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Programme on

"Optimal Transport"

April 15 – June 14, 2019

organized by

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Workshop 2
Optimal Transport in Analysis and Probability

June 3 – 7, 2019

- **Monday, June 3, 2019**

  09:30 – 10:00 **Opening & Registration**

  10:00 – 10:45 **Allen Tannenbaum**
  Optimal Mass Transport with Applications to the Robustness of Networks and Machine Learning

  10:45 – 11:15 *Coffee / Tea break*

  11:15 – 12:00 **Soumik Pal**
  On the difference between entropic cost and the optimal transport cost

  12:00 – 14:15 *Lunch break*

  14:15 – 15:00 **Alfred Galichon**
  Gale-Shapley meet Monge-Kantorovich

  15:00 – 15:45 **Sigrid Källblad**
  Stochastic control of measure-valued martingales with applications to robust finance

  15:45 – 16:15 *Coffee / Tea break*

  16:15 – 17:00 **Nizar Touzi**
  On the continuous-time Principal-Agent problem

  17:00 – 19:00 *Reception*

- **Tuesday, June 4, 2019**

  09:15 – 10:00 **Almut Burchard**
  How to take derivatives along displacement interpolants

  10:00 – 10:45 **Max Fathi**
  A new proof of the Caffarelli contraction theorem

  10:45 – 11:15 *Coffee / Tea break*

  11:15 – 12:00 **Bo'az Klartag**
  Complex Legendre Duality

- **Wednesday, June 5, 2019**

  09:15 – 10:00 **Giuseppe Savaré**
  Contraction and regularizing properties of heat flows in metric measure spaces

  10:00 – 10:45 **Karl-Theodor Sturm**
  Gradient estimates for the Neumann heat flow on non-convex domains of metric measure spaces

  10:45 – 11:15 *Coffee / Tea break*

  11:15 – 12:00 **Matias Delgadino**
  Mean field limit by Gamma convergence

- **Thursday, June 6, 2019**

  09:15 – 10:00 **José Carrillo**
  Primal dual methods for Wasserstein gradient flows

  10:00 – 10:45 **Jan Haskovec**
  Rigorous continuum limit for the discrete network formation problem

  10:45 – 11:15 *Coffee / Tea break*

  11:15 – 12:00 **Franca Hoffmann**
  Kalman-Wasserstein Gradient Flows

  12:00 – 14:00 *Lunch break*

  14:00 – 14:45 **Young-Heon Kim**
  The Monge problem in Brownian stopping optimal transport

  14:45 – 15:30 **Aaron Palmer**
  Stochastic Transport with Optimal Stopping

  15:30 – 16:00 *Coffee break*

  16:00 – 16:45 **Federico Glaudo**
  A criterion for the stability of optimal maps with applications to the random matching problem

- **Friday, June 7, 2019**

  09:15 – 10:00 **Codina Cotar**
  Equality of the Jellium and Uniform Electron Gas next-order asymptotic terms for Coulomb and Riesz potentials

  10:00 – 10:45 **Julio Backhoff**
  Bayesian learning with Wasserstein barycenters

  10:45 – 11:15 *Coffee / Tea break*

  11:15 – 12:00 **Guillaume Carlier**
  On the well-posedness of the multi-marginal Schrödinger system

All talks take place at ESI, Boltzmann Lecture Hall.

Abstracts

Julio Backhoff (University of Vienna)
Bayesian learning with Wasserstein barycenters

A novel paradigm for Bayesian learning is introduced based on optimal transport theory. Namely, we propose to use the Wasserstein barycenter of the posterior law on models as a predictive posterior, thus introducing an alternative to classical choices like the maximum a posteriori estimator and the Bayesian model average. We exhibit conditions granting the existence and statistical consistency of this estimator, and provide insight into its theoretical advantages. Finally, we introduce a novel numerical method which is ideally suited for the computation of our estimator. This method can be seen as a stochastic gradient descent algorithm in the Wasserstein space.

Almut Burchard (University of Toronto)
How to take derivatives along displacement interpolants

Displacement interpolation connects pairs of probability measures on a manifold by moving mass along geodesics whose endpoints are matched by an optimal transportation plan. The resulting paths play the role of geodesics in the space of probability measures. In this talk, I will describe recent work with Benjamin Schachter on differentiating functionals (such as the entropy or the Dirichlet integral) along displacement interpolants. Our main results are conditions for the existence of higher-order derivatives and recursive formulas for their computation.

Guillaume Carlier (Université Paris-Dauphine)
On the well-posedness of the multi-marginal Schrödinger system

Entropic regularization of optimal transport has become very popular in recent years in particular because it leads to fast numerical methods. It is interesting to note that the key ideas behind this approximation go back to Schrödinger in the 1930s. The analogue of Kantorovich potentials, called Schrödinger potentials are obtained by solving a system of nonlinear integral equations. The fact that this system is well posed is well-known for the two marginal case I will explain how to deal with the multi-marginal case by some global inverse functions arguments.

