ESI- Geometry for Higher Spin Gravity

Variational Bicomplex.

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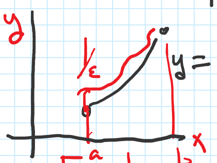
Overview

- Many thanks to Andreas and ESI for the opportunity.
- Lecture 1.
  - Gentle beginning.

    $$\int L(x,y,y')dx$$

  - Jets
  - The "free" variational bicomplex.
  - The first variational formula in the large.
  - Standard Applications
- Lecture 2.
  - Historical Remarks
  - Generalizations
  - Diverse Applications.
    - Brief Presentations.

- The Simplest Problem in CofV.



$$I[y] = \int_a^b L(x, y(x), y'(x))\,dx.$$

Find y(x) to minimize I relative to "nearby" curves.

$y_\varepsilon(x) = y_0(x) + \varepsilon h(x)$. (where $y_0(x)$ is a is a minimizer)

$f(\varepsilon) = I[y_\varepsilon]$

$\frac{d}{d\varepsilon} f(\varepsilon)\Big|_{\varepsilon=0} = 0$, for all $h(x)$.

$$\frac{df}{d\varepsilon} = \frac{d}{d\varepsilon}\int_a^b L(x, y_\varepsilon, y_\varepsilon')\,dx$$
$$= \int_a^b \left(\frac{\partial L}{\partial y} h + \frac{\partial L}{\partial y'} h'\right) dx = 0 \begin{cases} \text{at } \varepsilon = 0 \\ \text{for all } h \end{cases}$$
$$= \int_a^b \frac{\partial L}{\partial y} h + \frac{\partial L}{\partial y'} h'\,dx = 0 \text{ for all } h.$$

Integrate by parts. Be careful.

- $L, y(x) \in C^1$ $\boxed{\text{Gelfand Fomin}}$

Lemma: $\alpha, \beta \in C^0$

$\int_a^b \alpha(x) h(x) + \beta(x) h'(x)\,dx = 0$ for all.

$\Rightarrow \beta \in C^1$ $\quad \alpha(x) = \beta'(x)$

- $L, y(x) \in C^2$.

$$\frac{\partial L}{\partial y'} h' = \frac{d}{dx}\left(\frac{\partial L}{\partial y'} h\right) - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) h$$

First Variational Formula.

$$\frac{d}{d\varepsilon} I_\varepsilon = \int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'}\right] h(x)\,dx + \frac{\partial L}{\partial y'} h(x)\Big|_{x=a}^{x=b}$$

$E(L) = \frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'}$ ← Euler-Lagrange

Boundary Term

$$\frac{d}{d\varepsilon} I_\varepsilon\Big|_{\varepsilon=0} = \int_a^b E(L) h(x)\,dx + \frac{\partial L}{\partial y'} h(x)\Big|_{x=a}^{x=b}.$$

just as important as E(L) term

$L(x,y,y')$ is a total derivative if there is $F(x,y)$ such that:

$$L(x,y,y') = \frac{d}{dx}F(x,y) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y'$$

In this case.

$$I[y] = \int_a^b L(x,y,y')\,dx = \int_a^b \frac{dF}{dx}dx = F(x,y)\Big|_{x=a}^{x=b}.$$

$$\frac{d I[y_\varepsilon]}{d\varepsilon} = \text{"pure boundary"}.$$

no contribution to E(L) term

Proposition L is a total derivative if and only if $E(L) \equiv 0$ (null Lagrangian)

Proof:

$\Rightarrow$ $L = \frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y'$

$$\frac{\partial L}{\partial y} = \frac{\partial F}{\partial x\partial y} + \frac{\partial F}{\partial y y'}y'.$$

$$\frac{\partial L}{\partial y'} = \frac{\partial F}{\partial y} \Rightarrow E(L) = \frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \equiv 0$$

