Lorentzian IKKT matrix model with the mass term as a nonperturbative formulation of superstring theory

Talk at ESI workshop 2023
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Ref.)
Asano, JN, Piensuk, Yamamori, work in progress
Asano, Chou, JN, Piensuk, Tripathi, Yamamori, work in progress

0. Introduction

IKKT matrix model

Ishibashi-Kawai-Kitazawa-Tsuchiya,
Nucl.Phys.B 498 (1997) 467, hep-th/9612115 [hep-th]

- a nonperturbative formulation of superstring theory
  ``lattice gauge theory'' of everything (matter, force and space-time)

$$S_{\mathrm{b}} = -\frac{1}{4g^2}\mathrm{tr}([A_\mu, A_\nu][A^\mu, A^\nu])$$
$$S_{\mathrm{f}} = -\frac{1}{2g^2}\mathrm{tr}(\Psi_\alpha(\mathcal{C}\,\Gamma^\mu)_{\alpha\beta}[A_\mu, \Psi_\beta])$$

$N \times N$ Hermitian matrices   SO(9,1) symmetry

A_μ $(\mu = 0, \cdots, 9)$ Lorentz vector
Ψ_α $(\alpha = 1, \cdots, 16)$ Majorana-Weyl spinor

Lorentzian metric $\eta = \mathrm{diag}(-1, 1, \cdots, 1)$
is used to raise and lower indices.

- This action can be obtained by taking the zero-volume limit of supersymmetric Yang-Mills theory in 10 dimensions.

the Euclidean IKKT model

"Wick rotation" : $A_0 = -iA_{10}$

Aoki-Iso-Kawai-Kitazawa-Tada ('98)
Hotta-JN-Tsuchiya ('98)

$$Z_{\mathrm{E}} = \int dA\, d\Psi\, e^{-(S_{\mathrm{b}}+S_{\mathrm{f}})} = \int dA\, e^{-S_{\mathrm{b}}}\, \mathrm{Pf}\mathcal{M}(A)$$

$$S_{\mathrm{b}} \propto \mathrm{tr}\,(F_{\mu\nu})^2$$

positive semi-definite!

$$F_{\mu\nu} = -i\,[A_\mu, A_\nu] \;:\; \text{Hermitian}$$

Euclidean model is well defined without any cutoff.

Krauth-Nicolai-Staudacher ('98), Austing-Wheater ('01)

$\mathrm{Pf}\mathcal{M}(A)$: complex valued

➡ Fluctuation of the phase becomes milder for lower dimensional configs.

A possible mechanism for SSB of SO(10) J.N.-Venizzi ('00)

Emergence of 3d space suggested by complex Langevin simulation.

Anagnostopoulos, et al. JHEP 06 (2020) 069 , arXiv: 2002.07410 [hep-th]

Partition function of the Lorentzian IKKT model

Kim-JN-Tsuchiya Phys.Rev.Lett. 108 (2012) 011601, 1108.1540 [hep-th]

partition function

$$Z_{\mathsf{L}} = \int dA\, d\Psi\, e^{i(S_{\mathsf{b}}+S_{\mathsf{f}})} = \int dA\, e^{iS_{\mathsf{b}}}\, \mathrm{Pf}\mathcal{M}(A)$$

This seems to be natural from the connection to the worldsheet theory.

c.f.) $$S = \int d^2\xi \sqrt{g} \left(\frac{1}{4}\{X^\mu, X^\nu\}^2 + \frac{1}{2}\bar{\Psi}\gamma^\mu\{X^\mu, \Psi\} \right)$$

$$\xi_0 \equiv -i\xi_2$$

The worldsheet coordinates should also be Wick-rotated.

We will show that this model, after an appropriate regularization, has surprising properties due to the Lorentz symmetry, which forms a **non-compact group**.

Plan of the talk

0. Introduction
1. How to make the Lorentzian model well-defined
2. Surprising properties due to noncompact symmetry
3. Confirmation in the N=2 bosonic model

   Numerical simulation
   1/D expansion

4. Summary and discussions

1. How to make the Lorentzian model well-defined

Regularizing the Lorentzian model

- Unlike the Euclidean model,
  the Lorentzian model is NOT well defined as it is.

$$Z_{\mathsf{L}} = \int dA\, d\Psi\, e^{i(S_{\mathsf{b}}+S_{\mathsf{f}})} = \int dA\, e^{iS_{\mathsf{b}}} \underbrace{\mathrm{Pf}\,\mathcal{M}(A)}$$

pure phase factor

polynomial in A

real valued unlike Euclidean

- Introducing convergence factor

$$S_{\mathsf{b}} = \frac{1}{4} N \left\{ -2\,\mathrm{tr}(F_{0i})^2 + \mathrm{tr}(F_{ij})^2 \right\}$$

$$S_{\mathsf{b}}^{(\varepsilon)} = \frac{1}{4} N \left\{ -2e^{-i\varepsilon}\,\mathrm{tr}(F_{0i})^2 + e^{i\varepsilon}\,\mathrm{tr}(F_{ij})^2 \right\}$$

This corresponds to deforming the contour as
$$\begin{cases} A_0 &= \mathrm{e}^{-i\frac{3}{4}\varepsilon}\tilde{A}_0 \\ A_i &= \mathrm{e}^{i\frac{1}{4}\varepsilon}\tilde{A}_i \end{cases}$$

$$\varepsilon = \frac{\pi}{2} \iff \text{Euclidean}$$

$$\langle \mathcal{O}(A_0, A_i) \rangle_{\mathsf{L}} = \langle \mathcal{O}(\mathrm{e}^{-i\frac{3\pi}{8}}\tilde{A}_0, \mathrm{e}^{i\frac{\pi}{8}}\tilde{A}_i) \rangle_{\mathsf{E}}$$

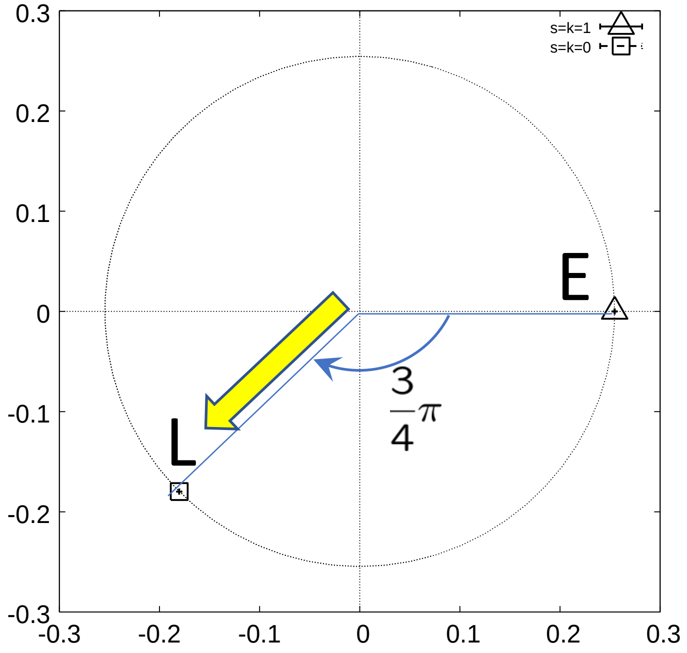
due to Cauchy's theorem

(Yuhma Asano '19, private communication)

