Paracontrolled calculus and regularity structures

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Two approaches to singular PDEs

Singular SPDEs contain ill-posed multiplications, e.g., generalized KPZ equation

$$\partial_t h = \partial_x^2 h + \underbrace{f(h)}_{\frac{1}{2}-}\underbrace{(\partial_x h)}_{-\frac{1}{2}-}{}^2 + \underbrace{g(h)}_{\frac{1}{2}-}\underbrace{\xi}_{-\frac{3}{2}-}$$

Multiplication $C^\alpha \times C^\beta \to C^{\alpha\wedge\beta}$ is well-posed iff $\alpha + \beta > 0$.
$\to$ We need **renormalizations**.

Two approaches

- Regularity structure (Hairer '14)
  $\to$ "Black box" theorem (Bruned-Hairer-Zamotti '19, Chandra-Hairer '16, & Bruned-Chandra-Chevyrev-Hairer '21)
- Paracontrolled calculus (Gubinelli-Imkeller-Perkowski '15)
  $\to$ High order PC (Bailleul-Bernicot '19)

Aim

- To show the equivalence between two approaches.
- To give an algebraic perspective to PC.

RS vs. PC

RS and PC are extensions of the **rough path theory** for SDEs

$$dX = F(X)dB.$$

- RS provides a pointwise description

$$X_t - X_s = F(X_s)(B_t - B_s) + O(|t-s|^{1-}).$$

- PC provides a spectral description

$$X = F(X) \varolessthan B + (C^{1-}).$$

($\varolessthan$: Bony's paraproduct

$$f \varolessthan g = \sum_{i<j-1} \rho(2^{-i}\nabla)f \cdot \rho(2^{-j}\nabla)g,$$

$\rho(2^{-i}\cdot)$ denotes a dyadic decomposition of 1.)

Goal: pointwise description $\Leftrightarrow$ spectral description

Main result (rough)

Rough path theory	RS		PC
Rough path	Model	$\Leftrightarrow$	Pararemainders
Controlled path	Modelled distribution	$\Leftrightarrow$	Paracontrolled distribution
Stochastic integral	Chandra-Hairer	Future work	No systematic theory

Theorem (Bailleul-H '21)

- *(JMSJ '21a) RS $\Rightarrow$ PC in a general algebraic setting.*
- *(JEP '21b) PC $\Rightarrow$ RS under additional (but harmless) assumptions, which are satisfied by* **another** *basis of Bruned-Hairer-Zambotti's algebra.*

Related researches

- Martin-Perkowski '20: modelled $\Leftrightarrow$ "paramodelled".
- Tapia-Zambotti '20: a geometry of the space of branched rough paths.

Algebraic settings

	Rough path theory	RS
Algebra	Connes-Kreimer	Regularity structure
Analysis	Rough path	Model
	Controlled path	Modelled distribution

Branched RP is a continuous path from $[0,T]$ to Butcher group, a character group on Connes-Kreimer algebra.

Generalization

Regularity structure = Hopf algebra T^+ + comodule T.

Hopf algebra T^+ = "Jointing trees" + "Spliting a tree"

= product $(\cdot : T^+ \otimes T^+ \to T^+)$ + coproduct $(\Delta^+ : T^+ \to T^+ \otimes T^+)$.

Comodule T = coproduct $(\Delta : T \to T \otimes T^+)$.

Concrete regularity structure

Definition

A *concrete regularity structure* (T^+, T) *consists of*

1. *Connected graded Hopf algebra* $T^+ = \bigoplus_{\alpha \in A^+} T^+_\alpha$.

$$A^+ \subset [0,\infty) \quad \textit{loc. fin.}, \quad \dim T_0^+ = 1, \quad \dim T_\alpha^+ < \infty,$$
$$T^+_{\alpha_1} \cdot T^+_{\alpha_2} \subset T^+_{\alpha_1+\alpha_2},$$
$$\Delta^+ : T^+ \to T^+ \otimes T^+, \quad \Delta^+ T^+_\alpha \subset \oplus_{\alpha_1+\alpha_2=\alpha}(T^+_{\alpha_1} \otimes T^+_{\alpha_2}).$$

2. *Graded right comodule* $T = \bigoplus_{\beta \in A} T_\beta$.

$$A \subset \mathbb{R} \quad \textit{loc. fin.}, \quad \inf A > -\infty, \quad \dim T_\beta < \infty,$$
$$\Delta : T \to T \otimes T^+, \quad \Delta T_\beta \subset \oplus_{\beta_1+\beta_2=\beta}(T_{\beta_1} \otimes T^+_{\beta_2}).$$

Some remarks

Polynomial regularity structure is an easy example of RS.