$\Leftarrow$ $E(L) = \frac{\partial L}{\partial y} - \partial_x \frac{\partial L}{\partial y'} = \cdots + \frac{\partial L}{\partial y'\partial y'}y'' \equiv 0$

0

$\Rightarrow \frac{\partial L}{\partial y'\partial y'} = 0 \Rightarrow L = A(x,y) + B(x,y)y'.$

$\Rightarrow \cdots$ $A = \frac{\partial F}{\partial x}$, $B = \frac{\partial F}{\partial y}$.

which means L is total derivative

Inverse Problem of C of V

$T = T(x, y, y', y'')$ (given)

When is there $L(x, y, y')$ such that

$T = E(L)$

$= \frac{\partial L}{\partial y} - \frac{\partial L}{\partial x \partial y'} - \frac{\partial L}{\partial y y'} y' - \frac{\partial L}{\partial y' \partial y'} y''$?

Proposition: $T = E(L)$ if and only if

* $\frac{\partial T}{\partial y'' \partial y''} = 0$ and

$\frac{\partial}{\partial y'} T = D_x \frac{\partial T}{\partial y''}$

Helmholtz Condition

More generally:

$f \cdot T = E(L)$

$f^\alpha_\beta T_\alpha = E_a(L)$; $\not{\exists}$ $T = E$.

Proof. $T = E(L)$

$\frac{\partial T}{\partial y'} = \frac{\partial L}{\partial y \partial y'} - \frac{\partial L}{\partial x \partial y' \partial y'} - \frac{\partial L}{\partial y \partial y' \partial y'} y'$

$\Rightarrow \quad - \frac{\partial L}{\partial y \partial y'} - \frac{\partial L}{\partial y' \partial y' \partial y'} y''$.

$= \frac{d}{dx} \frac{\partial L}{\partial y' \partial y'}$

$= \frac{d}{dx} \frac{\partial T}{\partial y''}$.

$\Leftarrow$ $L = \int_0^1 y \cdot T(x, ty, ty', ty'')\, dt$

But wait.

called Vainburg formula.

This L is 2^{nd} order!

So to completely solve inv. problem we need to prove that int. by part can reduce the order of L to 1.

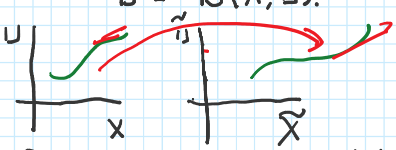
Symmetries of Lagrangians

Conside a transformation

$$\tilde{x} = A(x)$$
$$\tilde{u} = B(x,u).$$

x, y

x, u



Curves are mapped to curves, so

$$\tilde{u}_{\tilde{x}} = C(x,u,u_x).$$

At the infinitesimal level this becomes the Lie prolongation formula.

$$X = a(x)\partial_x + b(x,u)\partial_u$$

$$pr^1 X = a(x)\partial_x + b(x,u)\partial_u + \left(\frac{d}{dx}b - \frac{da}{dx}u_x\right)\frac{\partial}{\partial u_x}$$

Lie derivative of the Lagrangian 1-form

$$\lambda = L(x,u,u_x)\,dx$$

$$\mathcal{L}_{pr^1 X}\lambda = pr^1 X \lrcorner\, d\lambda + d(pr^1 X \lrcorner\, \lambda).$$

$$= pr^1 X \lrcorner \left(\frac{\partial L}{\partial u}du + \frac{\partial L}{\partial u_x}du_x\right)dx + d(aL)$$

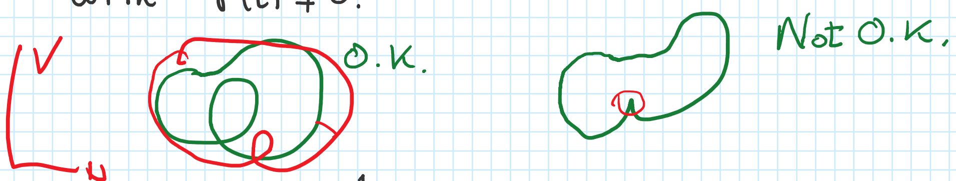
$$= \dots$$

$$= (b - a u_x)\,E(L)\,dx + \frac{d}{dx}\eta \cdot dx$$

$$\eta = aL + (b - a u_x)\frac{\partial L}{\partial u_x}.$$

$\Rightarrow$ Noether's Theorem

Let $\gamma(t) = (u(t), v(t))$ be a closed smooth curve.
with $\dot{\gamma}(t) \neq 0$.



