Lie equations, Cartan bundles, Tanaka theory and differential invariants (II)

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Pseudogroups and Differential Invariants

Let $\mathcal{G}$ be a Lie pseudogroup (group if finite-dim) acting on a differential equation $\mathcal{E}$ by symmetries. In particular, $\mathcal{G}$ can be the gauge group of $\mathcal{E}$. The goal is to understand the quotient

$$\bar{\mathcal{E}} = \mathcal{E}/\mathcal{G},$$

in particular to compute its dimensional characteristics.

For this we pass to (absolute scalar) differential invariants, i.e. functions constant on (global) orbits of the action. If the group is algebraic and transitive on M (the base of $\mathcal{E}$) then the invarinats can (and will) be assumed rational in jets.

Denoting by $\mathfrak{g}$ the Lie algebra (sheaf) corresponding to $\mathcal{G}$, the algebra of differential invarians $\mathcal{A} = \cup_k \mathcal{A}_k$ (k is the jet-order; functions $I \in \mathcal{A}_k$ assumed polynomial in jets of order $k \geq k_0$) is:

$$I \in \mathcal{A} \;\Leftrightarrow\; \varphi^* I = I \;\forall \varphi \in \mathcal{G} \;\overset{\text{con}}{\Leftrightarrow}\; L_X I = 0 \;\forall X \in \mathfrak{g}.$$

Lie-Tresse approach

Hilbert's nullstellensatz makes a bijective correspondence between algebraic varieties and ideals of functions vanishing on them. Similarly, there is a bijection between algebraic differential equations and differential ideals, defining them.

While straightforward generalization of tools, like Gröbner basis, may lead to uncomputable objects (infinite size), the CK theorem guarantees finite generation of regular systems $\mathcal{E}$ of PDEs.

D-module structure for $\mathcal{A}$ is given by invariant derivarions, i.e. vector fields on $\mathcal{E}^{(\infty)}$ $\equiv$ first ord operators ∇ in total derivatives:

$$\varphi^* \circ \nabla = \nabla \circ \varphi^* \ \forall \varphi \in \mathcal{G}.$$

Example (Invariants of curves on the plane)

$M = \mathbb{R}^2(x,y)$, $\mathcal{E} = J^\infty(M,1) \overset{\text{loc}}{\simeq} J^\infty(\mathbb{R},\mathbb{R})$, $G = SO(2) \ltimes \mathbb{R}^2$.

$$k = \frac{y_{xx}}{(1+y_x^2)^{3/2}},\ \frac{d}{ds} = \frac{1}{\sqrt{1+y_x^2}}\frac{d}{dx} \ \Rightarrow\ \mathcal{A} = \left\langle K = k^2, \nabla = k\frac{d}{ds} \right\rangle$$

Global version: Finiteness

Consider an algebraic action of a pseudogroup $\mathcal{G}$ on a formally integrable irreducible differential equation $\mathcal{E}$ over $M = J^0$ such that $\mathcal{G}$ acts transitively on M.

Theorem (BK & V.Lychagin "Global Lie-Tresse" 2016)

There exists a number l and a Zariski closed invariant proper subset $\mathcal{S}_l \subset \mathcal{E}^l$ such that the algebra $\mathcal{A}$ of differential invariants on $\mathcal{E}^\infty$ separates orbits from $\mathcal{E}^\infty \setminus \pi_{\infty,l}^{-1}(\mathcal{S}_l)$ and is finitely generated in the following sense.
There exists a finite number of rational differential invariants $I_1, \dots, I_t \in \mathcal{A}$ and a finite number of rational invariant derivations $\nabla_1, \dots, \nabla_s : \mathcal{A} \to \mathcal{A}$ such that any function from $\mathcal{A}$ is a polynomial of differential invariants $\nabla_J I_i$, where $\nabla_J = \nabla_{j_1} \cdots \nabla_{j_r}$ for some multi-indices $J = (j_1, \dots, j_r)$, with coefficients being rational functions of I_i.

Solution to the equivalence problem

In addition to the above, the orbits of $\mathcal{G}^k$ on $\mathcal{E}^k \setminus \pi_{k,l}^{-1}(S_l)$ are regular, i.e. closed, have the same dimension and algebraically fiber the space. In other words, for $k \geq l$ there exists a rational geometric quotient

$$\big(\mathcal{E}^k \setminus \pi_{k,l}^{-1}(\mathcal{S}_l)\big)/\mathcal{G}^k \simeq \mathcal{Q}_k.$$

The projective limit $\mathcal{Q} = \lim\limits_{\infty \leftarrow k} \mathcal{Q}_k$ has the structure of diffiety, i.e. locally: differential equation. This means that there are finitely many D-relations between the generators (I_i, ∇_J), interpreted as invariant differential equations, which generate all diff syzygies.

