Def.: Gopakumar-Vafa invariants

- Given a SUSY-rep. $\mathcal{R}$,

$$\mathcal{R} \overset{BPS}{=} [2(0,0)+(\tfrac{1}{2},0)] \otimes \sum_{j_L, j_R} N^{j_L, j_R}_{[e]} (j_L, j_R)$$

from M2s wrapped on curves in fixed class $[e]$,

$$Tr_R(\underbrace{(-1)^{2j_R}}_{\text{s.t.}} \mathcal{R}) = \sum_{g=0}^{\infty} \boxed{n^g_e} [2(0)+(\tfrac{1}{2})]^{g+1}$$

s.t. non-BPS multiplets contribute nothing!

genus $g$ GV-invariants

$n^g: H_2(X,\mathbb{Z}) \longrightarrow \mathbb{Z}$

ex.: • $\mathcal{R}$ = hypermultiplet:

$$\Rightarrow Tr_R((-1)^{2j_R}\mathcal{R}) = 2(0)+(\tfrac{1}{2})$$

$$\Rightarrow n^0_e = 1, \quad n^{g>0}_e = 0$$

• $\mathcal{R}$ = BPS vectormultiplet:

$$\Rightarrow Tr_R((-1)^{2j_R}\mathcal{R}) = -4(0) - 2(\tfrac{1}{2})$$

$$\Rightarrow n^0_e = -2, \quad n^{g>0}_e = 0$$

- M-theory on $X$ has
  - gauge group $U(1)^{b_2(X)}$
  - gravity multiplet (metric, graviphoton, $2\times$ spinor)
  - $b_2(X)-1$ massless vector multiplets (real scalar, spinor, real vector)
  - $h^{2,1}(X)+1$ massless hypermultiplets (4 real scalars, spinor)

<u>moduli space</u>: $\mathcal{M} = \mathcal{M}_V \times \mathcal{M}_H$

$$\hat{J} = \sum_{a=1}^{b^2} \frac{t^a}{\mathrm{Vol}(X)^{1/3}} [D_a]$$

$$\mathrm{Vol}(X) = \frac{1}{6}\int_X J^3$$

$\rightsquigarrow$ Kähler moduli $t^a$
= projective coordinates on $\boxed{\mathcal{M}_V}$.

- $h^{2,1}$ complex structure moduli
- $2h^{2,1}+2$ "Wilson lines" $\int_{\text{3-cycles}} C_3$
- $C_{\mu\nu\rho}$, $\mathrm{Vol}(X)$

$\boxed{\text{our goal: use GVs to characterize } \mathcal{M}_V}$

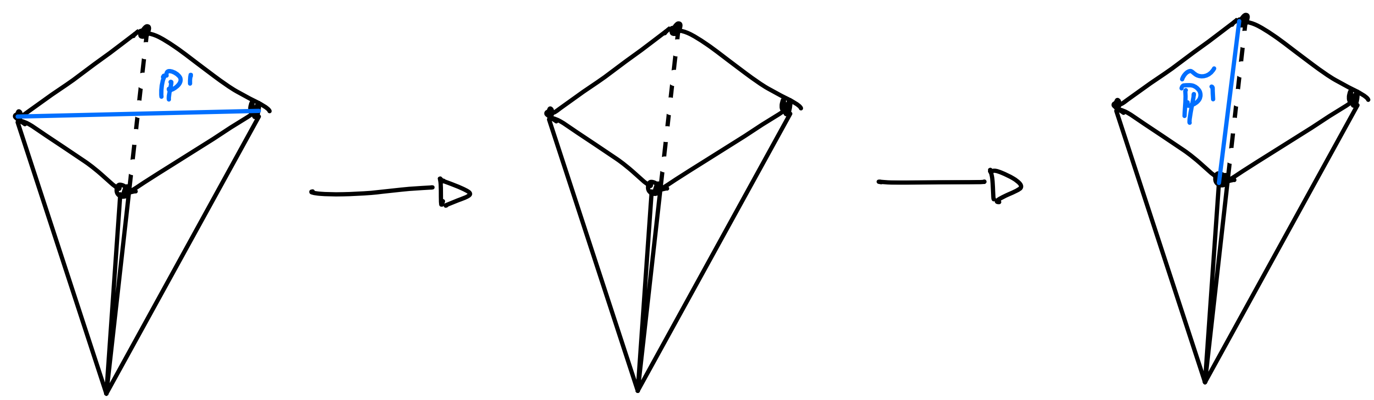
- As $\mathrm{Vol}(X)$ parameterizes $\mathcal{M}_H$, can evaluate $\mathcal{M}_V$ in limit $\mathrm{Vol}(X) \to \infty$