- $T^+ = T = \mathbb{R}[X_1, \dots, X_d]$.
- $X^k := \prod_{i=1}^d X_i^{k_i}$, where $k = (k_i)_{i=1}^d \in \mathbb{N}^d$.
- Product $X^k \cdot X^\ell = X^{k+\ell}$.
- Coproduct $\Delta X^k = \sum \binom{k}{\ell} X^\ell \otimes X^{k-\ell}$.

Character group

Since T^+ is a Hopf algebra, the set G of algebra morphisms $g : T^+ \to \mathbb{R}$ forms a group with

- Product $(g_1 * g_2)(\tau) := (g_1 \otimes g_2)\Delta\tau$.
- Inverse $g^{-1} := g \circ S$, S is the antipode of T^+.

$G \curvearrowright T$ by

$$\Gamma_g \tau := (\mathrm{id} \otimes g)\Delta\tau.$$

Original RS by Hairer consists of the pair (T, G).

Models

	Rough path theory	RS
Algebra	Connes-Kreimer	Regularity structure
Analysis	Rough path	Model
	Controlled path	Modelled distribution

Definition

The space $\mathcal{M}$ consists of the pair $M = (g, \Pi)$ such that

- $g : \mathbb{R}^d \ni x \mapsto g_x \in G$ *is a continuous map such that*

$$g_{yx}(\tau) := (g_y * g_x^{-1})(\tau) = O(|y-x|^\alpha), \quad \tau \in T_\alpha^+.$$

- $\Pi : T \to \mathcal{S}'(\mathbb{R}^d)$ *is a bounded operator such that*

$$\Pi_x \tau(y) := (\Pi \otimes g_x^{-1})\Delta\tau(y) = O(|y-x|^\beta), \quad \tau \in T_\beta$$

(in distributional sense).

Modelled distributions

	Rough path theory	RS
Algebra	Connes-Kreimer	Regularity structure
Analysis	Rough path	Model
	Controlled path	Modelled distribution

Definition

For $\gamma \in \mathbb{R}$ and any model $M = (g, \Pi)$, the space $\mathcal{D}^\gamma(g)$ consists of all maps $f : \mathbb{R}^d \to T$ such that

$$\big(f(y) - \Gamma_{g_{yx}} f(x)\big)|_{T_\alpha} = O(|y-x|^{\gamma-\alpha}), \quad \alpha < \gamma.$$

Reconstruction operator is a continuous linear operator $\mathcal{R} = \mathcal{R}^M : \mathcal{D}^\gamma(g) \to \mathcal{D}'(\mathbb{R}^d)$ such that

$$\mathcal{R}f(y) = \big(\Pi_x f(x)\big)(y) + O(|y-x|^\gamma), \quad f \in \mathcal{D}^\gamma(g).$$

$\exists_1$ if $\gamma > 0$ and $\exists$ if $\gamma \neq 0$ (Hairer '14 & Caravenna-Zambotti '20).

From RS to PC

We establish the counterparts of models and modelled distributions in PC.

(g,Π), $\mathcal{D}^\gamma(g)$ (pointwise) $\Longrightarrow$ PC counterparts (spectral)

$\nwarrow \quad \nearrow$

(T, T^+)

Notations

- Fix a homogeneous basis $\mathcal{B}^{(+)}$ of $T^{(+)}$.
- For any $\tau, \sigma \in \mathcal{B}^{(+)}$, we define the element $\tau/\sigma \in T^+$ by

$$\Delta^{(+)}\tau = \sum_{\sigma \in \mathcal{B}^{(+)}} \sigma \otimes (\tau/\sigma).$$

Ex. Bruned-Hairer-Zamotti '19 (applicable to all semilinear parabolic SPDEs)

- $\mathcal{B}^{(+)}$ consists of rooted decorated trees.
- In the expansion of $\Delta^{(+)}\tau$, σ is a subtree of τ and τ/σ is a quotient graph.