$$I(\gamma) = \frac{1}{2\pi}\int_0^1 \frac{\dot{u}\ddot{v} - \ddot{u}\dot{v}}{\dot{u}^2 + \dot{v}^2}\,dt$$

$\hookrightarrow L = L(t, u, v, \dot{u}, \dot{v})$

Calculate $E(L) = 0$.

$\Rightarrow \frac{d}{d\varepsilon} I(\gamma_\varepsilon) = 0, \quad \dot{\gamma}_\varepsilon \neq 0.$

$\Rightarrow I(\gamma)$ is an isotopy invariant of planar immersed curves.

Null Lagrangians $\sim$ deformation invariants

Let K be the Gaussian curvature for a metric g on surface.

$$(x,y) \longrightarrow \vec{R}(x,y) = [u(x,y), v(x,y), w(x,y)].$$

$$K \sim \det\begin{bmatrix} R_{uu}\cdot R_{vv} & R_{uu}\cdot R_u & R_{uu} R_v \\ R_{vv}\cdot R_u & E & F \\ R_{vv}\cdot R_v & F & G \end{bmatrix} - \det\begin{bmatrix} R_{uv}\cdot R_{uv} & \cdots & \\ \cdots & \cdots & \cdots \\ & & \end{bmatrix}$$

$$\frac{\delta K}{\delta u} = \frac{\delta K}{\partial v} = \frac{\delta K}{\delta w} = ?$$

Fine Structure. $\quad \frac{\partial(u_x, u_y)}{\partial(x,y)}, \quad \frac{\partial(u_x, v_y)}{\partial(x,y)}, \ldots,$

Summary

1. We have the beginning of an exact sequence

$F(x,u) \longrightarrow L = \frac{d}{dx} F$.

$L \longrightarrow T = E(L)$

$T \longrightarrow H(T)$

$F \longrightarrow E\left(\frac{dF}{dx}\right) = 0$

$L \longrightarrow H(E(L)) = 0$.

2. The "operators" in this are "natural"
   - commute $\mathcal{L}_X$

3. $\mathcal{L}_X \lambda = X \text{"}\lrcorner\text{"} E(\lambda) + \frac{d}{dx} X \text{"}\lrcorner\text{"} \lambda$.

$\approx$ Cartan like formula for $\mathcal{L}_X \lambda$

Exercise 1. ✓

Repeat for

$L = L(x, u, u_x, u_{xx})$.

Exercise 2.

Repeat for

$L = L(x, y, u, u_x, u_y)$.

Motivation.

$\Rightarrow$ efficient variational calculus.

Motivations

1. The variational bicomplex is a double complex $\Omega^{*,*}(J^\infty(E), d_H, d_V)$ of differential forms on the infinite jet bundle of $\pi: E \to M$.
2. $\Rightarrow$ very effective variational calculus (variational prin, symmetries, cons. laws, Ham. oper)
3. $\Rightarrow$ Euler-Lagrange complex

$$\to \mathrm{Div} \to E/L \to \mathrm{Helm} \to$$

4. $\Rightarrow$ cohomology readily computed.
5. $\Rightarrow$ global first variational formula.
6. $\Rightarrow$ various versions of Noether's theorem, ("Taken's Problem")
7. general framework for conservation laws
8. general framework for characteristic class

$$\left.\begin{array}{l}7.\\8.\end{array}\right\} \Omega^{*,*}_G(R, d_H, d_V).$$

Jets

Let $\pi : E \to M$ be a fibered manifold

$$\begin{array}{ccc} E & & (x^i, u^\alpha) \\ \downarrow & & \downarrow \pi \\ M & & (x^i) \end{array} \qquad s : (x^i) \to (x^i, u^\alpha = s^\alpha(x^i)).$$

s

$J^k(E)$ = bundle of jets of local sections

= equiv. classes of local sections

$s_1 \sim s_2$ at $x_0 \in M$ if the coordinate values s_1, s_2 and their derivatives to order k agree.