Solutions u to $\mathcal{E}$, with graphs outside $\mathcal{S}$, are separated so: Consider the generting set $\mathfrak{I} = (I_i, \nabla_j I_i)$ of cardinality N. Denote $M_u = (j_\infty u)(M) \subset J^\infty$, and call signature of u the restriction

$$\mathfrak{I}_u = \mathfrak{I}|_{M_u}.$$

Then u_1 and u_2 are $\mathcal{G}$-equivalent iff their signatures $\mathfrak{I}_{u_1}$ and $\mathfrak{I}_{u_2}$ coincide as unparametrized submanifolds in $\mathbb{R}^N$.

Counting the invariants

Number of differential invariants of order $\leq k$ is $s_k = \dim \mathcal{Y}_k$. The difference $h_k = s_k - s_{k-1}$ is the number of "pure order" k differential invariants. The Poincaré function is defined by

$$P(z) = \sum_{k=0}^{\infty} h_k z^k.$$

V.Arnold conjectured $P(z)$ to be rational. This was verified for $\mathcal{G} = \mathrm{Diff}_{\mathsf{loc}}(M)$ by R.Sarkisyan and for general pseudogroups satisfying GLT by BK and V.Lychagin.

Representing

$$P(z) = \frac{R(z)}{(1-z)^d},$$

for some polynomial $R(z)$, $R(1) \neq 0$, the functional dimension and functional rank for the quotient equation are expressed as follows:

$$d = -\lim_{z\to 1} \frac{\log P(z)}{\log(1-z)}, \qquad c = R(1) = \lim_{z\to 1} P(z)(1-z)^d.$$

Example: conformal structures

For conformal structures $[g]$ on M^n one has:

$$P(z) = \begin{cases} \frac{z^3(1+z)(1+5z-8z^2+3z^3)}{(1-z)^3}, & \text{for } n=3, \\ \frac{(n+1)nz-2(n+z)}{2z(1-z)^n} + \frac{n}{z} + \big(1+\binom{n}{2}+nz\big)(1-z^2), & \text{for } n>3. \end{cases}$$

This means in the case $n=3$:
$h_0=h_1=h_2=0$, $h_3=1$, $h_4=9$ and $h_k=k^2-4$ for $k\geq 5$.

In the case $n>3$:
$h_0=h_1=0$, $h_2=\frac{n^2(n^2-1)}{12}-n^2-1$,
$h_3=\frac{1}{24}n(n^4+2n^3-5n^2-14n-32)$ and for $k\geq 4$:
$h_k=(\binom{n+1}{2}-1)\cdot\binom{n+k-1}{k}-n\cdot\binom{n+k}{k+1}=\frac{n(k-1)}{2}\binom{n+k-1}{k+1}-\binom{n+k-1}{k}$.

In particular there is precisely 1 conformal invariant of order 3 in the case $n=3$ (from Cotton tensor) and 3 conformal invariants of order 2 in the case $n=4$ (from Weyl tensor).

Cartan approach to the equivalence problem

The Cartan method of constructing invariant coframes (with bookkeeping the freedom) starts with a geometric structure q recast as an EDS (first order PDE system) and it leads to the tower of bundles

$$M \leftarrow \mathcal{P}_0 \leftarrow \mathcal{P}_1 \leftarrow \dots$$

The starting construction of 0-frames is a principal bundle, but then the frames are principal (with abelian group) only seqentially $\mathcal{P}_k \to \mathcal{P}_{k-1}$ but not as bundles over M.

For structures of finite type the number of steps is finite and the final bundle $\mathcal{P} \to M$ carries a canonical absolute parallelism $e_i \in \mathcal{D}(\mathcal{P})$, $i = 1, \dots, \dim \mathcal{P}$ (equivariant $\Rightarrow$ Cartan connection) allowing to solve the equivalence problem:

$$\{e_i, e_j\} = c_{ij}^k e_k \;\Rightarrow\; \text{Diff Invariants: } c_{ij}^k,\; \text{Inv Derivations: } e_k.$$

In general, c_{ij}^k are defined not on M but on $\mathcal{P}$, and are covariants. They can be pushed down to M only for Cartan connections.