Model $\Rightarrow$ Pararemainders

For technical reasons, we consider the Hölder space with polynomial weights. We omit the details here.

Theorem (Bailleul-H '21a)

Let $M = (g, \Pi) \in \mathcal{M}$. There exist continuous linear maps

$$[\cdot]^g : T^+ \to C(\mathbb{R}^d), \quad [\cdot]^M : T \to \mathcal{S}'(\mathbb{R}^d).$$

such that

- *For any $\tau \in T^+_\alpha$, one has $[\tau]^g \in C^\alpha$, and*

$$g(\tau) = \sum_{\eta \in \mathcal{B}^+,\, 0<|\eta|<\alpha} g(\tau/\eta) \otimes [\eta]^g + [\tau]^g.$$

- *For any $\sigma \in T_\beta$, one has $[\sigma]^M \in C^\beta$, and*

$$\Pi\sigma = \sum_{\zeta \in \mathcal{B},\, |\zeta|<\beta} g(\sigma/\zeta) \otimes [\zeta]^M + [\sigma]^M.$$

Modelled distribution $\Rightarrow$ Paracontrolled distribution

Proposition (Bailleul-H '21a)

Let $\gamma \in \mathbb{R}$ *and* $M = (g, \Pi) \in \mathcal{M}$. *For any modelled distribution*

$$f = \sum_{\tau \in \mathcal{B},\, |\tau| < \gamma} f_\tau \tau \in \mathcal{D}^\gamma(g),$$

one has

$$f_\sigma = \sum_{\tau \in \mathcal{B},\, |\sigma| < |\tau| < \gamma} f_\tau \otimes [\tau/\sigma]^g + [f_\sigma]^g, \quad \sigma \in \mathcal{B},$$

with $[f_\sigma]^g \in C^{\gamma - |\sigma|}$. *Moreover, the reconstruction* $\mathcal{R}^M f$ *is of the form*

$$\mathcal{R}^M f = \sum_{\tau \in \mathcal{B},\, |\tau| < \gamma} f_\tau \otimes [\tau]^M + [f]^M,$$

where $[f]^M \in C^\gamma$.

These formulas give an algebraic perspective to the paracontrolled systems (Gubinelli-Imkeller-Perkowski '15, Bailleul-Bernicot '19, etc.).

From PC to RS

Next we show

(g,Π), $\mathcal{D}^\gamma(g)$ (pointwise)	$\Longleftarrow$	$[\tau]^g$, $[\sigma]^M$, $[f_\sigma]^g$ (spectral)
$\nwarrow$	(T,T^+)	$\nearrow$

To show this, we need some additional (but harmless) assumptions on $\mathcal{B}^{(+)}$.

Assumption (rough)

- (T,T^+) *is freely generated by a finite set, "polynomials", and "derivatives".*
- (g,Π) *canonically applies to the polynomials.*

"BHZ algebra = Connes-Kreimer algebra + polynomials + derivatives".
(cf. Bruned-Hairer-Zambotti '19, Bruned-Manchon '21.)

Polynomials and derivatives

Assumption A

Let $\mathcal{B}^{(+)}$ *be a homogeneous basis of* $T^{(+)}$.