$j^k(s)$

E

M

(x, u)

$\downarrow$

x

$s_1(x) = 1 + 2x - 3x^2 + 4x^3$

$s_2(x) = 1 + 2x - 3x^2 + 7x^3$

$s_1 \sim s_2$ at $x = 0$ on J^2 not J^3.

$s : U \to E$; $\quad j^k(s) : U \to J^k(E)$

(sections lift) $\quad x_0 \to [s(x_0)]^{(k)}$!

Example: $\mathbb{R} \times \mathbb{R}^2 \quad (x, u, v)$

$$\begin{array}{cc} \downarrow & \downarrow \\ \mathbb{R} & x \end{array}$$

$J^2(\mathbb{R}, \mathbb{R}^2) \sim (x, u, v, u_x, v_x, u_{xx}, v_{xx})$

$\dot u \quad \dot v \quad \ddot u \quad \dot v$

Example: $\mathbb{R}^2 \times \mathbb{R} \quad (x, y, u)$

$$\begin{array}{cc} \downarrow & \downarrow \\ \mathbb{R}^2 & (x, y) \end{array}$$

$J^2(\mathbb{R}^2, \mathbb{R}) \sim (x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})$

Example: $S^2(T^*M) : E$, $\det g \neq 0$

$\downarrow$

M

Curvature: $R : J^2(S^2 T^* M) \to T^1_3(M)$

$I(g) = \int_M Pf(R(g))$

integral of local inv.

Example:

$\varphi : S^3 \to S^2 \qquad \varphi^*(\nu) = d\theta$

$\int_{S^3} \theta \wedge d\theta.$

Contact System

$\pi: E \to M$

$\pi^k: J^k(E) \to M$ $\cancel{*}$

$\pi^k_\ell: J^k(E) \to J^\ell(E)$.

$\mathcal{C}(J^k(E)) = \{ \omega \in \Omega^*(J^k(E)) \mid j(s)^*(\omega) = 0 \}$

for all $s: U \to E$ contact ideal.

When is $\sigma: U \to J^k(E)$ given by $\sigma = j^k(s)$.

iff. $\sigma^*(\mathcal{C}) = 0$.

Example:

$J^2(\mathbb{R}, \mathbb{R}^2)$ $(x, u, v, \dot{u}, \dot{v}, \dots)$

$\theta^u = du - \dot{u}\,dx \quad \theta^v = dv - \dot{v}\,dx$

$\dot{\theta}^u = d\dot{u} - \ddot{u}\,dx \quad \dot{\theta}^v = d\dot{v} - \ddot{v}\,dx$

$d\theta^u = dx \wedge \dot{\theta}^u \quad d\theta^v = dx \wedge \dot{\theta}^v$ ✓

$d\dot{\theta}^u = dx \wedge d\ddot{u} \quad d\dot{\theta}^v = dx \wedge d\ddot{v}$

$J^3(\mathbb{R}^2, \mathbb{R})$

$\theta = du - u_x\,dx - u_y\,dy$

$\theta_x = du_x - u_{xx}\,dx - u_{xy}\,dy$

$\theta_y \quad - {}_y - u_{xy}\,dx - u_{yy}\,dy$

$\theta_{xx} = du_{xx} - u_{xxx}\,dx - u_{xxy}\,dy$.

$\vdots$

$d\theta = dx \wedge \theta_x + dy \wedge \theta_y$

$d\theta_x = dx \wedge \theta_{xx} + dy \wedge \theta_{xy}$

$d\theta_{xx} = dx \wedge du_{xx}$.

