Dirac fields on Kerr spacetime and the Hawking radiation III

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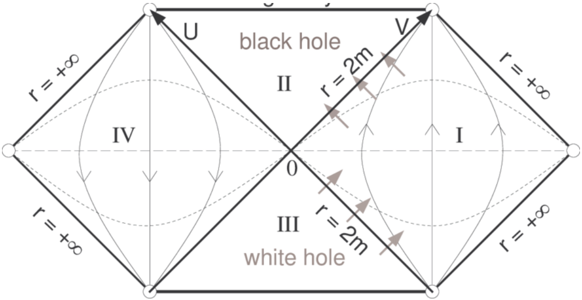
Part III: The Unruh state for fermions and its Hadamard property (with C. Gérard and M. Wrochna)

(picture : PhD Giuseppe Gaetano Luciano)

Choice of states

∂_U is generator of null geodesics on the past horizon.



The acceleration of the integral lines of $v_{\mathcal{H}}$ on the past horizon is $a = \kappa_+$!

III.1 Propagators

Characteristic manifold

The principal symbol of $\mathbb{D}$ is the section $\sigma_{\mathbb{D}} \in C^\infty(T^*M \setminus o; \mathrm{End}(\mathbb{S}^*, \mathbb{S}))$ given by

$$\sigma_{\mathbb{D}}(x,\xi) = \Gamma(g^{-1}(x)\xi), \quad (x,\xi) \in T^*M \setminus o.$$

Lemma

The Weyl operator $\mathbb{D}$ is pre-normally hyperbolic, i.e., there exists a differential operator $\mathbb{D}'$ such that $(\sigma_{\mathbb{D}} \circ \sigma_{\mathbb{D}'})(x,\xi) = (\xi \cdot g^{-1}(x)\xi)\mathbf{1}$.

The characteristic manifold of $\mathbb{D}$ is defined as

$$\mathrm{Char}(M) = \{(x,\xi) \in T^*M \setminus o \,:\, \sigma_{\mathbb{D}}(x,\xi) \text{ is not invertible}\}.$$

By the Lemma,

$$\mathrm{Char}(M) = \{(x,\xi) \in T^*M \setminus o \,:\, \xi \cdot g^{-1}(x)\xi = 0\} =: \mathcal{N}.$$

Its two connected components are

$$\mathcal{N}^\pm := \mathcal{N} \cap \{(x,\xi) \in T^*M \setminus o \,:\, \pm v\cdot\xi > 0, \quad \forall v \in T_xM \text{ future directed time-like}\}. \tag{1}$$

Propagators

An adaptation of an argument due to Dimock to the case of general pre-normally hyperbolic operators gives the existence and uniqueness of retarded and advanced propagators, $\mathbb{G}_{\mathrm{ret}}$ and $\mathbb{G}_{\mathrm{adv}}$. Recall that $\mathbb{G}_{\mathrm{ret/adv}}$ is by definition a two-sided inverse of $\mathbb{D}$ (on test sections) such that
$\operatorname{supp}\mathbb{G}_{\mathrm{ret/adv}}v \subset J_{\pm}(\operatorname{supp} v), \quad v \in C_{\mathrm{c}}^{\infty}(M;\mathbb{S})$, where $J_{\pm}(K)$ stands for the causal future/past of $K \subset M$. The Pauli-Jordan or causal propagator is the difference $\mathbb{G} = \mathbb{G}_{\mathrm{ret}} - \mathbb{G}_{\mathrm{adv}}$.
We have

$$(v_1|\mathbb{G}v_2)_M = -(\mathbb{G}v_1|v_2)_M, \;\; v_i \in C_{\mathrm{c}}^{\infty}(M;\mathbb{S}), \tag{2}$$

i.e. $\mathbb{G}^* = -\mathbb{G}$ for the pairing $(\cdot|\cdot)_M$ defined in (**??**).