  $\Rightarrow$ $\mathcal{M}_V$ is not corrected by powers $\ell_P / \mathrm{Vol}(\text{cycles})^{\frac{1}{\dim}}$

- $\boxed{\mathcal{M}_V = (\text{fundamental domain of}) \text{ extended Kähler cone } \mathcal{K}_{[X]}}$

  $\to$ understanding $\mathcal{M}_V$ $\Leftrightarrow$ understanding bi-rational geometry of CY 3-folds!

- Building block of bi-rational transformations: simple conifold flop





$t = \mathrm{Vol}(\mathbb{P}^1)$, $t > 0$ — $t = 0$ conifold singularity — $t = -\mathrm{Vol}(\widetilde{\mathbb{P}}^1)$, $t < 0$

- $N(\mathbb{P}^1 \subseteq X) \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$

$\Rightarrow$ M2-brane wrapped on $\mathbb{P}^1$ admits no internal deformations

$$H^0(\mathcal{O}(-1)) = 0 .$$

Since M2 on $\mathbb{P}^1$ is BPS particle, spontaneously breaks $\boxed{\text{4 translations in } \mathbb{R}^4 \text{ \& 2 complex supercharges}}$

$\rightsquigarrow$ 4 Goldstone modes $x^\mu$ & 2 "Goldstinos" $\lambda^\alpha$

quantizing the fermionic zeromodes $\{\lambda^\alpha, \lambda^\dagger_\beta\} = \delta^\alpha_\beta$

gives $\{|0\rangle, \lambda^\dagger_\alpha |0\rangle, \lambda^\dagger_\alpha \lambda^\dagger_\beta |0\rangle\}$

$$= 2(0,0) + (\tfrac{1}{2},0)$$

$\Rightarrow$ $\boxed{\text{single BPS hypermultiplet (and nothing else!)}}$ cf. Strominger '95

- For $C$ Riemann surface of genus $g$, with <u>smooth & compact</u> moduli space $\mathcal{M}$, get

(I) $g$ additional fermionic zeromodes

$$\Rightarrow [2(0,0) + (\tfrac{1}{2})] \longrightarrow [2(0,0) + (\tfrac{1}{2})]^{g+1}$$

upon canonical quantization

(II) $\dim \mathcal{M}$ additional bosonic modes

+ fermionic superpartners

$\Rightarrow$ SUSY quantum mech. on $\mathcal{M}$

ground state Hilbert space $\mathcal{H}_{BPS} \simeq H^{\bullet}_{dR}(\mathcal{M})$

& $SU(2)_R$ is identified with

$$SU(2) \subseteq \underbrace{SL(2)}_{\text{Lefschetz } SL(2) \text{ on } H^{\bullet}_{dR}(\mathcal{M})}$$

$\Rightarrow$ get factor $\sum_{j=0}^{d/2} a_j (0, j)$ ← Lefschetz decomp. of $H^{\bullet}_{dR}(\mathcal{M})$

$$\Rightarrow R = \left[2(0,0) + (\tfrac{1}{2},0)\right]^{g+1} \otimes \sum_j a_j (0,j)$$

$$\Rightarrow \mathrm{Tr}_R (-1)^{2j_R} R = \left[2(0,0) + (\tfrac{1}{2},0)\right]^{g+1} \cdot \sum_j (-1)^{2j} (2j+1) a_j$$

$$\Rightarrow \boxed{n^g_e = \sum_j (-1)^{2j} (2j+1) a_j = (-1)^{\dim_{\mathbb{C}} \mathcal{M}} \chi(\mathcal{M})}$$

(But note that smooth $\mathcal{M}$ are rare... )

- Simple special cases:

(A) Flop $\longrightarrow$ $\mathcal{M} = \{pt\}$ $\Rightarrow$ $n_e^0 = 1$ ✓

(B) $\mathbb{P}^1 \hookrightarrow D$ $\Rightarrow$ $\mathcal{M}_{\mathbb{P}^1\text{-fiber}} = R_g$

$$\begin{array}{c} D \\ \downarrow \\ R_g \end{array}$$

$$H_{dR}(R_g) = H^0 \oplus H^1 \oplus H^2$$

$H^1 \longrightarrow 2g$ $SU(2)_R$-singlets

$H^0, H^2 \longrightarrow$ single $(\tfrac{1}{2})$-rep.

$$\Rightarrow \mathcal{R} = [2(0,0) + (\tfrac{1}{2},0)] \otimes [2g \cdot (0,0) + (0,\tfrac{1}{2})]$$

$\Rightarrow$ $2g$ hypermultiplets

+ single (complex) vectormultiplet

"W-boson"

$\Rightarrow$ as $\mathrm{Vol}(\mathbb{P}^1) \longrightarrow 0$, get

$U(1) \longrightarrow SU(2)$ enhancement

with $g$ adjoint hypermultiplets

Aspinwall '96
Katz, Morrison, Plesser '96

$$\boxed{n_e^0 = -\chi(R_g) = 2 - 2g}$$

# Part II: GVs determine birational geometry

## The GV formula

- M-theory on $X \times S^1$ = type IIA string theory on $X$

$$\mathcal{M}^{IIA} = \mathcal{M}_V^{IIA} \times \mathcal{M}_H^{II}$$

dim $= h^{1,1}(X)$

$\Rightarrow$ local complex coord.: $z^a = i t^a - b^a$

$$y_c \equiv \sum_a z^a \omega_a = i y - B$$

- $Vol(X)$ is now part of $\mathcal{M}_V^{IIA}$ $\Rightarrow$ expect pert. + non-pert. corr. in $\alpha'$-expansion

but $g_s^{IIA}$ is in hypermultiplet

$\Rightarrow$ tree-level exactness in $g_s^{IIA}$-expansion.

Non-pert.: euclidean worldsheets wrapped on curves correct $\mathcal{M}_V^{IIA}$

$$\sim e^{-2\pi Vol(\mathcal{C})}$$

- string worldsheet = M2-brane on $S^1$

$\Rightarrow$ at large $Vol(S^1) \gg Vol(X)^{1/6}$

worldline of BPS-particle along $S^1$ !

- Gopakumar-Vafa computed corr. to 4d effective action from BPS particles of the 5d theory: [GV '98 ×2, Dedushenko-Witten '14]

$$\delta S = \int \frac{d^4x\, d^4\theta}{(2\pi)^4} \sum_{q \in \mathrm{Mori}(X)} \sum_{g=0}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{g-1}}{k} n_q^g \, e^{2\pi i k \int_X g^c \wedge q}$$

$$\times \frac{\left(\frac{\pi}{8} W\right)^2}{\sin^{2-2g}\left(k\sqrt{\left(\frac{\pi}{8}W\right)^2}\right)}$$

$W$ = anti-self dual part of graviphoton

$$= -\int \frac{d^4x\, d^4\theta}{(2\pi)^4} \sum_{q \in \mathrm{Mori}(X)} n_q^0 \sum_{k=1}^{\infty} \frac{1}{k^3} \left(e^{2\pi i \int g^c \wedge q}\right)^k + \mathcal{O}(W^2)$$

$$= -\int \frac{d^4x\, d^4\theta}{(2\pi)^4} \sum_{q \in \mathrm{Mori}(X)} n_q^0 \, \mathrm{Li}_3\left(e^{2\pi i \int g^c \wedge q}\right) + \dots$$

$$= -i \int d^4x\, d^4\theta \sum_{g=0}^{\infty} \underbrace{\mathcal{F}_g(z^a, z^0)}_{\text{genus } g \text{ topological string free energy}} (W^2)^g$$

genus $g$ topological string free energy } BCOV '94, Antoniadis-Gava-Narain-Taylor '94