Relation between the two approaches

The essential difference between the jet and Cartan approaches is where the invariants live. The following commutative diagram helps:

$$\begin{array}{ccc}
\mathcal{P} & \longleftarrow & \pi_\infty^*\mathcal{P} \\
\Big\downarrow{\scriptstyle \rho} & & \Big\downarrow \\
M & \underset{\pi_\infty}{\longleftarrow} & \mathcal{E}^\infty \subset J^\infty\pi
\end{array}$$

Initially the Cartan invariants are functions on

$$\pi_\infty^*\mathcal{P} = \{(\omega, q_\infty) \in \mathcal{P} \times \mathcal{E}_\infty \mid \rho(\omega) = \pi_\infty(q_\infty)\}$$

and they suffice to solve the equivalence problem.

Projecting the algebra of invariants on the Cartan bundle to the base we obtain the algebra of absolute differential invariants consisting of $\mathcal{G}$-invariant functions on $\mathcal{E}_\infty$. This is achieved either by normalization of the frame or by invariantization of the invariants on $\mathcal{P}$ with respect to the structure group.

Example: pseudo-Riemannian metrics (M, g)

The Lie equation $\mathrm{Lie}(g)$ is Frobenius after one prolongation: the algebraic Sternberg prolongation is $\mathfrak{so}(p,q)^{(1)} = 0$. $\mathrm{Lie}(g)$ is compatible iff g is flat. The invariants are derived from the curvature tensor and its covariant derivatives by contractions.

With the jet approach, the pseudo-group $\mathcal{G} = \mathrm{Diff}_{\mathsf{loc}}(M)$ is lifted to the bundle S^2T^*M and prolonged. The Poincaré function is

$$P(z) = \begin{cases} \frac{z^2(1-z+2z^2-z^3)}{(1-z)^2}, & \text{for } n = 2, \\ \frac{n}{z} + \binom{n}{2} \cdot (1-z^2) - \frac{1}{(1-z)^n} \cdot \left(\frac{n}{z} - \binom{n+1}{2}\right), & \text{for } n > 2. \end{cases}$$

(Singularity at $z = 0$ is inessential and is used for brevity.)

The Cartan approach builds an orthonormal frame bundle $\rho : \mathcal{P} \to M$ with the principal group $P = O(p,q)$, $p + q = n$. The structure relations are as follows

$$d\sigma = \omega \wedge \sigma, \quad d\omega + \omega \wedge \omega = R\sigma \wedge \sigma.$$

For non-flat g the compatibility $d^2 = 0$ reduces the structure group.

Nonholonomic structures

A sub-Riemannian structure (M, Δ, g) is a Riemannian metric g on a non-holonomic distribution Δ. In this case for $q = (\Delta, g)$ the equation $\mathcal{E} = \mathrm{Lie}(q)$ is also of finite type, however even in the most symmetric case its differential closure $\bar{\mathcal{E}}$ is computed through several prolongation-projection steps.

Example $(S^{2n+1} \subset \mathbb{C}^{n+1}, \Delta = TS^{2n+1} \cap J(TS^{2n+1}), g_{FS})$

For initial Lie equation $\mathcal{E} \subset J^1(TM)$ on $M = S^{2n+1}$ its 1-symbol is $\mathfrak{g} = \mathfrak{co}(2n) \ltimes \mathbb{R}^{2n} \subset \mathrm{End}(TM)$. For the prolongation-projection $\mathcal{E}' = \pi_{2,1}(\mathcal{E}^{(1)})$ its 1-symbol is $\mathfrak{g}' = \mathfrak{cu}(n) \ltimes \mathbb{R}^{2n}$. Note that both $\mathfrak{g}$ and $\mathfrak{g}'$ have infinite type (nontrivial characteristics). Next prolongation-projection $\mathcal{E}'' = \pi_{2,1}(\mathcal{E}'^{(1)})$ has 1-symbol $\mathfrak{g}'' = \mathfrak{u}(n)$, and this $\mathcal{E}''$ is formally integrable (compatible, finite type).

On the Cartan side, to overcome this difficulty, one introduces a filtration of the tangent space TM, changing the prolongation.

Algebraic prolongations

Given a distribution Δ its weak derived flag is given by

$$\Delta_0 = 0,\ \Delta_1 = \Delta,\ \Delta_2 = [\Delta, \Delta_1],\ \Delta_3 = [\Delta, \Delta_2], \dots$$

We assume Δ completely non-holonomic, i.e. $\exists \nu$: $\Delta_\nu = TM$.

Further on let $\mathfrak{g}_i = \Delta_{-i}/\Delta_{-i-1}$ and $\mathfrak{m} = \oplus_{i<0}\mathfrak{g}_i$. At every point $x \in M$ the space $\mathfrak{m}(x)$ has a natural structure of graded nilpotent Lie algebra (bracket induced by commutators). Δ is called strongly regular if the type of $\mathfrak{m} = \mathfrak{m}_x$ is independent of $x \in M$.