Theorem (Duistermaat-Hörmander)

$$WF(\mathbb{G})' \subset \mathcal{C} = \{(X_1, X_2) \in \mathcal{N} \times \mathcal{N};\; X_1 \sim X_2\}.$$

Cauchy problem

If S is a space-like Cauchy surface, the Cauchy problem

$$\begin{cases} \mathbb{D}\phi = 0, \\ r_S\phi = \varphi \in C^\infty_{\mathrm{c}}(S; \mathbb{S}^*_S), \end{cases}$$

where $\mathbb{S}^*_S$ is the restriction of $\mathbb{S}^*$ to S and $r_S\phi = \phi_{|S}$, has a unique solution $\phi =: \mathbb{U}_S\varphi \in \mathrm{Sol}_{\mathrm{sc}}(M)$.
For all $\phi \in \mathrm{Sol}_{\mathrm{sc}}(M)$ one has:

$$\phi(x) = -\int_S \mathbb{G}(x,y)\Gamma(g^{-1}\nu)(y)\phi(y) i_l^*(d\mathrm{vol}_g)(y),$$

$(S = \mathrm{Ker}\,\nu,\ \nu\cdot l = 1)$. Choosing $l = n, \nu = -gn$, where n the future directed vector field normal to S, this can be rewritten as

$$\phi(x) = -\int_S \mathbb{G}(x,y)\Gamma(n(y))\phi(y)\, d\mathrm{vol}_h(y),$$

where h is the induced Riemannian metric on S.

Equivalent Hilbert spaces

We now recall other Hilbert spaces unitarily equivalent to $\mathrm{Sol}_{\mathrm{L}^2}(M)$. Let Σ be a smooth space-like Cauchy surface.

Proposition

The following maps are unitary

$$\left(\frac{C_c^\infty(M;\mathbb{S})}{\mathbb{D}C_c^\infty(M;\mathbb{S}^*)}, \imath\mathbb{G}\right) \xrightarrow{\mathbb{G}} \left(\mathrm{Sol}(M), (\cdot|\cdot)_{\mathbb{D}}\right) \xrightarrow{r_\Sigma} \left(C_c^\infty(\Sigma;\mathbb{S}_\Sigma^*), \nu_\Sigma\right),$$

where $\overline{\varphi}_1 \cdot \nu_\Sigma \varphi_2 = \imath \int_\Sigma \bar{\varphi}_1 \cdot \Gamma(n)\varphi_2 \, d\mathrm{vol}_h, \ \varphi_i \in C_c^\infty(\Sigma;\mathbb{S}_\Sigma^*)$.

As a consequence of the Proposition we have the identities

$$(\phi_1|\phi_2)_{\mathbb{D}} = (v_1|\imath\mathbb{G}v_2)_M = (\imath\mathbb{G}v_1|v_2)_M, \text{ for } \phi_i = \mathbb{G}v_i, \ v_i \in C_c^\infty(M;\mathbb{S}),$$

which extends to

$$(v|\phi)_M = (\imath\mathbb{G}v|\phi)_{\mathbb{D}}, \ v \in C_c^\infty(M;\mathbb{S}), \ \phi \in \mathrm{Sol}_{\mathrm{L}^2}(M). \qquad (3)$$

III.2 Algebraic quantization of the Weyl equation

CAR algebras

Let $(\mathcal{Y}, \nu)$ be a pre-Hilbert space. We denote by $CAR(\mathcal{Y}, \nu)$ the unital complex $*$-algebra generated by elements $\psi(y)$, $\psi^*(y)$, $y \in \mathcal{Y}$, with the relations

$$\psi(y_1 + \lambda y_2) = \psi(y_1) + \overline{\lambda}\psi(y_2),$$

$$\psi^*(y_1 + \lambda y_2) = \psi(y_1) + \lambda\psi^*(y_2), \; y_1, y_2 \in \mathcal{Y}, \lambda \in \mathbb{C},$$

$$[\psi(y_1), \psi(y_2)]_+ = [\psi^*(y_1), \psi^*(y_2)]_+ = 0,$$

$$[\psi(y_1), \psi^*(y_2)]_+ = \overline{y}_1 \cdot \nu y_2 \mathbf{1}, \; y_1, y_2 \in \mathcal{Y},$$

$$\psi(y)^* = \psi^*(y), \; y \in \mathcal{Y},$$

where $[A, B]_+ = AB + BA$ is the anti-commutator.

States

Definition (States)

A linear functional ω over the C^*- algebra $\mathcal{U}$ is defined to be positive if

$$\omega(A^*A) \geq 0, \forall A \in \mathcal{U}.$$

A positive linear functional over a C^*- algebra $\mathcal{U}$ with $\|\omega\| = 1$ is called a state.