- $\boxed{\text{only } \mathcal{F}_0 \text{ determines } \mathcal{M}_V^{IIA}}$

- including pert. $\alpha'$ corr., (setting $z^0 = 1$),

$$\mathcal{F}_0 = -\frac{1}{3!}\mathcal{K}_{abc} z^a z^b z^c + \frac{1}{2} a_{ab} z^a z^b + \frac{c_a}{24} z^a + \frac{\zeta(3)\chi(X)}{2(2\pi i)^3} - \frac{1}{(2\pi i)^3}\underbrace{\sum_q n_q^0 \, Li_3\left(e^{2\pi i z_q}\right)}_{\text{worldsheet-inst. corrections}}$$

triple-int. form

$$c_a = \int_X c_2(TX) \wedge \omega_a$$

- How does this help us understanding $\mathcal{K}_{[X]}$?

$\mathcal{F}_0 = \mathcal{F}_0(z^a)$ is holomorphic

$\rightsquigarrow$ can find all Kähler chambers via analytic continuation [Gendler, Heidenreich, McAllister, JM, Rudelius '22]

<u>Note:</u> For flop across face of Kähler cone, $n^0_{\mathcal{C}^0}$ isolated $\mathbb{P}^1$'s shrink, in class $[\mathcal{C}^0]$ (or also $n \cdot [\mathcal{C}^0]$ with $n = 1, \dots, n_{max} \leq 6$ $\longrightarrow$ higher length flops Katz-Morrison '92)

$$\rightsquigarrow \mathcal{F}_0 = \text{degree-3 poly} - \frac{1}{(2\pi i)^3} Li_3\left(e^{2\pi i \vec{q}_0 \cdot \vec{z}}\right) + \text{exponentially suppressed}$$

Jonquière '1889:

$$\frac{Li_s(e^{2\pi i z})}{(2\pi i)^s} + (-1)^s \frac{Li_s(e^{-2\pi i z})}{(2\pi i)^s} = \frac{\zeta(1-s,z)}{\Gamma(s)}$$

$$\overset{s=3}{=} -\frac{z}{12} + \frac{z^2}{4} - \frac{z^3}{6}$$

$$\Rightarrow -\frac{1}{3!}\mathcal{K}_{abc} z^a z^b z^c + \frac{1}{2} a_{ab} z^a z^b + \frac{c_a}{24} z^a - \frac{n^0_{q^0} Li_3(e^{2\pi i q^0 z})}{(2\pi i)^3}$$

$$= -\frac{1}{3!}\mathcal{K}'_{abc} z^a z^b z^c + \frac{1}{2} a'_{ab} z^a z^b + \frac{c'_a}{24} z^a - \frac{n^0_{q^0} Li_3(e^{-2\pi i q^0 z})}{(2\pi i)^3}$$

with
$$\boxed{\begin{aligned} \mathcal{K}'_{abc} &= \mathcal{K}_{abc} - n^0_{q^0} q^0_a q^0_b q^0_c \\ c_a &= c'_a + 2 n^0_{q^0} q^0_a \end{aligned}}$$

This matches how $\mathcal{K}, c_2$ transform across flop!

& in flopped phase, $-q^0_a \simeq -[e^0]$ is effective, $[e^0]$ isn't.

Useful corollary:

- For every $[e^0]$ that shrinks across flop in *any* $X \in [X]_{b.r.}$, in *every* $X' \in [X]_{b.r.}$ $\exists\, s_{X'} \in \{-1,+1\}$ s.t. $s[e^0]$ is effective
  $\Rightarrow K_{[X]}$ is hyperplane arrangement!

Now, let $X \in [X]_{b.r.}$

& denote a rational ray $r$ in $H_2$ as

- <u>GV-nilpotent</u> if $n^0_e \neq 0$ at most for finitely many $e \in r \cap H_2(X,\mathbb{Z})$
- <u>GV-potent</u> otherwise