This is based on a joint work with Maxime Laborde (McGill, Montréal)

José Carrillo (Imperial College London)
Primal dual methods for Wasserstein gradient flows

Combining the classical theory of optimal transport with modern operator splitting techniques, I will present a new numerical method for nonlinear, nonlocal partial differential equations, arising in models of porous media, materials science, and biological swarming. Using the JKO scheme, along with the Benamou-Brenier dynamical characterization of the Wasserstein distance, we reduce computing the solution of these evolutionary PDEs to solving a sequence of fully discrete minimization problems, with strictly convex objective function and linear constraint. We compute the minimizer of these fully discrete problems by applying a recent, provably convergent primal dual splitting scheme for three operators. By leveraging the PDE's underlying variational structure, our method overcomes traditional stability issues arising from the strong nonlinearity and degeneracy, and it is also naturally positivity preserving and entropy decreasing. Furthermore, by transforming the traditional linear equality constraint, as has appeared in previous work, into a linear inequality constraint, our method converges in fewer iterations without sacrificing any accuracy. Remarkably, our method is also massively parallelizable and thus very efficient in resolving high dimensional problems. We prove that minimizers of the fully discrete problem converge to minimizers of the continuum JKO problem as the discretization is refined, and in the process, we recover convergence results for existing numerical methods for computing Wasserstein geodesics. Finally, we conclude with simulations of nonlinear PDEs and Wasserstein geodesics in one and two dimensions that illustrate the key properties of our numerical method.

Codina Cotar (University College London)
Equality of the Jellium and Uniform Electron Gas next-order asymptotic terms for Coulomb and Riesz potentials

Abstract: We consider two sharp next-order asymptotics problems, namely the asymptotics for the minimum energy for optimal point configurations and the asymptotics for the many-marginals Optimal Transport, in both cases with Riesz costs with inverse power-law long-range interactions. The first problem describes the ground state of a Coulomb or Riesz gas, while the second appears as a semiclassical limit of the Density Functional Theory energy modelling a quantum version of the same system. Recently the second-order term in these expansions was precisely described, and corresponds respectively to a Jellium and to a Uniform Electron Gas model.

The present work shows that for inverse-power-law interactions with power $s \in [d-2, d)$ in d dimensions, the two problems have the same minimum value asymptotically. For the Coulomb case in $d = 3$, our result verifies the physicists' long-standing conjecture regarding the equality of the second-order terms for these two problems. Furthermore, our work implies that, whereas minimum values are equal, the minimizers may be different. Moreover, provided that the crystallization hypothesis in $d = 3$ holds, which is an extension of Abrikosov's conjecture originally formulated in $d = 2$, then our result verifies the physicists' conjectured ≈ 1.4442 lower bound on the famous Lieb-Oxford constant. Our work also rigorously confirms some of the predictions formulated by physicists, regarding the optimal value of the Uniform Electron Gas second-order asymptotic term.

Additionally, we show that on the whole range $s \in (0, d)$, the Uniform Electron Gas second-order constant is continuous in s. Besides, our method provides a novel and robust alternative technique to the screening method of Sandier and Serfaty for the next order term in the Coulomb and Riesz gases problems.

This is based on joint works with Mircea Petrache, available on arxiv: https://arxiv.org/abs/1707.07664

Matias Delgadino (Imperial College London)
Mean field limit by Gamma convergence

In this work we give a proof of the mean-field limit for ?-convex potentials using a purely variational viewpoint. We take advantage that all evolutions of the involved quantities can be written as gradient flows of functionals at different levels: in the set of symmetric probability measures on N variables and in the set of probability measures on probability measures. This basic fact allows us to rely on ?-convergence tools for gradient flows to finish the proof by identifying the limits of the different terms in the Evolutionary Variational Inequalities (EVIs) associated to each gradient flow. The ?-convexity of the potentials is crucial to identify uniquely the limits and in order to derive the EVIs at each description level of the interacting particle system.

This is joint work with J.A. Carrillo and G. Pavliotis.

Max Fathi (University of Toulouse)
A new proof of the Caffarelli contraction theorem

The Caffarelli contraction theorem states that the Brenier map sending the Gaussian measure onto a uniformly log-concave probability measure is lipschitz. In this talk, I will present a new proof, using entropic regularization and a variational characterization of lipschitz transport maps.

Based on joint work with Nathael Gozlan and Maxime Prod'homme.