Definition (Quasi-free states)

A state ω on $CAR(Y, \nu)$ is a (gauge invariant) quasi-free state if

$$\omega\left(\Pi_{i=1}^n \psi^*(y_i) \Pi_{j=1}^m \psi(y'_j)\right) = 0, \quad \text{if } n \neq m,$$
$$\omega\left(\Pi_{i=1}^n \psi^*(y_i) \Pi_{j=1}^n \psi(y'_j)\right) = \sum_{\sigma \in S_n} sgn(\sigma) \Pi_{i=1}^n \omega(\psi^*(y_i)\psi(y'_{\sigma(i)})).$$

Quasi-free states

A quasi-free state ω on $\mathrm{CAR}(\mathcal{Y},\nu)$ is determined by its covariances $\lambda^{\pm} \in L_{\mathrm{h}}(\mathcal{Y},\mathcal{Y}^{*})$ (hermitian form on $\mathcal{Y}$), defined by

$$\omega(\psi(y_1)\psi^{*}(y_2)) =: \overline{y}_1 \lambda^{+} y_2, \quad \omega(\psi^{*}(y_2)\psi(y_1)) =: \overline{y}_1 \lambda^{-} y_2, \quad y_1, y_2 \in \mathcal{Y}.$$

A pair of Hermitian sesquilinear forms $\lambda^{\pm}$ on $\mathcal{Y}$ are the covariances of a quasi-free state on $\mathrm{CAR}(\mathcal{Y},\nu)$ iff

$$\lambda^{\pm} \geq 0, \quad \lambda^{+} + \lambda^{-} = \nu.$$

It follows that $\lambda^{\pm}$ uniquely extend to the completion $\mathcal{Y}^{\mathrm{cpl}}$ of $\mathcal{Y}$ for ν.

Definition

The quasi-free state ω on $\mathrm{CAR}(\mathcal{Y},\nu)$ is a pure state iff there exist projections $\pi^{\pm}$ on $\mathcal{Y}^{\mathrm{cpl}}$ such that

$$\lambda^{\pm} = \nu \circ \pi^{\pm}.$$

Note that $\pi^{\pm}$ are selfadjoint for ν and $\pi^{+} + \pi^{-} = \mathbb{1}$.

III.3 Hadamard states for the Weyl equation

Spacetime covariances

A quasi-free state ω is specified a pair of *spacetime covariances* (or two-point functions if one speaks of the associated Schwartz kernels), i.e. a pair of operators $\mathbb{\Lambda}^{\pm}$ satisfying:

$$
\begin{aligned}
&i) \quad \mathbb{\Lambda}^{\pm}\colon C_{\rm c}^{\infty}(M;\mathbb{S}) \to \mathcal{D}'(M;\mathbb{S}^*) \text{ is linear continuous,}\\
&ii) \quad \mathbb{\Lambda}^{\pm} \geq 0,\\
&iii) \quad \mathbb{\Lambda}^{+} + \mathbb{\Lambda}^{-} = \imath\mathbb{G},\\
&iv) \quad \mathbb{D}\,\mathbb{\Lambda}^{\pm} = \mathbb{\Lambda}^{\pm}\,\mathbb{D} = 0.
\end{aligned}
$$

Alternatively, one can define the state ω by its solution space covariances, i.e. operators $C^{\pm} \in B(\mathrm{Sol}_{\mathrm{L}^2}(M))$ such that

$$C^{\pm} \geq 0, \quad C^{+} + C^{-} = \mathbf{1}.$$

$C^{\pm}$ is pure if $(C^{\pm})^2 = C^{\pm}$. By Proposition 1, the two types of covariances are related as follows:

$$\bar{v}\cdot\mathbb{\Lambda}^{\pm} v = (\mathbb{G}v|C^{\pm}\mathbb{G}v)_{\mathbb{D}}, \ v \in C_{\rm c}^{\infty}(M;\mathbb{S}).$$

Hadamard states

Definition (Radzikowski)

A quasi-free state ω on $\mathrm{CAR}(M)$ is a Hadamard state if it satisfies (the Hadamard condition):

$$\mathrm{WF}(\mathbb{A}^{\pm})' \subset \mathcal{N}^{\pm} \times \mathcal{N}^{\pm},$$

where $\mathcal{N}^{+}$ and $\mathcal{N}^{-}$ are the two components of the characteristic set defined in (1).

Here $(x, \xi, y, \eta) \in \mathrm{WF}(\mathbb{A})' \Leftrightarrow (x, \xi, y, -\eta) \in \mathrm{WF}(\mathbb{A})$. Recall that $\mathrm{WF}(\mathbb{A}^{\pm} u) \subset {}_M\mathrm{WF}(\mathbb{A}^{\pm})' \cup \mathrm{WF}(\mathbb{A}^{\pm})'(\mathrm{WF}(u))$, where ${}_M\Gamma = \{(x_1, \xi_1) \in T^*M \setminus o;\ \exists x_2 \text{ such that } (x_1, \xi_1, x_2, 0) \in \Gamma\}$.