The Tanaka algebra $\mathfrak{g} = \mathrm{pr}(\mathfrak{m}) = \oplus_{i=-\nu}^{+\infty}\mathfrak{g}_i$ of Δ is the graded Lie algebra given by the rule $\mathfrak{g}_- = \oplus_{i<0}\mathfrak{g}_i = \mathfrak{m}$, and for $k \geq 0$:

$$\mathfrak{g}_k = \{u \in \bigoplus_{i<0} \mathfrak{g}_{k+i} \otimes \mathfrak{g}_i^* : u([X,Y]) = [u(X), Y] + [X, u(Y)],\ X, Y \in \mathfrak{m}\}.$$

Another possibility is to reduce $\mathfrak{g}_0$ and prolong $\mathfrak{g} = \mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0)$ or perform higher order reductions.

Tanaka theory in nutshell

Tanaka constructed a tower of fiber bundles (nonholonomic frames)

$$M \leftarrow \mathcal{P}_0 \leftarrow \mathcal{P}_1 \leftarrow \dots$$

The starting construction of 0-frames is a principal bundle, with Lie algebra of the structure group equal $\mathfrak{g}_0$ (possible reduction). Next, similar to the Cartan method, further frame bundles are principal (with group $\mathfrak{g}_k$, $k > 0$) only seqentially $\mathcal{P}_k \to \mathcal{P}_{k-1}$ but not as bundles over M.

Again, for structures of finite type the number of steps is finite and the final bundle $\mathcal{P} \to M$ carring a canonical absolute parallelism (equivariant $\Rightarrow$ Cartan connection) allowing to solve the equivalence problem. In particular, the symmetry group is bounded so:

$$\dim \mathcal{G} \leq \dim \mathfrak{g} = \sum \dim \mathfrak{g}_i.$$

Finite type is determined by the same rank 1 condition but for the algebra $\mathfrak{g}_0' = \{\phi \in \mathfrak{g}_0 : [\phi, \mathfrak{g}_i] = 0 \ \forall i < -1\}$ (characteristics).

Filtered Lie equations

On the jet side, the counter-part is given through the theory of filtered jets. A filtered structure $\mathcal{F}$ on the manifold M is given by a non-holonomic vector distribution Δ and a finite number of successive reductions of the generalized frame bundles $\mathcal{P}_k$.
Filtration on TM induces pointwise filtration on the maximal ideal in the algebra of functions, whence on diff operators and jets.

Theorem (BK 2013)

The symmetry algebra $\mathcal{S}$ (possibly infinite-dimensional) of a filtered structure $\mathcal{F}$ has the natural filtration with the associated grading $\mathfrak{s}$ naturally injected into $\mathfrak{g}(x)$ for any regular point $x \in M$. In particular,

$$\dim \mathcal{G} \leq \sup_M \dim \mathfrak{g}(x).$$

Provided $\mathcal{F}$ is of finite type or is analytic, we have:

$$\dim \mathcal{G} \leq \inf_M \dim \mathfrak{g}(x).$$

Example: N-extended Poincaré structures

Let $(\mathbb{V}, g)$ be a metric vector space and $\mathbb{S}$ be a spin module. Let $\mathfrak{g}_{-2} = \mathbb{V}$, $\mathfrak{g}_{-1} = \underbrace{\mathbb{S} \oplus \cdots \oplus \mathbb{S}}_{N}$ and $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$.

Then $\mathfrak{m} \oplus \mathfrak{so}(\mathbb{V})$ is the N-extended Poincaré algebra.
Brackets $\Lambda^2 \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ were classified by D.Alekseevsky-V.Cortes.

The prolongation $\mathfrak{g} = \mathrm{pr}(\mathfrak{m})$ was computed by A.Altomani-A.Santi. It equals $\mathfrak{m} \oplus \mathfrak{g}_0$, $\mathfrak{g}_0 = \mathfrak{so}(\mathbb{V}) \oplus \mathbb{R} \oplus \mathfrak{g}_0'$, except for the cases $A_n/P_{2,n-1}$, C_n/P_2, F_4/P_4, $E_6/P_{1,6}$, where the prolongation is the corresponding semisimple Lie algebra $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \cdots \oplus \mathfrak{g}_2$.

This gives the bound for the symmetry algebra of the corresponding nonholonomic geometry.
One could also compute the submaximal symmetry dimension $\mathfrak{S}$: in the parabolic case this is due to the joint work with D.The; in non-exceptional case through the filtered deformation.

Thanks for your attention!