Alfred Galichon (New York University)
Gale-Shapley meet Monge-Kantorovich

Federico Glaudo (ETH Zurich)
A criterion for the stability of optimal maps with applications to the random matching problem

Let M be a 2-dimensional compact Riemannian manifold with volume measure m such that $\mathrm{m}(M) = 1$. Given a random family of points $(X_1, \ldots, X_n)$ m-uniformly distributed on M, the semi-discrete random matching problem concerns the study of the optimal transport cost/map from m to the empirical measure $\frac{1}{n}\sum \delta_{X_i}$.

In 2016 L. Ambrosio, F. Stra and D. Trevisan proved that the expected value of $W_2^2(\mathrm{m}, \frac{1}{n}\sum \delta_{X_i})$ is asymptotic to $\frac{\log(n)}{4\pi n}$ as $n \to \infty$. We will show how a refined contractivity property of the heat flow makes it possible to simplify the proof and to generalize the result. Moreover we will describe how this approach, together with a new quantitative stability criterion for optimal transport maps, is strong enough to capture also some properties of the optimal map (and not only of the optimal cost).

This is joint work with L. Ambrosio and D. Trevisan.

Jan Haskovec (KAUST)
Rigorous continuum limit for the discrete network formation problem

Motivated by recent papers describing the formation of biological transport networks we study a discrete model proposed by Hu and Cai consisting of an energy consumption function constrained by a linear system on a graph. For the spatially two-dimensional rectangular setting we prove the rigorous continuum limit of the constrained energy functional as the number of nodes of the underlying graph tends to infinity and the edge lengths shrink to zero uniformly. The proof is based on reformulating the discrete energy functional as a sequence of integral functionals and proving their Γ-convergence towards a continuum energy functional.

Franca Hoffmann (Caltech)
Kalman-Wasserstein Gradient Flows

We study a class of interacting particle systems that may be used for optimization. By considering the mean-field limit one obtains a nonlinear Fokker-Planck equation. This equation exhibits a novel gradient structure in probability space, based on a modified Wasserstein distance which reflects particle correlations: the Kalman-Wasserstein metric. This setting gives rise to a methodology for calibrating and quantifying uncertainty for parameters appearing in complex computer models which are expensive to run, and cannot readily be differentiated. This is achieved by connecting the interacting particle system to ensemble Kalman methods for inverse problems.

This is joint work with Alfredo Garbuno-Inigo (Caltech), Wuchen Li (UCLA) and Andrew Stuart (Caltech).

Sigrid Källblad (KTH Stockholm)
Stochastic control of measure-valued martingales with applications to robust finance

Motivated by robust pricing problems in mathematical finance, we consider in this talk a specific martingale optimal transport problem. Our approach is based on reformulating the problem as an optimisation problem over so-called measure-valued martingales (MVMs) enabling the problem to be addressed by use of dynamic programming. In the emerging stochastic control problem MVMs appear as weak solutions to a specific SDE for which we prove existence of solutions; we then show that our control problem satisfies the Dynamic Programming Principle and relate the value function to a certain HJB-type equation. A key motivation for the study of control problems featuring MVMs is that a number of interesting probabilistic problems can be formulated as such optimisation problems; we illustrate this by applying our results to optimal Skorokhod embeddings as well as robust pricing problems.

The talk is based on joint work with A. Cox, M. Larsson and S. Svaluto.

Young-Heon Kim (University of British Columbia)
The Monge problem in Brownian stopping optimal transport

We discuss a recent progress in an optimal Brownian stopping problem, called the optimal Skorokhod embedding problem, which is an active research area especially in relation to mathematical finance. Given two probability measures with appropriate order, the problem considers the stopping time under which the Brownian motion carries one probability measure to the other, while minimizing the transportation cost. We focus on the cost given by the distance between the initial and the final point. A strong duality result of this optimization problem is obtained, which enables us to prove that the optimal stopping time is given by the first hitting time to a barrier determined by the optimal dual solutions.

The main part of this talk is based on joint work with Nassif Ghoussoub (UBC) and Aaron Palmer (UBC).

Bo'az Klartag (Weizmann Institute)
Complex Legendre Duality

The Legendre transform appears in the theory of the real Monge-Ampere equation and optimal transport in Euclidean spaces, since it intertwines between the convex potential of the optimal transport map and the convex potential of the inverse map. The theory of complex interpolation of normed spaces suggests an interesting interplay in the complex setting: Namely, in even dimensions the Legendre transform can also be viewed as a local symmetry of the complex Monge-Ampere equation. We introduce complex generalizations of the classical Legendre transform, which are additional local symmetries of the complex Monge-Ampere equation, each with its own unique fixed point. These new Legendre-type transforms give explicit local isometric symmetries for the Mabuchi metric on the space of Kahler metrics around any real analytic Kahler metric, answering a question originating in Semmes' work.

Joint work with B. Berndtsson, D. Cordero-Erausquin and Y.A. Rubinstein.