Proposition

Suppose that $\mathrm{WF}((C^{\pm})^{\frac{1}{2}}\phi) \subset \mathcal{N}^{\pm} \quad \forall \phi \in \mathrm{Sol}_{\mathrm{L}^2}(M)$. *Then the state* ω *is a Hadamard state.*

Oscillatory test functions 1

Let $\Omega \subset \mathbb{R}^n$ be an open set. For $x \in \Omega$, $q = (x, \xi) \in T^*\Omega \setminus o$ and $\chi \in C_c^\infty(\Omega)$ we denote

$$v_q^\lambda(x) := \chi(x) \mathrm{e}^{-\imath\lambda x \cdot \xi}, \quad \lambda \geq 1.$$

We then extend the definition to manifolds by chart diffeomorphism pullback. We will say that a function v_q^λ of this form is an oscillatory test function at $q_0 = (x_0, \xi_0)$ if $v_q^\lambda(x_0) \neq 0$.

Definition

We say that w_q^λ is a *generalized oscillatory test function at* $q_0 = (x_0, \xi_0) \in T^*M \setminus o$ if it is of the form $w_q^\lambda = A^* v_q^\lambda$, where $A \in \Psi^0(M)$ is properly supported and elliptic at q_0, and v_q^λ is an oscillatory test function at q_0.

Oscillatory test functions 2

Lemma

Let $\mathcal{X} \subset \mathcal{D}'(M)$ and let $\Gamma \subset T^*M \setminus o$ be closed. Then $\mathrm{WF}(u) \subset \Gamma$ for all $u \in \mathcal{X}$ iff for all non-zero $q_0 \in T^*M \setminus \Gamma$ there exists a generalized oscillatory test function w_q^λ at q_0 such that for all $u \in \mathcal{X}$ and $N \in \mathbb{N}$,

$$|(w_q^\lambda|u)_M| \leq C_{u,N}\lambda^{-N}, \ \lambda \geq 1,$$

uniformly for q in a neighborhood of q_0 in $T^*M \setminus o$.

Oscillatory test functions 3

Lemma

Suppose that for any $q_0 \in \mathcal{N}^{\mp}$ there exists a generalized oscillatory test function v_q^λ at q_0 such that if $\phi_q^\lambda = \mathbb{G} v_q^\lambda$ one has

$$\|(C^\pm)^{\frac{1}{2}} \phi_q^\lambda\|_{\mathbb{D}} \leq C_N \lambda^{-N}, \ \forall N \in \mathbb{N}$$

uniformly for q in a neighborhood of q_0 in $T^*M \backslash o$. Then ω is a Hadamard state.

Proof. It suffices to prove that $\mathrm{WF}(\mathbb{A}^\pm)' \cap \Delta \subset \mathcal{N}^\pm \times \mathcal{N}^\pm$, where $\Delta \subset T^*M \times T^*M$ is the diagonal. If $q_0 \in \mathcal{N}^\mp$ and v_q^λ are as in the lemma, we have

$$\bar{v_q}^\lambda \cdot \mathbb{A}^\pm v_q^\lambda = (\phi_q^\lambda | C^\pm \phi_q^\lambda)_{\mathbb{D}} = \|(C^\pm)^{\frac{1}{2}} \phi_q^\lambda\|_{\mathbb{D}}^2 \in O(\lambda^{-N}), \ N \in \mathbb{N},$$

so $(q_0, q_0) \notin \mathrm{WF}(\mathbb{A}^\pm)'$, which by the remark above implies that ω is a Hadamard state.

Proof of the proposition

Let $q_0 \in \mathcal{N}^{\mp}$ and $N \in \mathbb{N}$. By the hypothesis we have $\mathrm{WF}((C^{\pm})^{\frac{1}{2}}\phi) \subset \mathcal{N}^{\pm}$ for all $\phi \in \mathrm{Sol}_{\mathrm{L}^2}(M)$. We then use Lemma 8 to see that there exists a generalized oscillatory test function v_q^{λ} at q_0 such that $\sup_{\lambda \geq 1} \lambda^N |(v_q^{\lambda}|((C^{\pm})^{\frac{1}{2}}\phi)_M| < \infty \;\; \forall \phi \in \mathrm{Sol}_{\mathrm{L}^2}(M)$.
Applying the uniform boundedness principle to the family of linear forms

$$T_{\lambda} : \mathrm{Sol}_{\mathrm{L}^2}(M) \ni \phi \mapsto \lambda^N (v_q^{\lambda}|((C^{\pm})^{\frac{1}{2}}\phi)_M \in \mathbb{C}$$

we obtain that

$$\sup_{\lambda \geq 1, \|\phi\|_{\mathbb{D}}=1} \lambda^N |(v_q^{\lambda}|((C^{\pm})^{\frac{1}{2}}\phi)_M| < \infty.$$