<u>Lemma:</u> Let $r$ be a ray generated by flop curve $[e^0]$. Then,

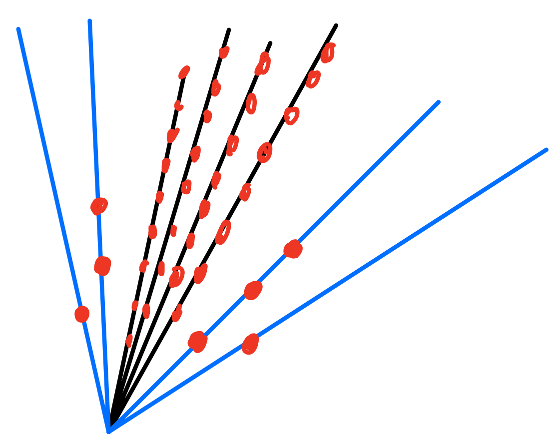
a) $r$ is GV-nilpotent

b) $r \cap \mathcal{M}' = \{0\}$

where $\mathcal{M}' := \overline{\operatorname{span}_{\mathbb{R}_+}\{[e] \not\propto [e^0] : n^0_{[e]} \neq 0\}}$

$\Rightarrow$ flop rays strictly outside

$\mathcal{M}_\infty :=$ cone generated by all GV-potent rays



What can happen along facet of $\mathcal{K}_X$?

Theorem [P.M.H. Wilson '92]:

A) Flop

B) $\mathbb{P}^1$-fibered surface $\longrightarrow R_g$

C) surface $\longrightarrow$ point

$\left.\right\}$ finite distance

D) $\mathrm{Vol}(X) \longrightarrow 0$ (at infinite dist.)

+ for general complex structure, $g=0$ in B)

claim: $\boxed{\text{only for A) get GV-nilpotent ray outside } \mathcal{M}_\infty \text{, assuming generic CS.}}$

argument: C, D feature GV-potent rays b.c. they generate CFTs respectively are at $\infty$-dist. (cf. distance conjecture)

B has $n_e^0 = -2 \Rightarrow$ nilpotent, but $SU(2)$-instanton corr. to 4d $\mathcal{N}=2$ SYM encode infinitely many states. "$\square$"

<u>Corollary</u>: a) $\mathcal{M}_\infty^\vee = \mathcal{K}_{ext.}$,

b) nilpotent rays outside $\mathcal{M}_\infty$ are the combinatorial object governing birational geometry.

c) $\mathrm{Mori}(X)\big|_{\substack{\text{generic}\\ \text{CS}}} = \mathrm{span}_{\mathbb{R}_+}\{[e] \text{ s.t. } n_e^0 \neq 0\}$

$\Rightarrow$ can <u>compute</u> (extended) Kähler cones using $g=0$ GV-invariants.

<u>caveat</u>: • in practice, can compute GV-invariants only to finite degree:

Let $v \in K_X \cap H^2(X,\mathbb{Z})$ be co-prime

$\to$ define $d_v : H_2(X,\mathbb{Z}) \longrightarrow \mathbb{Z}$

$q \longmapsto \langle q, v\rangle$

& set $M_d := \{ q \in \mathrm{Mori}(X) \cap H_2(X,\mathbb{Z}) : d_v(q) \leq d \}$ is finite $\forall\, d \in \mathbb{N}$.

<u>e.g.</u>: defining $K_X^d$ from $\mathcal{M}_d$ and $n^0(\mathcal{M}_d)$, $\lim_{d\to\infty} K_X^d = K_X$

- But at any finite maximal degree, no certificate whether $K_X^d = K_X$

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- In many cases, can combine this algorithm with toric geometry to do better:
  - Let $X \subseteq V$ be $CY_3$ hypersurface in toric variety $V$.
  - Assume "favorability": $H^2(X,\mathbb{Z}) \cong H^2(V,\mathbb{Z})$
  - We have $\mathrm{Mori}(X) \subseteq \mathrm{Mori}(V)$

$$\Leftrightarrow \quad K_X \supseteq K_V$$

suppose $\exists$ curve $\mathcal{C}$, $n^0_{\mathcal{C}} \neq 0$, along all extremal rays of $\mathrm{Mori}(V)$

$$\Rightarrow \mathrm{Mori}(X) = \mathrm{Mori}(V)$$

  - Above one can do better by

$$\mathrm{Mori}(V) \longrightarrow \bigcap_i \mathrm{Mori}(V_i)$$

with $V_i \xrightarrow{\text{flip}} V_j$ whose exceptional locus does not intersect $X$.