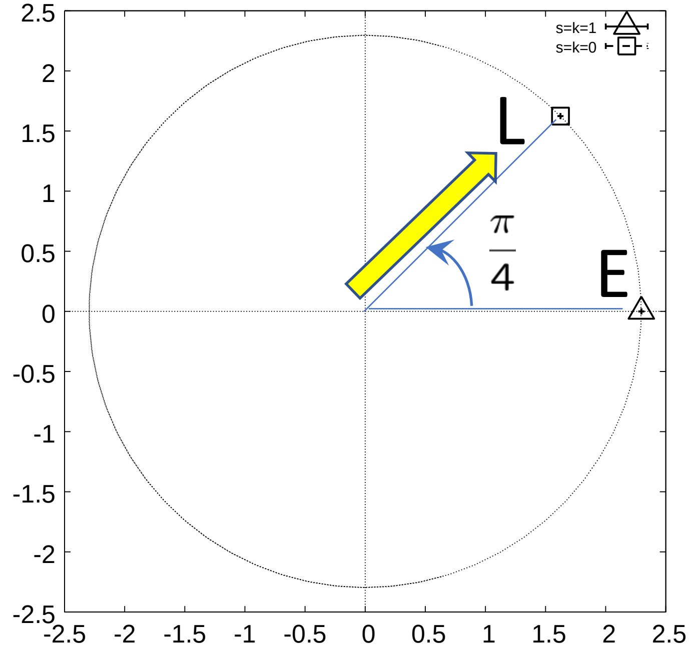
Confirmation of the equivalence by CL simulation

10D bosonic model

$$\left\langle \frac{1}{N}\mathrm{tr}(A_0)^2 \right\rangle_{\mathrm{L}} = e^{-\frac{3\pi}{4}i} \left\langle \frac{1}{N}\mathrm{tr}(\tilde{A}_0)^2 \right\rangle_{\mathrm{E}}$$



$$\left\langle \frac{1}{N}\mathrm{tr}(A_i)^2 \right\rangle_{\mathrm{L}} = e^{\frac{\pi}{4}i} \left\langle \frac{1}{N}\mathrm{tr}(\tilde{A}_i)^2 \right\rangle_{\mathrm{E}}$$



The emergent space-time is complex and has Euclidean signature!

⇒ Can we regularize the Lorentzian IKKT model in a different manner?

Introducing a Lorentz invariant mass term

$$Z = \int dA \, e^{i(S_{\mathrm{b}}+S_{\mathrm{m}})} \mathrm{Pf}\mathcal{M}(A)$$

Anagnostopoulos-Azuma-Hatakeyama-Hirasawa-J.N.-Papadoudis-Tsuchiya, work in progress

$$S_{\mathrm{m}} = \frac{1}{2} N\gamma \left\{ e^{i\varepsilon} \mathrm{tr}(A_0)^2 - e^{-i\varepsilon} \mathrm{tr}(A_i)^2 \right\}$$

convergence factor

c.f.) $S_{\mathrm{b}}^{(\varepsilon)} = \frac{1}{4} N \left\{ -2e^{-i\varepsilon} \mathrm{tr}(F_{0i})^2 + e^{i\varepsilon} \mathrm{tr}(F_{ij})^2 \right\}$

contour deformation to a model with SO(D)

$\gamma < 0$	$\varepsilon \to -0$	○
$\gamma > 0$	$\varepsilon \to +0$	× (leads to unbounded action)

By choosing $\gamma > 0$, we can define the Lorentzian IKKT model in such a way that it is inequivalent to the Euclidean IKKT model.

Indeed, we will see surprising properties of the model for $\gamma > 0$.

Classical solutions

$$Z = \int dA\, e^{i(A^4+\gamma A^2)}$$
$$= \int dA\, e^{i\gamma^2(\tilde{A}^4+\tilde{A}^2)}$$

$$A_\mu = \sqrt{|\gamma|}\tilde{A}_\mu$$

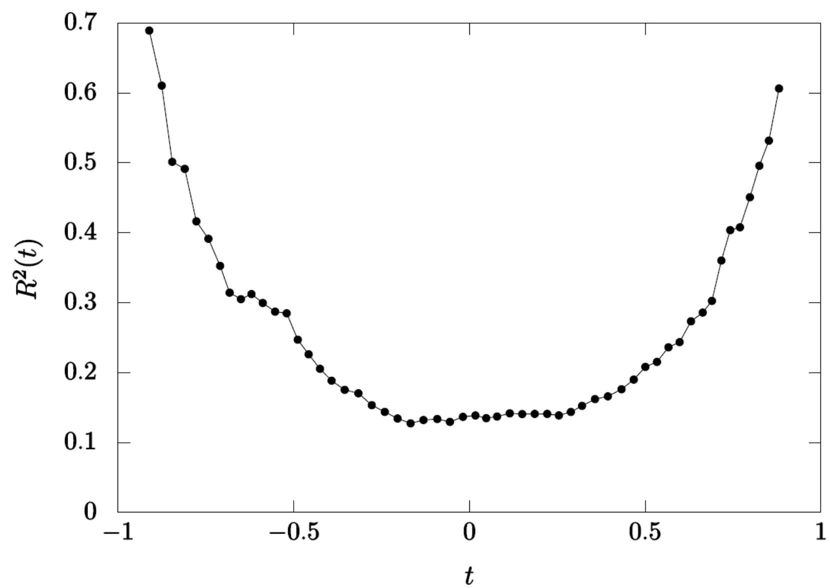
$$\gamma^2 \Leftrightarrow \frac{1}{\hbar}$$

Classical solutions dominate at large $|\gamma|$.

Eq. of motion : $[A^\nu, [A_\nu, A_\mu]] - \gamma A_\mu = 0$

Hatakeyama-Matsumoto-J.N.-Tsuchiya-Yosprakob, *PTEP* 2020 (2020) 4, 043B10

- $A_\mu = 0$ is always a solution.
  (trivial saddle)
- Typical Hermitian A_μ solutions show expanding behavior for $\gamma > 0$
  Not true for $\gamma < 0$!
- Space-time dimensionality cannot be determined classically.



Non-trivial saddle point with expanding behavior may dominate the path integral due to its large entropy in the $N \to \infty$, $\gamma \to 0$ limits.

A historical remark

Mass term is introduced to obtain interesting classical solutions.

- H. C. Steinacker, *Gravity as a quantum effect on quantum space-time*, *Phys. Lett. B* **827** (2022) 136946, [arXiv:2110.03936].
- H. C. Steinacker, *Cosmological space-times with resolved Big Bang in Yang-Mills matrix models*, *JHEP* **02** (2018) 033, [arXiv:1709.10480].
- S.-W. Kim, J. Nishimura, and A. Tsuchiya, *Expanding universe as a classical solution in the Lorentzian matrix model for nonperturbative superstring theory*, *Phys. Rev.* **D86** (2012) 027901, [arXiv:1110.4803].
- S.-W. Kim, J. Nishimura, and A. Tsuchiya, *Late time behaviors of the expanding universe in the IIB matrix model*, *JHEP* **10** (2012) 147, [arXiv:1208.0711].
- H. C. Steinacker, *Quantized open FRW cosmology from Yang-Mills matrix models*, *Phys. Lett.* **B782** (2018) 176–180, [arXiv:1710.11495].
- M. Sperling and H. C. Steinacker, *Covariant cosmological quantum space-time, higher-spin and gravity in the IKKT matrix model*, *JHEP* **07** (2019) 010, [arXiv:1901.03522].