Denoting $\phi_q^{\lambda} = \mathbb{G} v_q^{\lambda}$ and using also (3) this gives

$$\begin{aligned} \|(C^{\pm})^{\frac{1}{2}}\phi_q^{\lambda}\|_{\mathbb{D}} &= \sup_{\|\phi\|_{\mathbb{D}}=1} |((C^{\pm})^{\frac{1}{2}}\phi_q^{\lambda}|\phi)_{\mathbb{D}}| \\ &= \sup_{\|\phi\|_{\mathbb{D}}=1} |(\phi_q^{\lambda}|(C^{\pm})^{\frac{1}{2}}\phi)_{\mathbb{D}}| = \sup_{\|\phi\|_{\mathbb{D}}=1} |(v_q^{\lambda}|(C^{\pm})^{\frac{1}{2}}\phi)_M| \in O(\lambda^{-N}) \end{aligned}$$

which by Lemma 9 implies that ω is a Hadamard state. □

Motivation and examples of Hadamard states

Hadamard states look microlocally like vacuum states on Minkowski, they also permit to renormalize the quantum energy momentum tensor.

Examples if (M, g) has time-like Killing vector field ∂_t:

(1) $C^{\pm} = \mathbf{1}_{\mathbb{R}^{\pm}}(D_t)$ is the vacuum w.r.t. ∂_t (it is pure)

(2) $C^{\pm} = (1 + e^{\mp\beta D_t})^{-1}$ is the thermal state at temperature $T = \beta^{-1}$ w.r.t. ∂_t (it is mixed)

One can take more general functions $\chi_{\pm} \in L^{\infty}(\mathbb{R})$ such that

$$\chi_{\pm} \geq 0,\ \chi_{+} + \chi_{-} = 1, \quad \chi_{\pm}(\lambda) \in \mathcal{O}(\lambda^{-N}) \text{ in } \mathbb{R}^{\mp},$$

$\operatorname{sing\,supp} \chi_{\pm}$ compact. The associated states are mircolocally passive. Non existence theorems by Kay, Wald and Pinamonti, Sanders, Verch for a Hadamard state associated to a Killing field that is not everywhere timelike.

Existence of the Unruh state, first version

Consider $\not{D}\phi = 0$. We define a pure state on $\mathrm{M}_{\mathrm{I}\cup\mathrm{II}}$ by taking:

on $\mathcal{H}$ we take $\mathbf{1}_{\mathbb{R}_\pm}(-D_U)$ ("Kay–Wald vacuum")
on $\mathscr{I}^-$ we take $\mathbf{1}_{\mathbb{R}_\pm}(D_{t^*})$ (asymptotic vacuum)

Theorem (Gérard-H-Wrochna '20)

For $|a|\mathfrak{m}^{-1} << 1$, the so-obtained Unruh state is pure and Hadamard in $\mathrm{M}_{\mathrm{I}\cup\mathrm{II}}$. Its restriction to M_{I} is asymptotically thermal with respect to $v_{\mathcal{H}}$ at the past horizon $\mathcal{H}^-$ with temperature equal to the Hawking temperature $T_{\mathrm{H}} = \frac{\kappa_+}{2\pi}$.

Remark: ∂_U is *not* Killing! Yet Hadamard condition and symmetries of the problem impose this choice. Recall Hadamard condition:

$$\mathrm{WF}(C^{\pm}\phi) \subset \mathcal{N}^{\pm} \quad \text{for all solutions of } \not{D}\phi = 0$$

Interpretation: $:\phi^2:$ doesn't blow up at $\mathcal{H}^+$, "smooth" extendability across $\mathcal{H}^+$.