1. *There exists a generating set* $\mathcal{G}_\circ^+ \subset \mathcal{B}^+$ *such that, each element* $\tau \in \mathcal{B}^+$ *is uniquely written by*
$$\tau = X^k \prod_{n=1}^{N} (\tau_n / X^{k_n}),$$
where $k, k_1, \ldots, k_N \in \mathbb{N}^d$ *and* $\tau_1, \ldots, \tau_n \in \mathcal{G}_\circ^+$, *up to ordering. Moreover, the splitting map* Δ^+ *has an inductive structure (e.g. scale of the graph).*
2. *There exists a subset* $\mathcal{B}_\bullet \subset \mathcal{B}$ *such that, each element* $\sigma \in \mathcal{B}$ *is uniquely written by*
$$\sigma = X^k \eta,$$
where $k \in \mathbb{N}^d$ *and* $\eta \in \mathcal{B}_\bullet$.
3. *Any nonpolynomial element of* $\mathcal{B}^{(+)}$ *has noninteger homogeneity.*
4. $g_x(X^k) = x^k$, *and* $\Pi(X^k\eta)(x) = x^k(\Pi\eta)(x)$.

BHZ algebra

- $\mathcal{B}_\bullet$: all strongly conform trees with $\mathfrak{n} = 0$ at those roots.
- $\mathcal{G}_\circ^+$: all "planted" trees with $\mathfrak{e} = 0$ at the edges leaving from their roots.

To get PC $\Rightarrow$ RS, we additionally need

Assumption B

For any $\tau \in \mathcal{B}_\bullet$, its coproduct $\Delta\tau$ does not have terms of the form $\sigma \otimes X^k$ with $k \neq 0$.

BHZ algebra does not seem to satisfy this assumption. However,

Proposition (Bailleul-H '21b)

There is another basis of BHZ algebra which satisfies Assumption B.

We exchange $\mathfrak{n}$-decoration for the convolution with $x^k \partial^\ell K(x)$ ($K_\mathfrak{t}$ is the integral kernel, e.g. heat kernel).

Model $\Leftarrow$ Pararemainders

Theorem (Bailleul-H '21b)

Under Assumption A, the subfamilies

$$\{[\tau]^g \in C^{|\tau|}\,;\, \tau \in \mathcal{G}_\circ^+\}, \quad \{[\sigma]^M \in C^{|\sigma|}\,;\, \sigma \in \mathcal{B}_\bullet,\, |\sigma| < 0\}.$$

are sufficient to determine the original model $M = (g, \Pi)$. This inverse map is continuous, so one obtains a homeomorphism

$$\mathcal{M} \simeq \prod_{\tau \in \mathcal{G}_\circ^+} C^{|\tau|} \times \prod_{\sigma \in \mathcal{B}_\bullet,\, |\sigma|<0} C^{|\sigma|}$$

cf. Admissible models (Hairer '14) are determined by only

$$\{[\sigma]^M \in C^{|\sigma|}\,;\, \sigma \in \mathcal{B}_\bullet,\, |\sigma| < 0\},$$

since then T^+ and T are interwined.

Modelled distribution $\Leftarrow$ Paracontrolled distribution

Theorem (Bailleul-H '21b)

Assume that $\gamma \neq 0$ and $\gamma - |\tau| \notin \mathbb{N}$ for any $\tau \in \mathcal{B}$. Under Assumption B, the subfamily

$$\{[f_\sigma]^g\,;\,\sigma \in \mathcal{B}_\bullet,\ |\sigma| < \gamma\}$$

is sufficient to determine the original modelled distribution $f \in \mathcal{D}^\gamma(g)$. This inverse map is continuous, so one obtains a homeomorphism

$$\mathcal{D}^\gamma(g) \simeq \prod_{\tau \in \mathcal{B}_\bullet,\, |\tau| < \gamma} C^{\gamma - |\tau|}.$$

Future works

- Renormalizations in the space $\prod \mathcal{C}^{|\tau|}$?
  - Chandra-Hairer '16: a systematic proof of the convergence of BPHZ models.
  - Bailleul-Bruned '21: BPHZ models $\leftrightarrow$ renormalizations of $[\tau]$'s
    $\to$ Fourier counterpart of Chandra-Hairer?
- Spectral approaches to...
  - SPDEs on Riemannian manifolds: Dahlqvist-Diehl-Driver '19, Bailleul-Bernicot '19.
  - Discrete approximations: Gubinelli-Perkowski '17, Hairer-Matetski '18, Erhard-Hairer '19.
- Application to real analysis.
  - H '20 '21: algebraic perspective of the "iterated commutator estimate".