Should be on everyone's top 10 list of important formulas!

Properties of the Contact System

1. Maps lift:

$$\begin{tikzcd} E \arrow[r, "\Phi"] \arrow[d] & E \arrow[d] \\ M \arrow[r, "\varphi"] & M \end{tikzcd}$$

$pr^k \Phi : J^k(E) \to J^k(E)$

uniquely determined by

$pr^k \Phi(\mathcal{C}) \subset \mathcal{C}$

2' Vector fields lift.

Axiomatic Development!!
$\mathcal{C}(J^k(E))$

2. Maps drop.

$\Sigma : J^k(E) \to J^k(E) \quad k \geq 2.$

$\Sigma^*(\mathcal{C}) \subset \mathcal{C}.$

$\Rightarrow \exists \tilde{\Sigma}$:

$$\begin{tikzcd} J^k(E) \arrow[r, "\Sigma"] \arrow[d, "\pi^k_{k-1}"'] & J^k(E) \arrow[d, dash, "\pi^k_{k-1}"] \\ J^{k-1}(E) \arrow[r, "\tilde{\Sigma}"] & J^{k-1}(E) \end{tikzcd}$$

[der $(\mathcal{C}^k)$ has Cauchy char.]

Infinite Jets

$\pi^\infty : J^\infty(E) \to M$ ✓

$\pi^\infty_k : J^\infty(E) \to J^k(E)$.

Key point:

Any form $\omega \in \Omega^p(J^\infty(E))$ is locally a form on some $J^k(E)$.

Coordinates:

$x^i, u^\alpha, u^\alpha_i, u^\alpha_{ij}, u^\alpha_{ijk} \cdots$

$u^\alpha_{ij}(j^\infty(s)(x_0)) = \frac{\partial s^\alpha}{\partial x^i \partial x^j}(x_0)$

Contact Forms $\quad C(J^\infty(E))$

$\theta^\alpha = du^\alpha - u^\alpha_i dx^i$

$\theta^\alpha_i = du^\alpha_i - u^\alpha_{ij} dx^j$

$$\boxed{d\theta^\alpha_{i_1\cdots i_k} = dx^j \wedge \theta^\alpha_{j i_1 \cdots i_k}}$$

Wait! Two Big Results!

Now $C(J^\infty(E))$ is generated by 1 forms $\sim$ "integrable" $\sim$ "Frobenius"

{ orders

$\mathrm{Tot}(J^\infty(E)) = \mathrm{Ann}(C(J^\infty(E)))$

$= \{ X \mid X \lrcorner \omega = 0 \quad \omega \in C(J^\infty(E)) \}$

$= \langle D_i = \frac{\partial}{\partial x^i} + u^\alpha_i \frac{\partial}{\partial u^\alpha} + u^\alpha_{ij} \frac{\partial}{\partial u^\alpha_j} + \cdots \rangle$.

$\frac{d u^\alpha_i}{d \omega} = u^\alpha_{ij}$

total "vector fields" (no local flows)

$[D_i, D_j] = 0 \qquad D_i(f)$

$\pi : E \to M$

$\pi^\infty : J^\infty(E) \to M$.

$\to \mathcal{C}(J^\infty(E))$.

$\to$ Tot.

$\to \mathrm{Vert} = \{ X \mid \pi^\infty_*(X) = 0 \}$.

$T_\sigma(J^\infty(E)) = \mathrm{Tot}_p \oplus \mathrm{Vert}$.

int. int.