2. Surprising properties due to noncompact symmetry

Rotational v.s. Lorentzian symmetries

- Gaussian integrals with rotational or Lorentz symmetry

$$Z = \int dx\, e^{i\gamma(x_1^2+\cdots+x_{D-1}^2+x_D^2)} \propto \gamma^{-D/2}$$

$$Z = \int dx\, e^{i\gamma(x_1^2+\cdots+x_{D-1}^2-x_0^2)} \propto \gamma^{-D/2}$$

$$x_0 = -i\, x_D$$

Wick rotation

- One might think that the two symmetries are not very different. However, this is actually NOT the case !

$$Z = \int \frac{dk}{2\pi} \int d^D x\, e^{ik(x_1^2+\cdots+x_{D-1}^2+x_D^2-1)} = \frac{1}{2}\,\mathrm{vol}\, S_{D-1} < \infty$$

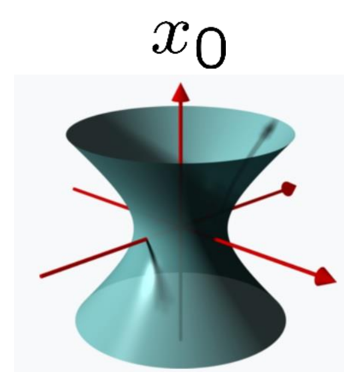
$$S_{D-1} = \{x \in \mathbb{R}^D \,|\, x_1^2 + \cdots + x_{D-1}^2 + x_D^2 = 1)$$

$$Z = \int \frac{dk}{2\pi} \int d^D x\, e^{ik(x_1^2+\cdots+x_{D-1}^2-x_0^2-1)} = \frac{1}{2}\,\mathrm{vol}\, H_{D-1}^{(1)} = \infty$$

$$H_{D-1}^{(1)} = \{x \in \mathbb{R}^D \,|\, x_1^2 + \cdots + x_{D-1}^2 - x_0^2 = 1)$$

hyperboloid of one sheet

Contour deformation is not possible !



Lorentz invariant IKKT model with a mass term

$$Z = \int dA \, e^{i(S_{\mathrm{b}}+S_{\mathrm{m}})} \mathrm{Pf} \mathcal{M}(A)$$

convergence factor

$$S_{\mathrm{m}} = \frac{1}{2} N\gamma \left\{ e^{i\varepsilon} \mathrm{tr}(A_0)^2 - e^{-i\varepsilon} \mathrm{tr}(A_i)^2 \right\}$$

contour deformation to an SO(D) model

$\gamma < 0$	$\varepsilon \to -0$	○
$\gamma > 0$	$\varepsilon \to +0$	× (leads to unbounded action)

According to the previous discussion, it is suggested that

$\gamma < 0 \Rightarrow Z < \infty$

$\gamma > 0 \Rightarrow Z < \infty$ or $Z = \infty$

We will show that the partition function indeed diverges for $\gamma > 0$ in the simplest case of $N = 2$ bosonic model.

3. Confirmation in the N=2 bosonic model
～numerical simulation

Classical solutions for N=2 bosonic model

classical EOM : $[A^\nu, [A_\nu, A_\mu]] - \gamma A_\mu = 0$

For N=2, we can obtain all the solutions up to symmetries.

$\gamma < 0$	$A_\mu = 0$		
$\gamma > 0$	$A_\mu = 0$	$A_\mu = \begin{cases} \sqrt{\frac{\gamma}{2}}\,\sigma_\mu & \mu = 1,2,3 \\ 0 & \text{otherwise} \end{cases}$	$A_\mu = \begin{cases} \sqrt{\gamma}\,\sigma_\mu & \mu = 1,2 \\ 0 & \text{otherwise} \end{cases}$
	(trivial solution)	(Pauli solution)	(squashed Pauli solution)

remaining symmetries

$SO(9,1) \times SU(2)$ unbroken	diagonal subgroup of $SO(3) \times SU(2)$	diagonal subgroup of $SO(2) \times U(1)$

Nontrivial solutions exist only for $\gamma > 0$.

Numerical simulation around the Pauli solution

$$Z = \int dA\, e^{i(S_{\mathrm{b}}+S_{\mathrm{m}})}$$

$$S_{\mathrm{m}} = \frac{1}{2} N\gamma \left\{ e^{i\varepsilon} \mathrm{tr}(A_0)^2 - \mathrm{tr}(A_i)^2 \right\}$$

introduced to regularize the divergence due to Lorentz symmetry

- initial configuration :

$$A_\mu = \begin{cases} \sqrt{\frac{\gamma}{2}}\,\sigma_\mu & \mu = 1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

(Pauli solution)

- results obtained by the generalized Lefschetz thimble method :

(sample configurations only on the thimble associated with the Pauli solution)

$$\left\langle \frac{1}{N} \mathrm{tr}\,(A_0)^2 \right\rangle \sim \frac{c}{\varepsilon}$$