Soumik Pal (University of Washington)
On the difference between entropic cost and the optimal transport cost

Consider the Monge-Kantorovich problem of transporting densities ρ_0 to ρ_1 on $\mathbb{R}^d$ with a strictly convex cost function. A popular relaxation of the problem is the one-parameter family called the entropic cost problem. The entropic cost J_h, $h > 0$, is significantly faster to compute and hJ_h is known to converge to the optimal transport cost as h goes to zero. We are interested the rate of convergence. We show that the difference between J_h and $1/h$ times the optimal cost of transport has a pointwise limit when transporting a compactly supported density to another that satisfies a few other technical restrictions. This limit is always given by the relative entropy of ρ_1 with respect to a Riemannian volume measure on $\mathbb{R}^d$ that measures the local sensitivity of the transport map. In the special case of the quadratic Wasserstein transport, this relative entropy is exactly one half of the difference of entropies of ρ_1 and ρ_0. Hence, in that case we complement the results of Adams et al., Duong et al., and Erbar et al., who obtain similar results under gamma convergence. More surprisingly, we demonstrate that this difference of two entropies (plus the cost) is also the limit for the Dirichlet transport introduced by Pal and Wong. The latter can be thought of as a multiplicative analog of the Wasserstein transport and corresponds to a non-local operator. It hints at an underlying gradient flow of entropy, in the sense of Jordan-Kinderlehrer-Otto, even when the cost function is not a metric. The proofs are based of Gaussian approximations to Schrödinger bridges as h approaches zero.

Aaron Palmer (University of British Columbia)
Stochastic Transport with Optimal Stopping

We extend the optimal transport problem to controlled diffusion processes with free end times. The dual problem can be viewed as a free boundary Hamilton-Jacobi-Bellman equation of second order. I will discuss the existence of maximizers to the dual problem in two qualitatively distinct regimes: when the drift is strong (the Lagrangian satisfies $L\ |v|^p$ for $1 < p < 2$) and when the drift is weak ($L\ |v|^p$ for $p > 1 + d/2$). With strong drift, we find that any target measure is reachable as a consequence of Hölder regularity for the dual potential. With weak drift, the density of the process has Sobolev regularity, and we require the initial and target measures to satisfy compatible regularity criteria. I will also discuss cases when the optimal stopping times are hitting times of a barrier (when the Lagrangian is monotonic in time), as well as a stopping time version of the Schrödinger bridge problem (when the Lagrangian is quadratic).

Joint work with S. Dweik, N. Ghoussoub, and Y.H. Kim.

Giuseppe Savaré (University of Pavia)
Contraction and regularizing properties of heat flows in metric measure spaces

We illustrate some novel contraction and regularizing properties of the Heat flow in metric-measure spaces that emphasize an interplay between Hellinger-Kakutani, Kantorovich-Wasserstein and Hellinger-Kantorvich distances. Contraction properties of Hellinger-Kakutani distances and general Csiszar divergences hold in arbitrary metric-measure spaces and do not require assumptions on the linearity of the flow.

When weaker transport distances are involved, we will show that contraction and regularizing effects rely on the dual formulations of the distances and are strictly related to lower Ricci curvature bounds in the setting of $RCD(K, N)$ metric measure spaces. (in collaboration with Giulia Luise)

Karl-Theodor Sturm (University of Bonn)
Gradient estimates for the Neumann heat flow on non-convex domains of metric measure spaces

We briefly recall the Eulerian and the Lagrangian approach to synthetic lower Ricci bounds on metric measure spaces due to Bakry-Emery and Lott-Sturm-Villani, resp., and present recent extensions to spaces with variable lower Ricci bounds. Our main results will be a gradient estimate for the heat flow with Neumann boundary conditions on domains of metric measure spaces obtained through "convexification" of the domains by means of subtle time changes. This improves upon previous results both in the case of non-convex domains and in the case of convex domains.

Allen Tannenbaum (Stony Brook University)
Optimal Mass Transport with Applications to the Robustness of Networks and Machine Learning

In the talk, we will describe some recent work on applications of optimal mass transport (OMT) theory to some key problems in machine learning and the robustness properties of networks, especially those connected to cancer. Some of the techniques will involve certain versions of OMT for Gaussian Mixture Models as well as for vector-valued data. Real world examples will be presented illustrating the theory.

Nizar Touzi (École Polytechnique, Paris)
On the continuous-time Principal-Agent problem

Motivated by the approach introduced by Sanninkov to solve principal-agent problems, we provide a solution approach which allows to address a wider range of problems. The key argument uses a representation result from the theory of backward stochastic differential equations. This methodology extends to the mean field game version of the problem, and provides a connexion with the P.-L. Lions optimal planning problem.