$[D_j, \frac{\partial}{\partial u^\alpha_{i_1 \dots i_k}}] = \delta^{(i_1}_j \partial^{i_2 \dots i_k)}_\alpha$

$\frac{\partial}{\partial u^\alpha_{ij}}$

The direct sum decomposition of $T(J^\infty(E))$ includes

$$\Omega^{(p)}(J^\infty(E)) = \bigoplus_{r+s=p} \Omega^{r,s}(J^\infty(E))$$

$\omega \in \Omega^{r,s}(J^\infty(E))$

$$\omega = A(x,u,\partial u) \, \underbrace{dx \dots dx}_{r} \wedge \underbrace{\theta \wedge \quad \theta}_{s}$$

horizontal degree

vertical degree.

$$\begin{array}{|c|c} \hline r,s+1 & \\ \hline r,s & r+1,s \\ \hline \end{array}$$

$d\omega =$

$= \underset{r+1,s}{d_H \omega} + \underset{r,s+1}{d_V \omega}$

Structure Equations

$d_H f = (D_i f) dx^i = \left(\frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial u^\alpha} u^\alpha_i + \frac{\partial f}{\partial u^\alpha_j} u^\alpha_{ij} \dots \right) dx^i$

$d_V f = \frac{\partial f}{\partial u^\alpha} \theta^\alpha + \frac{\partial f}{\partial u^\alpha_i} \theta^\alpha_i + \frac{\partial f}{\partial u^\alpha_{ij}} \theta^\alpha_{ij} \dots$

$d_H(dx^i) = 0$ $\qquad$ $d_V(dx^i) = 0$

$d_H \theta^\alpha_{j \dots j_k} = dx^i \wedge \theta^\alpha_{i j_1 \dots j_k}$ $\qquad$ $d_V(\theta^\alpha_{j_1 \dots j_k}) = 0$

$d = d_H + d_V$

Generalization

Keep $\mathcal{C}$ but throw away Vert.

Iparts

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$E \to M$ ✓

$J^\infty(E) \to M$. ✓

maps and vector field on E ✓

prol.

$$X = a^i \frac{\partial}{\partial x^i} + b^\alpha \frac{\partial}{\partial u^\alpha}$$

$$\mathrm{pr}X = \mathrm{tot}\,X + \mathrm{pr}X_{ev}$$

$$\mathrm{tot}\,X = a^i D_i$$

$$X_{ev} = (b^\alpha - u^\alpha_i a^i)\frac{\partial}{\partial u^\alpha}.$$

$\Omega^{r,s}$.

	0	1			
3	$\theta\wedge\theta\wedge\theta$	$dx\wedge\theta\wedge\theta\wedge\theta$	$\cdot$		$\nu\wedge\theta\wedge\theta$
2	$\theta\wedge\theta$	$dx\wedge\theta\wedge\theta$	$dx\wedge dx\wedge\theta\wedge\theta$		$\nu\wedge\theta\wedge\theta$
1	θ	$dx\wedge\theta$	$dx\wedge dx\wedge\theta$		$\nu\wedge\theta$
0	f	dx^i	$dx\wedge dx$	$\cdots$	$L\nu$

d

$dx^1 \cdots dx^n$

Lagrang.

$$\Omega^p = \bigoplus \Omega^{r,s}$$

$$d = d_H + d_V, \quad d_H^2 = 0, \quad d_H \circ d_V + d_V \circ d_H = 0, \quad d_V^2 = 0$$

But wait. There's more.

$$\mathrm{Iparts}: \Omega^{n,s} \to \Omega^{n,s}$$

$$\mathrm{Iparts}(\omega) = \theta^\alpha \wedge \left(\frac{\partial}{\partial u^\alpha} \lrcorner\, \omega - D_i \frac{\partial}{\partial u^\alpha_i} \lrcorner\, \omega + D_i D_j \frac{\partial}{\partial u^\alpha_{ij}} \lrcorner\, \omega - + \cdots \right)$$

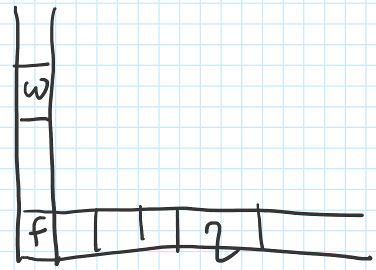
= Euler-Lagrange with "partial derivatives" with "hook".

$$\boxed{\begin{aligned} &\mathrm{Iparts} \circ d_H = 0 \\ &\mathrm{Iparts}^2 = \mathrm{Ipart}. \\ &\mathcal{L}_{\mathrm{pr}X}\, \mathrm{Iparts} = \mathrm{Iparts} \circ \mathcal{L}_{\mathrm{pr}X} \end{aligned}}$$

!!