$c \sim 14.7$

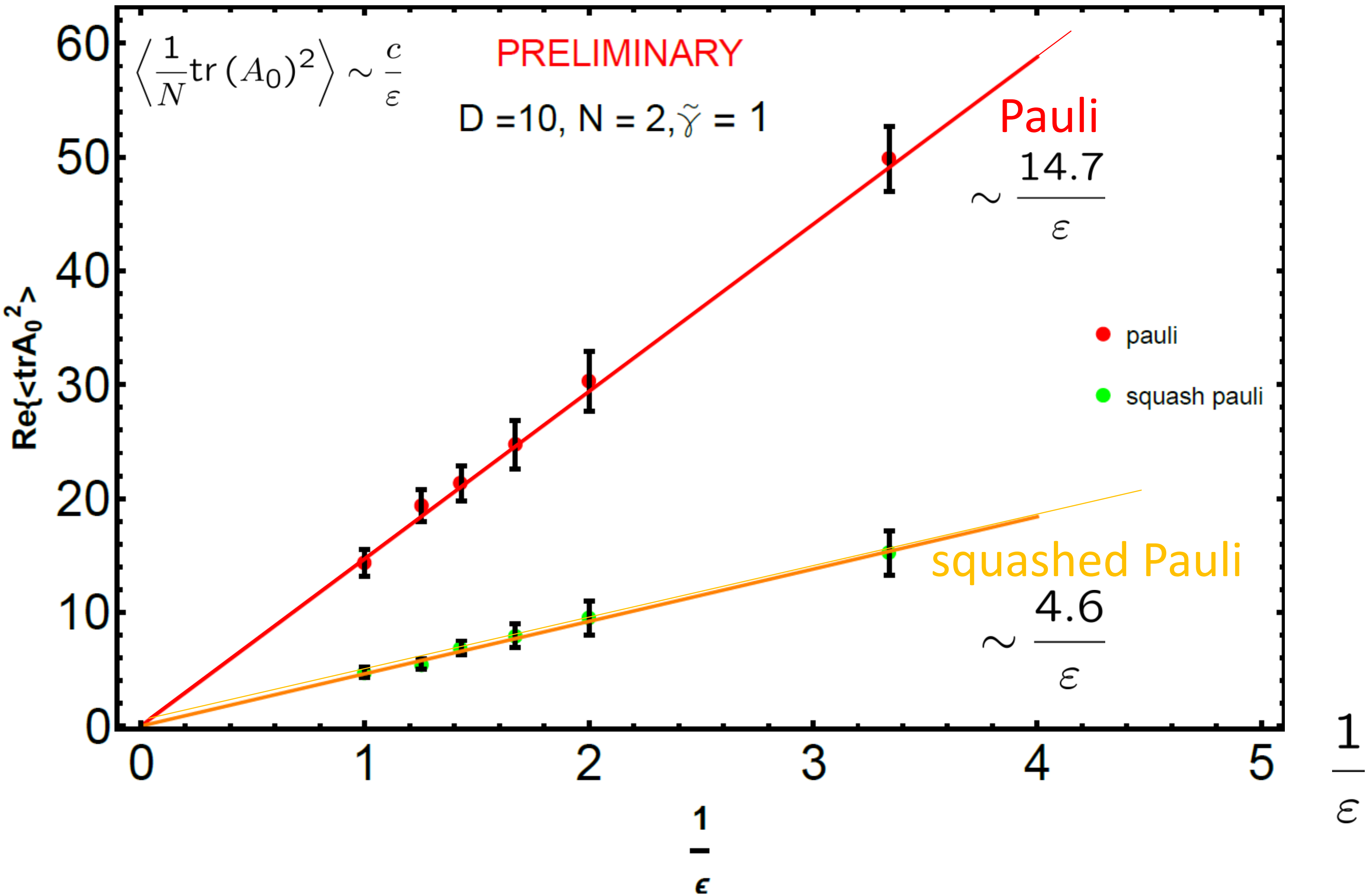
$c = \frac{3}{2} D = 15 \quad (1/D \text{ expansion})$

$$\left\langle -\frac{1}{N} \mathrm{tr}\,(A_0)^2 + \frac{1}{N} \mathrm{tr}\,(A_i)^2 \right\rangle = (\text{finite}) \sim \frac{3}{4}\gamma \quad (\text{at large } \gamma)$$

The divergence due to Lorentz symmetry is clearly confirmed.

Diverging behaviors for Pauli and squashed Pauli

$\tilde{\gamma} = 1$



Pauli has faster diverging behavior than squashed Pauli.

The divergence of partition function

$$\left\langle \frac{1}{N}\mathrm{tr}\,(A_0)^2 \right\rangle \sim -\frac{\partial}{\partial\varepsilon}\log Z$$

$$\left\langle \frac{1}{N}\mathrm{tr}\,(A_0)^2 \right\rangle \sim \frac{c}{\varepsilon} \quad \Rightarrow \quad Z \sim \varepsilon^{-c}$$

Pauli	$c \sim 14.7$	Partition function diverges faster for the Pauli thimble !
squashed Pauli	$c \sim 4.6$	

This implies that Pauli thimble dominates in the $N = 2$ bosonic model at $\gamma > 0$.

Note: This does not mean that the model is ill defined.
E.g., the expectation value $\langle \mathrm{tr}\,(A_\mu A^\mu) \rangle$ is finite.

3. Confirmation in the N=2 bosonic model

～ 1/D expansion

1/D expansion

$$A_\mu = \sum_{a=1}^{N^2-1} A_\mu^a t^a \qquad h_{ab} \sim A_\mu^a A^{\mu b}$$

Used in the Euclidean model without the mass term

Hotta-J.N.-Tsuchiya ('98)

$$\begin{aligned} Z &= \int dA\, e^{i(A^4+\gamma A^2)} \\ &= \int dh \int dA\, e^{i(h^2+hA^2+\gamma A^2)} \\ &= \int dh\, e^{ih^2-\frac{D}{2}\log\det K} \\ &= \int d\tilde{h}\, e^{-DS_{\text{eff}}[\tilde{h}]} \end{aligned}$$

$$\boxed{\begin{aligned} \tilde{h}_{ab} &= \frac{1}{\sqrt{D}} h_{ab} \\ \tilde{\gamma} &= \frac{1}{\sqrt{D}} \gamma \end{aligned}}$$

D appears here only as a parameter.

At large D with fixed $\tilde{\gamma}$,

$$\frac{\partial S_{\text{eff}}[\tilde{h}]}{\partial \tilde{h}_\mu} = 0 \quad \Rightarrow \quad \tilde{h} + iK^{-1} = 0$$

Large D saddles for N=2 bosonic model

Large D SPE : $\tilde{h} + iK^{-1} = 0$

For N=2, we can obtain all the relevant saddle points up to symmetries.

$\gamma < 0$	$\tilde{h} = v^{(+)}\mathbf{1}$		
$\gamma > 0$	$\tilde{h} = v^{(-)}\mathbf{1}$	$\tilde{h} = v^{(+)}\mathbf{1}$	$\tilde{h} = \tilde{\gamma}\,\mathrm{diag}\left(1, 1, \frac{i}{\tilde{\gamma}^2}\right)$

$$v^{(\pm)} \equiv \frac{\tilde{\gamma} \pm \sqrt{\tilde{\gamma}^2 + 8i}}{4}$$

Note that these are complex saddles！

remaining symmetries

SU(2) SU(2) U(1)

