g_{ab}(x,\rho)\, dx^a dx^b$$

- is flat if g_{ab} is conformally flat ☺☺☺
- is Ricci-flat if g_{ab} is Einstein ☕☕☕

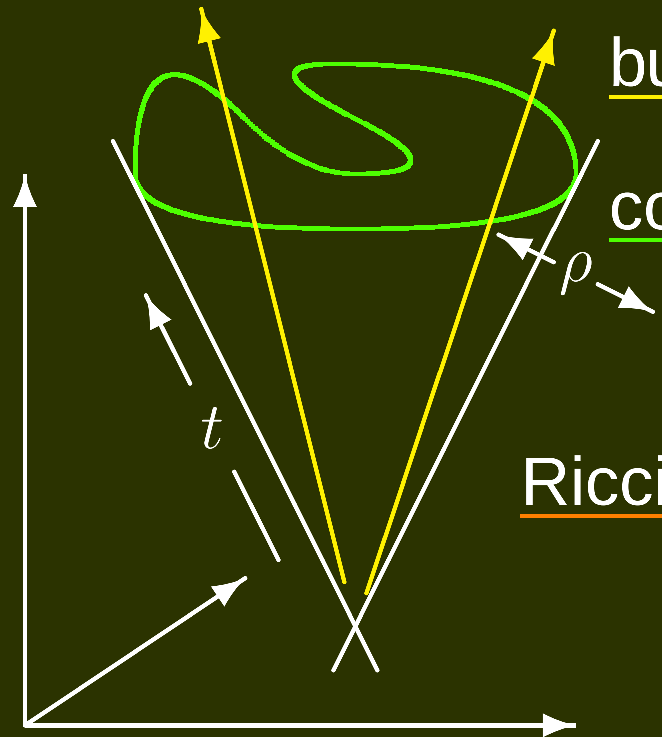
More generally, to force Ricci-flatness,

$$g_{ab}(x,\rho) = g_{ab} + 2\mathrm{P}_{ab}\,\rho + \left[\mathrm{P}_a{}^c\mathrm{P}_{bc} - \tfrac{1}{n-4}B_{ab}\right]\rho^2 + \mathrm{O}(\rho^3),$$

$$B_{ab} \equiv \nabla_c\nabla^c\mathrm{P}_{ab} - \nabla_a\nabla_b\mathrm{P}_c{}^c - 2W_{abcd}\mathrm{P}^{cd} + \mathrm{P}^{cd}\mathrm{P}_{cd}g_{ab} - n\mathrm{P}_a{}^c\mathrm{P}_{bc}$$

NB $n = 4$ is special $\qquad B_{ab} =$ Bach tensor!

Curved conformal geometry



- odd dimensions: no worries ☺☺☺
- even dimensions: worrying yet interesting ☠☠☠☺☺☺

NB Conformal invariance $\longleftarrow$ ϟϟwith careϟϟ

Applications

In all dimensions

- Conformal Laplacian (aka Yamabe operator)

$$f \mapsto \left(\Delta - \frac{n-2}{4(n-1)} R \right) f \quad \text{for } f \text{ of conformal weight } 1 - \frac{n}{2}$$

- The Cartan connection (Thomas's tractor bundle)

  next time

In even dimensions !!!

- Fefferman-Graham obstruction tensor (cf. Bach tensor)
- Branson's Q-curvature
- GJMS-operators

Dimension four

R.J. Riegert (1984)
A nonlocal action for the trace anomaly

T.P. Branson (1993)
The functional determinant

$$Q \equiv \left[-\tfrac{1}{6}\Delta R - \tfrac{1}{2}R^{ab}R_{ab} + \tfrac{1}{6}R^2\right]\, d\mathrm{vol}$$

$$\widehat{g}_{ab} = e^{2f} g_{ab} \quad \Rightarrow \quad \boxed{\widehat{Q} = Q + Pf}$$

$$Pf = \nabla_a[\nabla^a\nabla^b + 2R^{ab} - \tfrac{2}{3}Rg^{ab}]\nabla_b f$$ conformally invariant

$$\int_M Q = 8\pi^2\chi(M) - \frac{1}{4}\int_M \|W\|^2\, d\mathrm{vol}$$ conformally invariant

C.R. Graham, R. Jenne, L.J. Mason, and G.A.J. Sparling (1992)
Conformally invariant powers of the Laplacian I: Existence

END OF PART ONE

THANK YOU