At the edges

Rolle's Thm: $f'(x) = 0 \Rightarrow f = \text{constant}$.



$f \in \Omega^{0,0}(J^\infty(E))$, $d_H f = 0 \Rightarrow f = \text{constant}$

$\omega \in \Omega^{0,s}(J^\infty(E))$ $d_H \omega = 0 \Rightarrow \omega = 0$.

$\eta \in \Omega^{r,0}(J^\infty(E))$, $d_V \eta \; 0 \Rightarrow \eta$ = form on base M.

Proof: Locally f has order k.

$d_H f \; .r \Rightarrow D_i f = 0$

$[D_i, \frac{\partial}{\partial u_{k+1}}] = \frac{\partial}{\partial u_k}$.

$\Rightarrow f = f(x^i)$. Loc

Euler-Lagrange Complex

$$
\begin{tikzcd}
0 \arrow[r] & \Omega^{0,3} \arrow[r] & \Omega^{1,3} & & & \Omega^{n,3} \arrow[r, "I_p"] & \mathcal{F}^3 \arrow[r] & 0 \\
0 \arrow[r] & \Omega^{0,2} \arrow[r] \arrow[u] & \Omega^{1,2} & & \Omega^{n-1,2} & \Omega^{n,2} \arrow[r, "I_p"] \arrow[u] & \mathcal{F}^2 \arrow[r] & 0 \\
0 \arrow[r] & \Omega^{0,1} \arrow[r] \arrow[u] & \Omega^{1,1} & & \Omega^{n-1,1} \arrow[r] \arrow[u] & \Omega^{n,1} \arrow[r, "I_p"] \arrow[u] & \mathcal{F}^1 \arrow[r] & 0 \\
\mathbb{R} \arrow[r] & \Omega^{0,0} \arrow[r, "d_H"] \arrow[u] & \Omega^{1,0} \arrow[r] & \Omega^{2,0} \cdots & \Omega^{n-1,0} \arrow[r] \arrow[u] & \Omega^{n,0} \arrow[u] \arrow[ru] & &
\end{tikzcd}
$$

$E = I_p \circ d_V.$

$$0 \to \mathbb{R} \to \Omega^{0,0} \to \Omega^{1,0} \to \Omega^{2,0} \cdots \Omega^{n-1,0} \xrightarrow{D_{iv}} \Omega^{n,0} \xrightarrow{E} \mathcal{F}^1 \xrightarrow{\mathcal{H}} \mathcal{F}^2 \to \mathcal{F}^3 \cdots$$

Euler-Lagrange Complex.

$d_H f = 0 \Rightarrow f = \text{const.}$ $d_H \omega = 0 \Rightarrow \omega = 0$!!

Local Exactness

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1. The columns $\begin{array}{c}\Omega^{r,s+1}\\ \uparrow d_V\\ \Omega^{r,s}\end{array}$ are locally exact.

- "deRham" complex; $d_V u^\alpha_I = \theta^\alpha_I$

  $d\,\theta^\alpha_I = 0$

$\int_J$

- usual homotopy operators... $h_V^{r,s+1} \to h_V^{r,s}$.

$\omega = d_V h^{r,s} + h^{r,s+1} d_V, \quad \omega \in \Omega^{r,s}$.