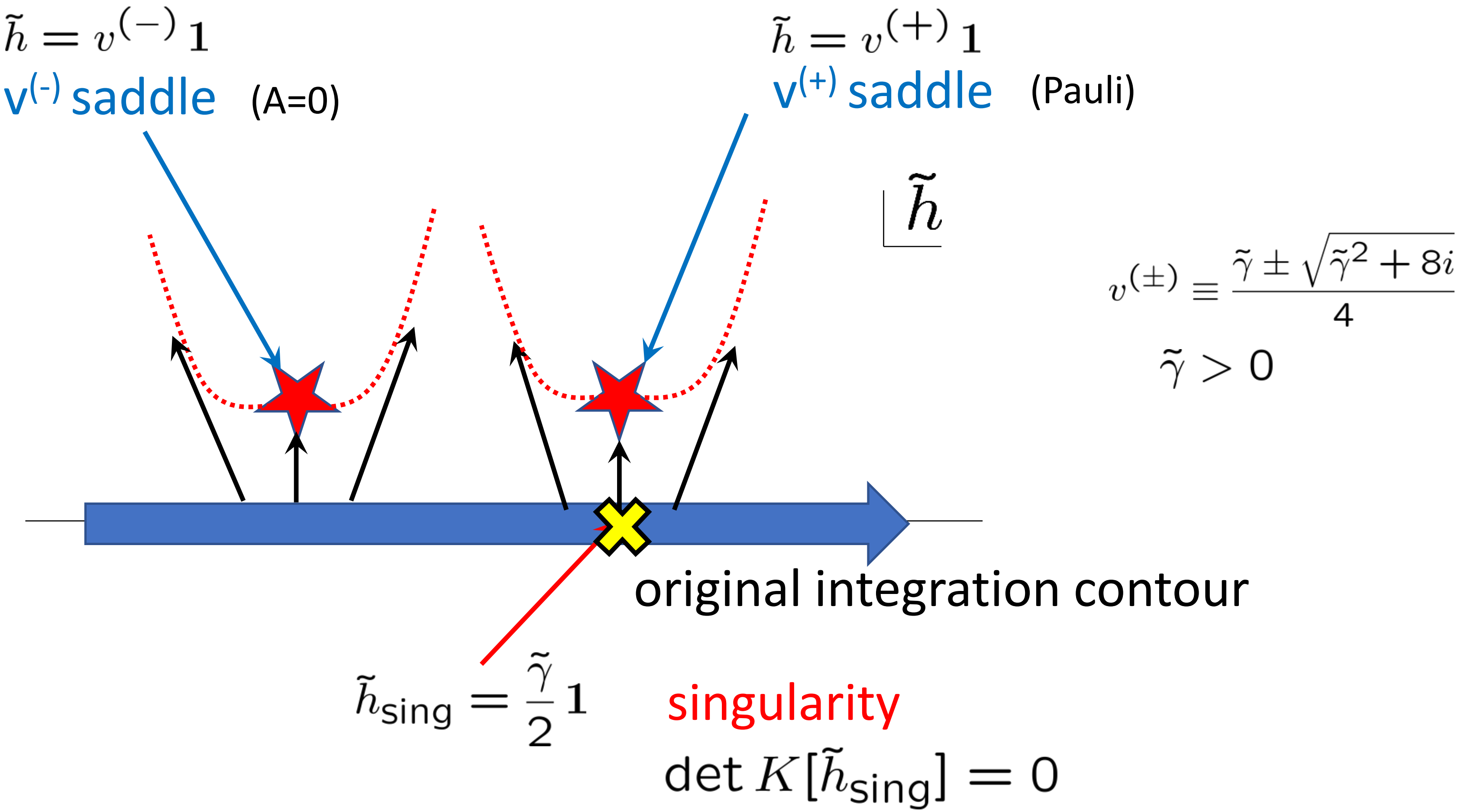
identification

trivial solution
SO(9, 1) × SU(2)
unbroken

Pauli solution
diagonal subgroup of
SO(3) × SU(2)

squashed Pauli solution
diagonal subgroup of
SO(2) × U(1)

Singularity on the real axis



This simply reflects the fact that a model like $Z = \int dA\, e^{i\,S}$ is not well defined because the integral is NOT absolutely convergent.

Also true for the SO(D) invariant case !

The case of SO(D) invariant model obtained by replacing $A_0 = iA_D$

$$S_{\mathrm{m}} = -\frac{1}{2} N\gamma\, e^{-i\varepsilon} \left\{ \mathrm{tr}(A_D)^2 + \mathrm{tr}(A_i)^2 \right\}$$

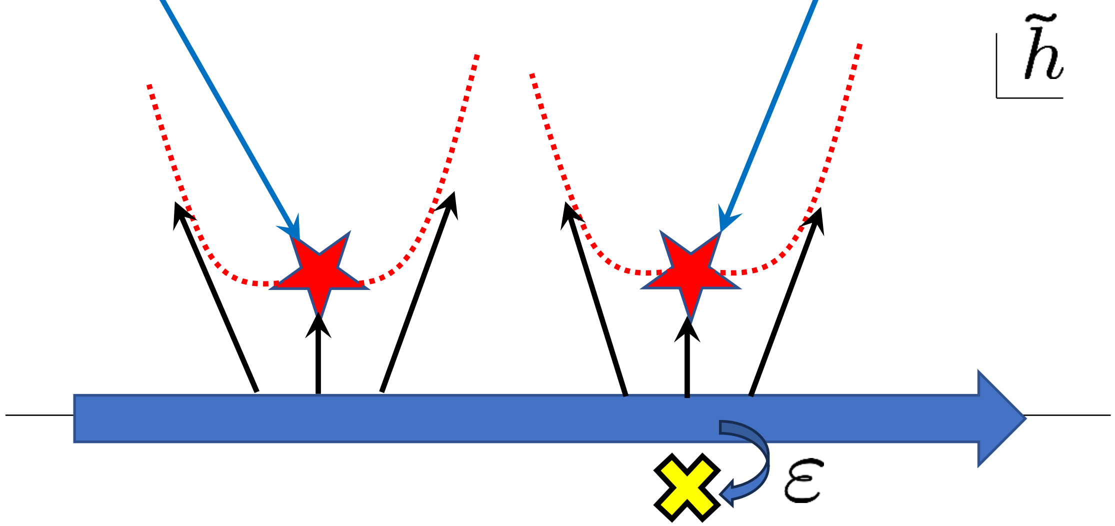
$$Z(v^{(+)}) \sim e^{-\frac{3}{8} i D \tilde{\gamma}^2} \left(\frac{\tilde{\gamma}}{2}\right)^{\frac{3}{2}D}$$ (large D)

$\tilde{h} = v^{(-)}\mathbf{1}$

$v^{(-)}$ saddle (A=0)

$\tilde{h} = v^{(+)}\mathbf{1}$

$v^{(+)}$ saddle (Pauli)



$$v^{(\pm)} \equiv \frac{\tilde{\gamma} \pm \sqrt{\tilde{\gamma}^2 + 8i}}{4}$$

$\tilde{\gamma} > 0$

The $v^{(+)}$ saddle becomes relevant and the associated partition function becomes finite in the $\varepsilon \to 0$ limit.

For $\tilde{\gamma} < 0$, the $v^{(+)}$ saddle becomes irrelevant since the singularity is shifted in the opposite direction.

consistent with the existence of the Pauli solution only for $\tilde{\gamma} > 0$.

The case of Lorentz invariant model

$$S_{\mathrm{m}} = \frac{1}{2} N\gamma \left\{ e^{i\varepsilon}\, \mathrm{tr}(A_0)^2 - e^{-i\varepsilon}\, \mathrm{tr}(A_i)^2 \right\}$$

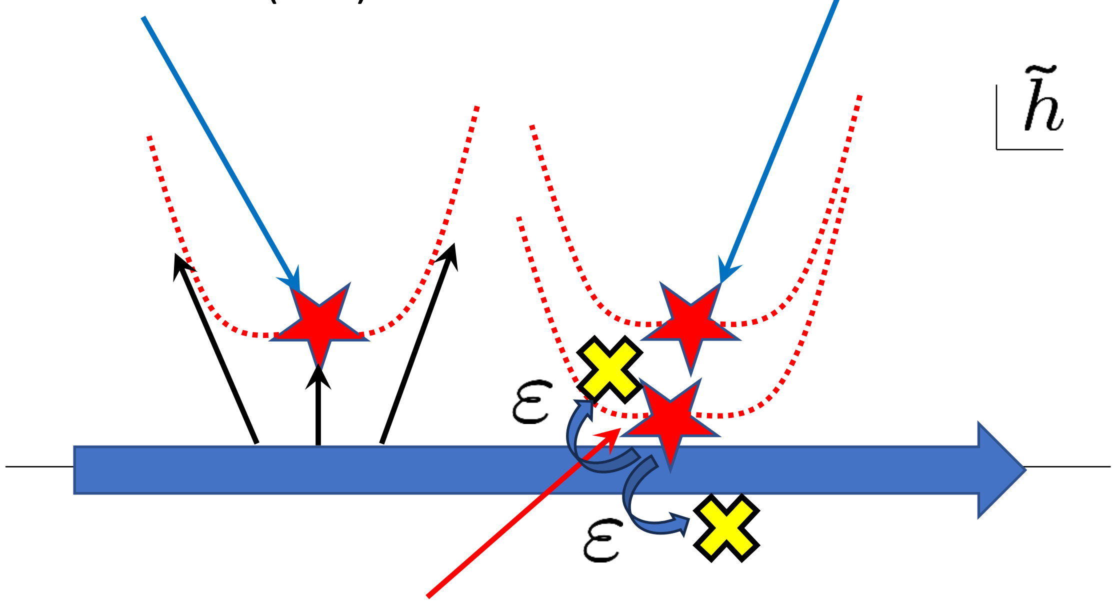
$\tilde{h} = v^{(-)}\mathbf{1}$

v$^{(-)}$ saddle (A=0)