Emergence of the Hawking temperature

For $\beta > 0$, let $\chi_\beta^\pm(s) = (1 + \mathrm{e}^{\mp\beta s})^{-1}$. Let $D_x = \imath^{-1}\partial_x$ acting in $L^2(\mathbb{R})$,

$$\chi_\infty^\pm(D_x) := \imath^* \circ \mathbf{1}_{\mathbb{R}^\pm}(D_x) \circ \imath$$

the restriction of $\mathbf{1}_{\mathbb{R}^\pm}(D_x)$ to $L^2(\mathbb{R}^+)$ ($\imath : L^2(\mathbb{R}^+) \to L^2(\mathbb{R})$ canonical embedding),

$$A = \frac{1}{2}(xD_x + D_x x) = \imath^{-1}(x\partial_x + \frac{1}{2})$$

the generator of dilations.

Lemma

On $L^2(\mathbb{R}^+)$ we have $\chi_\infty^\pm(D_x) = \chi_{2\pi}^\pm(A)$.

Proof of the lemma

Schwartz kernel of $\chi_\infty^\pm(D_x)$

$$\chi_\infty^\pm(D_x)(x,y) = \pm \imath (2\pi)^{-\frac{1}{2}}(x-y\pm \imath 0)^{-1}. \quad (4)$$

The Mellin transform $\mathcal{M}$ diagonalizes the generator of dilations, meaning that

$$\chi_\beta^\pm(A) = \mathcal{M}^{-1}\circ \chi_\beta^\pm(\sigma)\circ \mathcal{M}, \quad (5)$$

where $\chi_\beta^\pm(\sigma)$ denotes the operator of multiplication by $\chi_\beta^\pm$. A brief computation using the Mellin transform shows that the Schwartz kernel of (5) equals

$$\chi_\beta^\pm(A)(x,y) = \frac{1}{y}(\mathcal{M}^{-1}\chi_\beta^\pm)\left(\frac{x}{y}\right). \quad (6)$$

On the other hand,

$$(\mathcal{M}^{-1}\chi_{2\pi}^\pm)(x) = \pm \imath (2\pi)^{-\frac{1}{2}}(x-1\pm \imath 0)^{-1}$$

in the sense of distributions. Plugging this into (6) and comparing with (4) yields the result.

Construction of the Unruh state

- We restrict our discussion to block I.
- In $\mathrm{M_I}$ we construct the Unruh state on $\mathrm{Ker}_{L^2} \not{D}$ by

$$C^+ = \mathrm{P}_{\mathcal{H}^-} \chi_{\frac{2\pi}{\kappa_+}}(i^{-1}\mathcal{L}_{\mathcal{H}}) + \mathrm{P}_{\mathscr{I}^-} \mathbf{1}_{\mathbb{R}^+}(i^{-1}\mathcal{L}_{\mathscr{I}}),$$

  where $\mathrm{P}_{\mathcal{H}^-}$ and $\mathrm{P}_{\mathscr{I}^-}$ project to solutions that go to $\mathcal{H}^-$ and $\mathscr{I}^-$. $\mathcal{L}_{\mathcal{H}}$ and $\mathcal{L}_{\mathscr{I}}$ are Lie derivatives of spinors along the vector fields $v_{\mathcal{H}}$ and $v_{\mathscr{I}}$.
- Idea : estimate WF of $C^+\phi$ in terms of wavefront set on $\mathcal{H}^-$, $\mathscr{I}^-$ using reconstruction formulae :

$$\phi(x) = -\int_S \mathbb{G}(x,y)\Gamma(g^{-1}\nu)(y)\phi(y)i_l^*(d\mathrm{vol}_g)(y).$$

Here $\mathbb{G}$ is the causal propagator, $TS = \mathrm{Ker}\,\nu$, l transverse to S, $\nu \cdot l = 1$. By scattering theory this kind of formulae can be extended to L^2 solutions.

A key proposition

Proposition

Let (M, g) be an oriented and time oriented Lorentzian manifold of dimension n, and let $S \subset M$ be a null hypersurface equipped with a smooth density dm. For $u \in \mathcal{E}'(S)$ we define $\delta_S \otimes u \in \mathcal{E}'(M)$ by:

$$\int_M (\delta_S \otimes u)\varphi \, \mathrm{dvol}_g := \int_S u\varphi \mathsf{d}m, \quad \varphi \in C_{\mathrm{c}}^{\infty}(M).$$

Let also X be a vector field on M, tangent to S, null, future directed on S and suppose $G \in \mathcal{D}'(M \times M)$ satisfies $\mathrm{WF}(G)' \subset \{(q, q') \in \mathcal{N} \times \mathcal{N} \, : \, q \sim q'\}$. Then for any $u \in \mathcal{E}'(S)$ one has the implication:

$$\mathrm{WF}(u) \subset \{(y, \eta) \in T^*S \setminus o \, : \, \pm \eta \cdot X(y) \geq 0\}$$
$$\Rightarrow \mathrm{WF}(G(\delta_S \otimes u)) \cap \pi^{-1}(M \setminus S) \subset \mathcal{N}^{\pm}.$$