$$Z(v^{(+)}) \sim e^{-\frac{3}{8}iD\tilde{\gamma}^2} \left(\frac{\tilde{\gamma}}{2}\right)^{\frac{3}{2}D}$$ (large D)

$\tilde{h} = v^{(+)}\mathbf{1}$

v$^{(+)}$ saddle (Pauli)



$$v^{(\pm)} \equiv \frac{\tilde{\gamma} \pm \sqrt{\tilde{\gamma}^2 + 8i}}{4}$$

$\tilde{\gamma} > 0$

Convergence factor acts on space and time differently.

new saddle point appears near $\tilde{h}_{\mathrm{sing}} = \frac{\tilde{\gamma}}{2}\mathbf{1}$

$$Z_{\mathrm{new}} \sim e^{-\frac{3}{8}iD\tilde{\gamma}^2} (\tilde{\gamma}\varepsilon)^{-\frac{3}{2}D}$$ (large D)

diverges as $\varepsilon \to 0$

The new saddle point dominates in the $\varepsilon \to 0$ limit.

Transition at finite ε

$$\begin{cases} Z(v^{(+)}) \sim e^{-\frac{3}{8}iD\tilde{\gamma}^2} \left(\dfrac{\tilde{\gamma}}{2}\right)^{\frac{3}{2}D} \\ Z_{\text{new}} \sim e^{-\frac{3}{8}iD\tilde{\gamma}^2} (\tilde{\gamma}\,\varepsilon)^{-\frac{3}{2}D} \end{cases} \Rightarrow$$

$$|Z_{\text{new}}| > |Z(v^{(+)})|$$

for $\tilde{\gamma} < \tilde{\gamma}_{\text{c}} \equiv \sqrt{\dfrac{2}{\varepsilon}}$ (large D)

Probing this transition by calculating observables

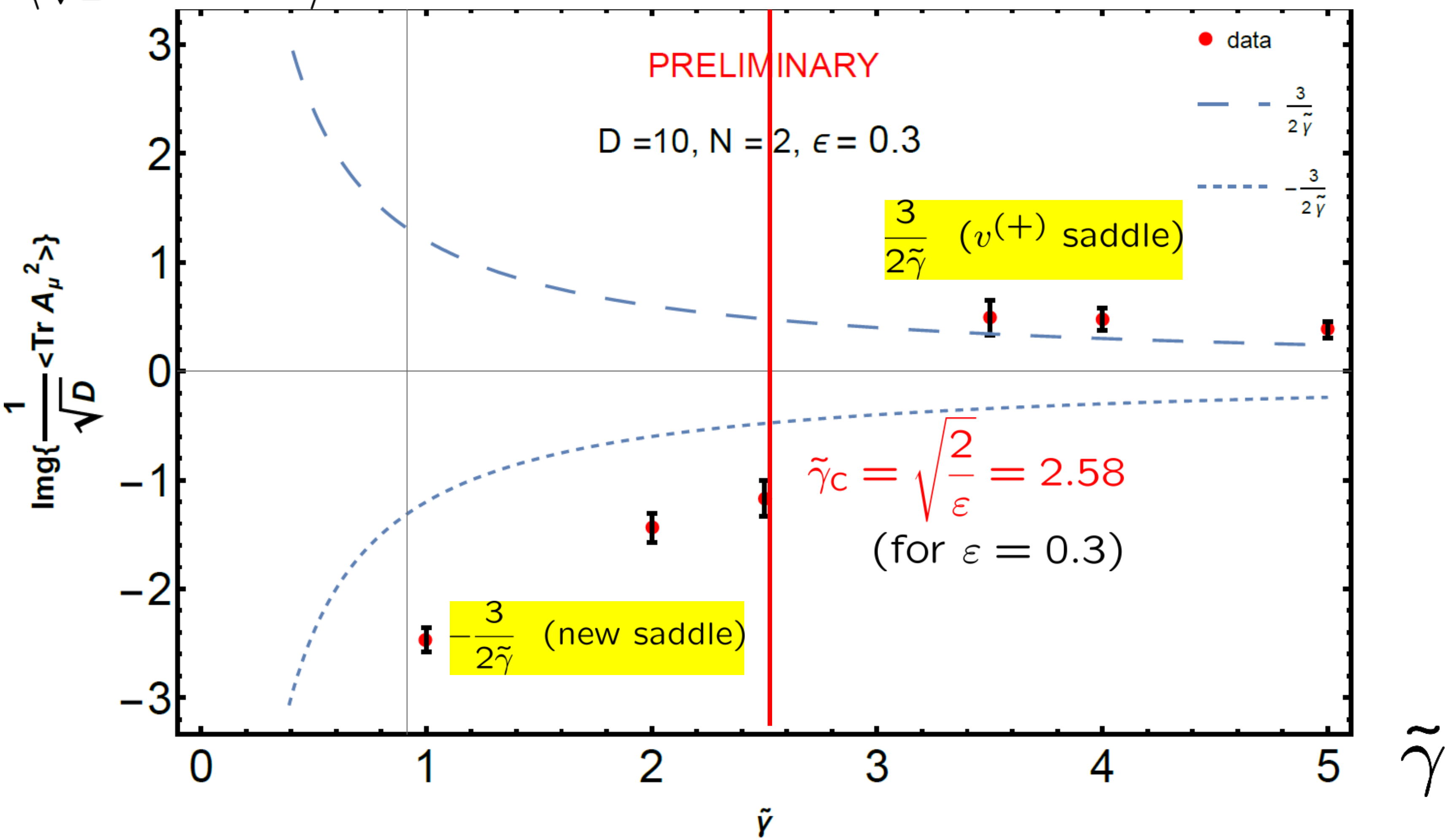
$$\left\langle \frac{1}{\sqrt{D}} \text{tr}\,(A_\mu A^\mu) \right\rangle = \frac{2\,i}{ND} \frac{\partial}{\partial \tilde{\gamma}} \log Z$$

$$= \begin{cases} \frac{3}{4}\tilde{\gamma} + i\frac{3}{2\tilde{\gamma}} & \text{for the } v^{(+)} \text{ saddle} \\ \frac{3}{4}\tilde{\gamma} - i\frac{3}{2\tilde{\gamma}} & \text{for the new saddle} \end{cases}$$

$$\text{Im} \left\langle \frac{1}{\sqrt{D}} \text{tr}\,(A_\mu A^\mu) \right\rangle = -\frac{3}{2\tilde{\gamma}} \to \frac{3}{2\tilde{\gamma}} \quad \text{at } \tilde{\gamma} \sim \sqrt{\frac{2}{\varepsilon}}$$