Application to our setting

Remark

1. *If* $\mathrm{WF}(G)'$ *has no points of the form* $(x_1, 0, x_2, \xi_2)$*, then*

$$\mathrm{WF}(G(v)) \subset \mathrm{WF}(G)'(\mathrm{WF}(v)).$$

2. *Recall that we have*

$$\mathrm{WF}(\mathbb{G})' \subset \{(q, q') \in \mathcal{N} \times \mathcal{N} \ : \ q \sim q'\}.$$

3. *Refinement of a strategy initiated by Moretti.*

Proof of the key proposition 1

We have

$$\mathrm{WF}(G(\delta_S \otimes u)) \subset \mathrm{WF}(G)'(\mathrm{WF}(\delta_S \otimes u)). \tag{7}$$

Denoting by $i : S \to M$ the canonical injection, we have:

$$\mathrm{WF}(\delta_S \otimes u) \subset (i^*)^{-1}(\mathrm{WF}(u)) \cup N^*S, \tag{8}$$

where $N^*S = \{(x, \xi) \in T^*M \setminus o \, : \, x \in S, \; \xi_{|T_xS} = 0\}$ is the conormal bundle to S.
Let now $(x_1, \xi_1) \in \mathrm{WF}(G(\delta_S \otimes u))$ with $x_1 \notin S$. By (7) there exists $(x_0, \xi_0) \in \mathrm{WF}(\delta_S \otimes u)$ such that $(x_1, \xi_1) \sim (x_0, \xi_0)$. Since $g_{|T_{x_0}S}$ is positive semi-definite with kernel $\mathbb{R}X(x_0)$, we can find $L \subset T_{x_0}S$ space-like with $T_{x_0}S = L \oplus \mathbb{R}X(x_0)$. The orthogonal $L^{\perp}$ is time-like and 2-dimensional, hence contains two null lines, $\mathbb{R}X(x_0)$ and $\mathbb{R}v$ for $v \in T_{x_0}M$ transverse to S. We can assume that $X(x_0) \cdot g(x_0)v = 1$ and v is future directed.

Proof of the key proposition 2

We fix a basis $(w_1, \dots, w_{n-2})$ of L and denote by $x = (y_1, y_2, y')$, $y' \in \mathbb{R}^{n-2}$ the coordinates in the basis $(v, X(x_0), w_1, \dots, w_{n-2})$ of $T_{x_0}M$.
We have then

$$x \cdot g(x_0) x = 2 y_1 y_2 - y' \cdot h y', \text{ where } h > 0,$$

and consequently, for $\xi_0 = (\eta_1, \eta_2, \eta')$ expressed in dual coordinates, we have

$$\xi_0 \cdot g(x_0)^{-1} \xi_0 = 2\eta_1 \eta_2 - \eta' \cdot h^{-1} \eta'. \tag{9}$$

Since $(x_0, \xi_0) \in \mathcal{N}$, we have $\xi_0 \cdot g^{-1}(x_0) \xi_0 = 0$. On the other hand, from (8) either $(x_0, \eta_2, \eta') \in \mathrm{WF}(u)$ or $\eta_2 = \eta' = 0$, i.e. $(x_0, \xi_0) \in N^*S$. Since h is positive definite and the l.h.s. of (9) vanishes, $\eta_2 = 0$ implies $\eta' = 0$. Therefore we have $\xi_0 = (\eta_1, \eta_2, \eta')$ with either

$$2\eta_1 \eta_2 - \eta' \cdot h \eta' = 0, \quad (x_0, \eta_2, \eta') \in \mathrm{WF}(u), \quad \eta_2 \neq 0, \tag{10}$$

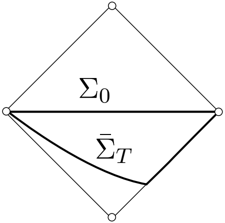
or $(x_0, \xi_0) \in N^*S$.

Proof of the key proposition 3

Let us first consider the second case. The fact that S is null is equivalent to $N^*S \subset \mathcal{N}$, which using the fact that N^*S is a Lagrangian submanifold of T^*M implies that N^*S is invariant under the bicharacteristic flow. Therefore the null bicharacteristic from (x_0, ξ_0) stays in N^*S, hence above S, and thus cannot reach the point (x_1, ξ_1) which is above $M \setminus S$.