Results for the Pauli thimble

$\mathrm{Im}\left\langle \frac{1}{\sqrt{D}} \mathrm{tr}\left(A_\mu A^\mu\right) \right\rangle$



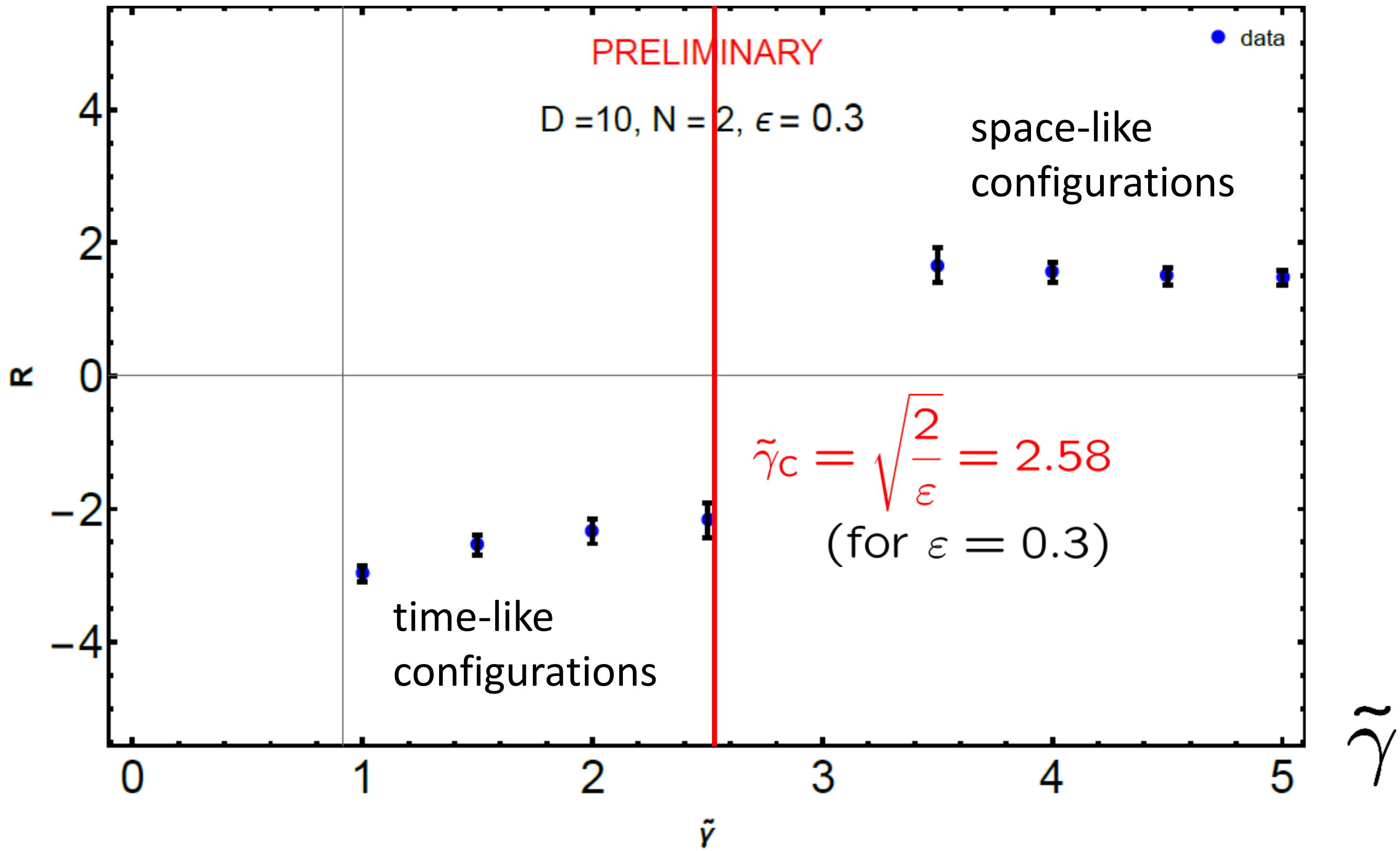
The transition is clearly confirmed by the thimble calculations.

What is this transition ?

time-like configs. v.s. space-like configs.

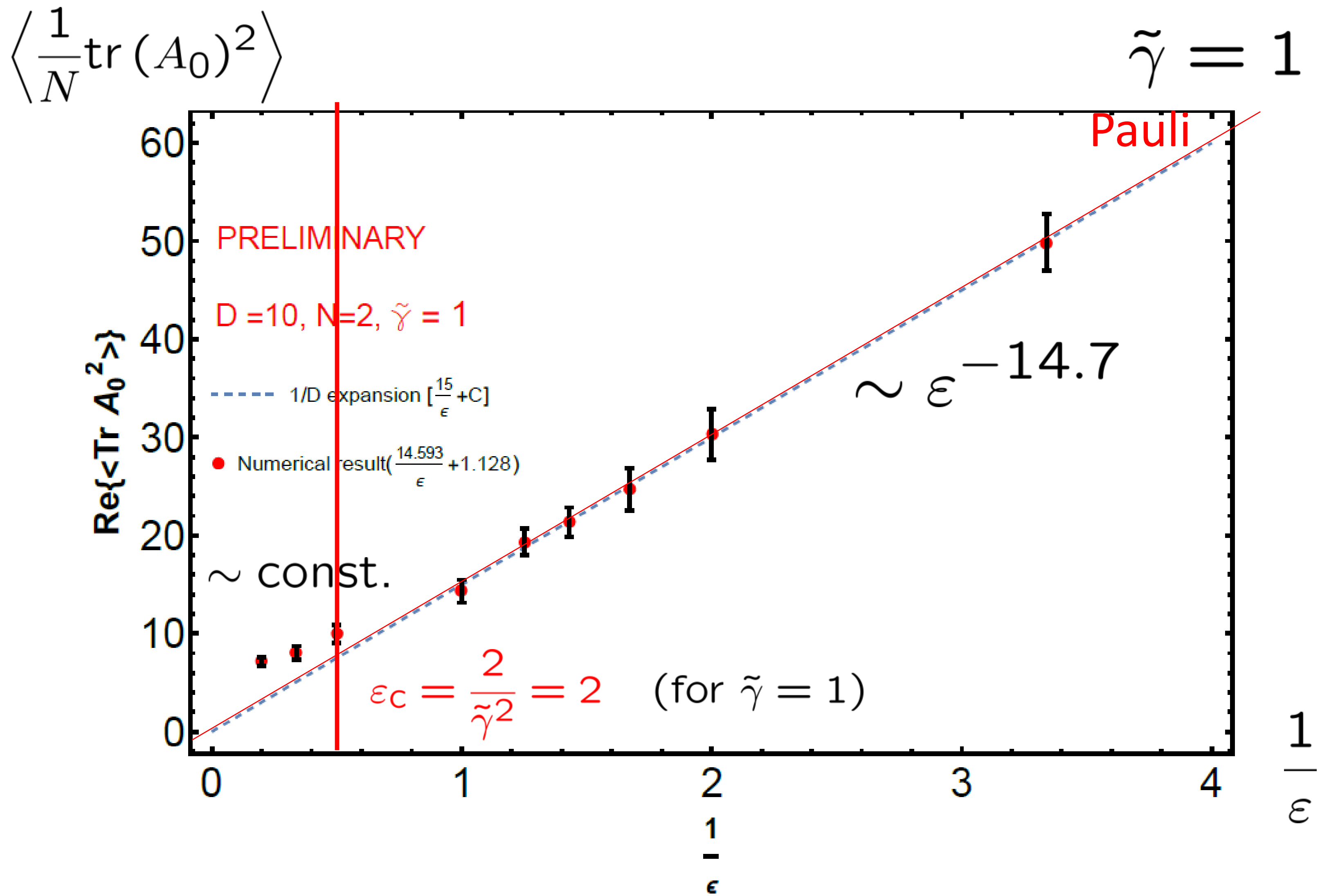
$$R = -\mathrm{tr}\,(A_0^\dagger A_0) + \mathrm{tr}\,(A_i^\dagger A_i)$$