Let us now consider the first case. Since by assumption $\mathrm{WF}(u) \subset \{(y, \eta) \in T^*S \setminus o \,:\, \pm\eta \cdot X(y) \geq 0\}$ we deduce from (10) that $\pm\eta_2 > 0$ and $\pm\eta_1 = \frac{1}{2}\eta_2^{-1}\eta' \cdot h\eta' > 0$. Therefore $(x_0, \xi_0) \in \mathcal{N}^{\pm}$ hence $(x_1, \xi_1) \in \mathcal{N}^{\pm}$ since $(x_1, \xi_1) \sim (x_0, \xi_0)$. This completes the proof of the proposition. □

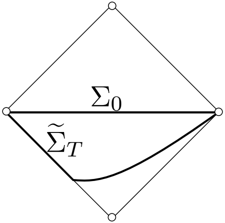
Choices of surfaces



Null geodesics that do not reach $\mathcal{H}^-$ nor $\mathscr{I}^-$ are still problematic.

- However, we can use special form $\chi_{\frac{2\pi}{\kappa_+}}(i^{-1}\mathcal{L}_{\mathcal{H}})$ and $\mathbf{1}_{\mathbb{R}+}(i^{-1}\mathcal{L}_{\mathscr{I}})$ to control wavefront set in region where $v_{\mathcal{H}}$ and $v_{\mathcal{H}}$ are time-like.
- If $\underline{|a|\mathfrak{m}^{-1} \ll 1}$, then all bad null geodesics reach a region where $v_{\mathcal{H}}$ and $v_{\mathscr{I}}$ are both time-like, so we can use propagation of singularities.

Choices of surfaces



Null geodesics that do not reach $\mathcal{H}^-$ nor $\mathscr{I}^-$ are still problematic.

- However, we can use special form $\chi_{\frac{2\pi}{\kappa_+}}(i^{-1}\mathcal{L}_{\mathcal{H}})$ and $\mathbf{1}_{\mathbb{R}+}(i^{-1}\mathcal{L}_{\mathscr{I}})$ to control wavefront set in region where $v_{\mathcal{H}}$ and $v_{\mathcal{H}}$ are time-like.
- If $\underline{|a|\mathfrak{m}^{-1} \ll 1}$, then all bad null geodesics reach a region where $v_{\mathcal{H}}$ and $v_{\mathscr{I}}$ are both time-like, so we can use propagation of singularities.

Summary

Definition

The *Unruh state* ω_{M} is the quasi-free state on $\mathrm{CAR}(\mathrm{M})$ with solution space covariances: $C^{\pm}_{\mathrm{M}} = \mathcal{S}^{-1}_{\mathrm{M}} \left(\mathbf{1}_{\mathbb{R}^{\pm}}(-\imath^{-1}\partial_U) \oplus \mathbf{1}_{\mathbb{R}^{\pm}}(\imath^{-1}\partial_{t^*}) \right) \mathcal{S}_{\mathrm{M}}$.

Theorem (Gérad-H-Wrochna '20)

1. *The Unruh state ω_{M} is a pure state.*
2. *The restriction $\omega_{\mathrm{M_I}}$ of ω_{M} to $\mathrm{M_I}$ has covariances* $C^{\pm}_{\mathrm{M_I}} = \mathcal{S}^{-1}_{\mathrm{M_I}} \left(\chi^{\pm}_{\mathscr{H}^-}(-\imath^{-1}\kappa_+(U\partial_U + \frac{1}{2})) \oplus \chi^{\pm}_{\mathscr{I}^-}(\imath^{-1}\partial_{t^*}) \right) \mathcal{S}_{\mathrm{M_I}}$ *for* $\chi^{\pm}_{\mathscr{I}^-}(\lambda) = \mathbf{1}_{\mathbb{R}^{\pm}}(\lambda)$, $\chi^{\pm}_{\mathscr{H}^-}(\lambda) = \left(1 + \mathrm{e}^{\mp T_{\mathrm{H}}^{-1}\lambda}\right)^{-1}$, *where* $T_{\mathrm{H}} = (2\pi)^{-1}\kappa_+$ *is the Hawking temperature.*

Theorem

There exists $0 < a_0 \leq 1$ such that if $|a|\mathfrak{m}^{-1} < a_0$ then the restriction $\omega_{\mathrm{M_{I\cup II}}}$ of the Unruh state ω_{M} to $\mathrm{M_{I\cup II}}$ is a Hadamard state.

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