$$\begin{cases} R < 0 & \text{time-like config.} \\ R > 0 & \text{space-like config.} \end{cases}$$



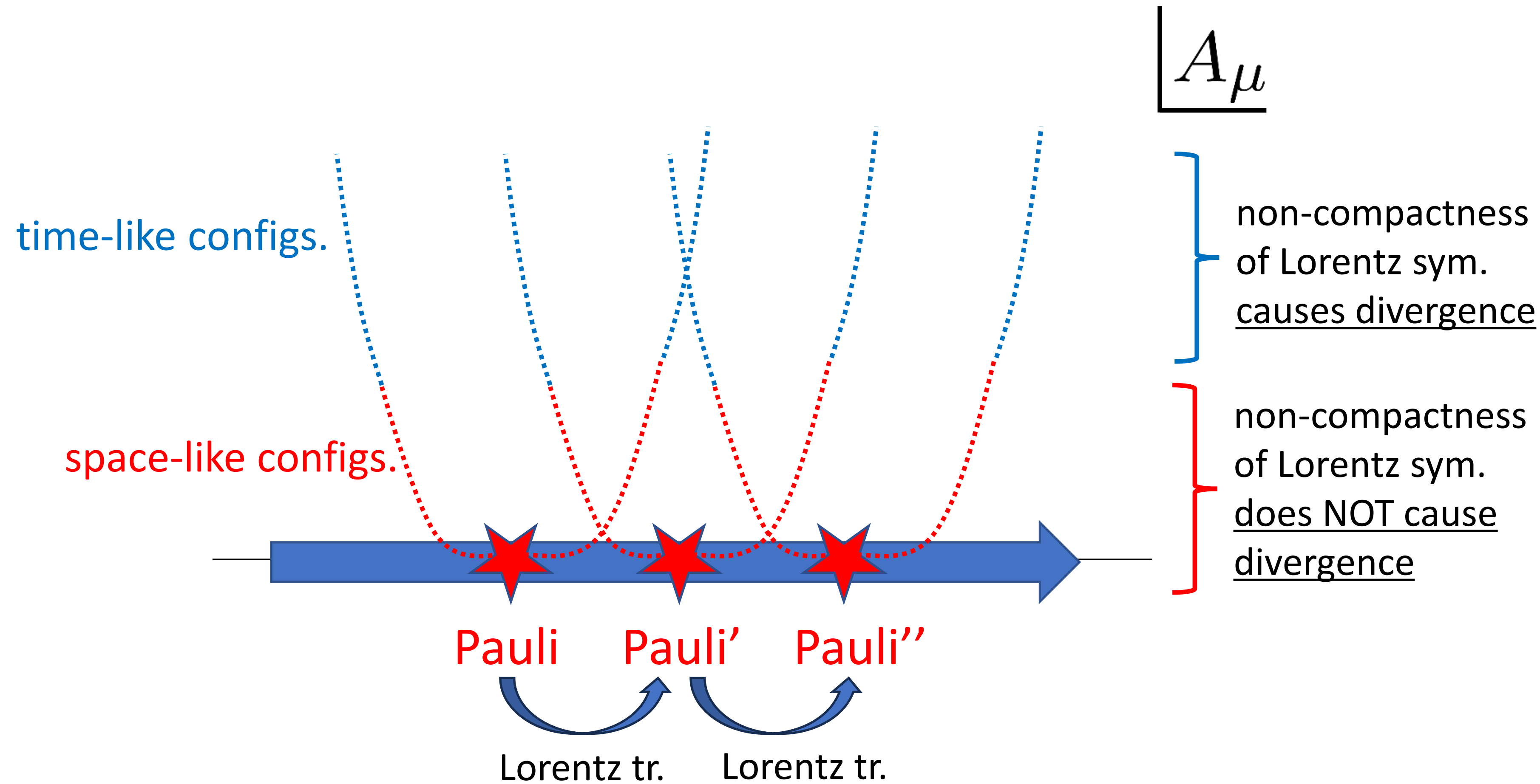
The divergence is caused by time-like configs !

No divergence for space-like configs.



For $\tilde{\gamma} = \infty$, we expect to see $\left\langle \frac{1}{N}\mathrm{tr}\,(A_0)^2 \right\rangle \to$ const. as $\varepsilon \to 0$.

The interpretation of the transition



$\varepsilon \to 0$ makes time-like configs dominate.

$\tilde{\gamma} \to \infty$ makes space-like configs dominate.

transition at $\tilde{\gamma}^2 \varepsilon = 2$

(large D)

4. Summary

Summary

- Lorentzian IKKT matrix model is not well defined as it is unlike the Euclidean version studied earlier.
- A naïve regularization makes it equivalent to the Euclidean model.
- We have proposed a regularization using a Lorentz invariant mass term, which makes it inequivalent to the Euclidean model.
- In the N=2 bosonic model, the partition function for the Pauli thimble diverges due to the non-compact Lorentz transformations of time-like configurations.
- As a result, the Pauli thimble dominates for any $\gamma > 0$.
- The SO(D) symmetric model obtained by replacing $A_0 = iA_D$ does not have these properties, which are, hence, of genuine Lorentzian nature.

Future prospects

- SUSY case

  1/D expansion cannot be applied (SUSY cannot be respected),
  but numerical simulation is doable. N=2 case is on-going.

- larger N

  The computational cost of the generalized Lefschetz thimble method
  grows with N as $O(N^6)$. But we may still do N=4,8,16,...

- Numerical solutions of the classical equation of motion

  Expanding behavior is obtained generically for $\gamma > 0$.
  The number of expanding directions is arbitrary.

  Hatakeyama-Matsumoto-J.N.-Tsuchiya-Yosprakob, *PTEP* 2020 (2020) 4, 043B10

- Numerical simulation based on the complex Langevin method

  The bosonic model simulation suggests (1+1)D expanding space-time.
  Pfaffian suppresses configs. with not more than 2 extended directions.

Kostas Anagnostopoulos' talk