2. The interior rows are exact.

- again $h_H^{r+1,s} \to h_H^{r,s}$ homotopy operators
- but these are local diff. operators built from $\frac{\partial}{\partial u^\alpha_I}$, D_i, (higher Euler operators).
- also a nice inductive argument:

  $d_H \omega = 0$ $\quad \omega$ of order k

  $\omega = d_H \eta + \tilde{\omega}, \quad \tilde{\omega}$ of order $k-1$

  $d_H \tilde{\omega} = 0$

At this point, no good control on orders

$\underbrace{d_H \omega}_{\text{order } k} = 0, \quad \omega = d_H \underbrace{\eta}_{\text{order} >> k}, \quad \eta = h_H^{r,s}(\omega)$

Variational Calculus

At this point we have a good variational calculus:

$\Omega^{r,s}(J^\infty(E))$

$d = d_H + d_V$.

$prX = X_{tot} + prX_{ev}$.

$EL = I_p \circ d_V : \Omega^{n,0} \to \Omega^{n,1}$

Helm: $I_p \circ d_V : F^1 \to F^2$.

Commutator Rules, Lie derivative rules.

Ex.

- $[prX, prY] = pr[X,Y]$, $\mathcal{L}_{prX} EL(\lambda) = EL(\mathcal{L}_{pr}\lambda)$.
- $[totX, totY] = tot[X,Y]$,
- $\mathcal{L}_{prX_{ev}} \omega = prX_{ev} \lrcorner\, d_V \omega + d_V(prX_{ev} \lrcorner\, \omega)$
- $EL(\mathcal{L}_{prX}\lambda) = \mathcal{L}_{prX} EL(\lambda)$.

All coded in DG software.

with(DifferentialGeometry): with(JetCalculus);
[AssignTransformationType, AssignVectorType, DifferentialEquationData, EulerLagrange, EvolutionaryVector,
FindLagrangian, FractionalPower, GeneralizedLieBracket, GeneratingFunctionToContactVector, HigherEulerOperators, HorizontalExteriorDerivative, HorizontalHomotopy, IntegrationByParts, JetFrameData, LinearizedOperator, Noether, ProjectedPullback, ProjectionTransformation, Prolong, PushforwardTotalVector, TotalDiff, TotalJacobian, TotalVector, VerticalExteriorDerivative, VerticalHomotopy, ZigZag]

n $\int K$ Gauss curv

$\int K_n$ $Pf(\Omega)$.

From Yesterday.

$$\begin{array}{c} E \\ \downarrow \\ M. \end{array}$$

$J^\infty(E)$.

$\theta^\alpha_{i \ldots i_k}$

$\delta\varphi$

dx

$$\boxed{\Omega^p(J^\infty(E)) = \bigoplus \Omega^{r,s}}$$

$d = d_H + d_V$

$$\begin{array}{ccccccc}
\Omega^{0,2} & & \Omega^{1,2} & & \Omega^{n,2} & \to & F^2 \\
\uparrow d_V & & \uparrow & & \uparrow & & \\
\Omega^{0,1} & \xrightarrow{d_H} & \Omega^{1,1} & & \Omega^{n,1} & \to & F^1 \\
\uparrow & & \uparrow & & \uparrow & \nearrow E. & \\
\Omega^{0,0} & \to & \Omega^{1,0} & \cdots \xrightarrow{d_H} & \Omega^{n,0} & &
\end{array}$$

I_p.

λ of any order
for variat..
problems.

$F^1 \subset \Omega^{n,1}$ $\quad \Delta = \Delta_a \theta^\alpha dx^1 .. dx^n$

Theorem. The interior rows of the variational bicomplex are globally exact. $\omega \in \Omega^{r,s}(J^\infty(E))$, $d_H \omega = 0$ $\Rightarrow$ $\omega = d_H \eta$ everywhere

"no cohomology".

Proof $0 \quad d_H(1) = 0$

* $0 \longrightarrow \Omega^{0,s} \xrightarrow{d_H} \Omega^{1,s} \longrightarrow \cdots$

Use Bott-Tu: $\begin{cases} \text{generalized Mayer-Vietoris sequence} \\ \text{good cover} \end{cases}$

* Introduce connections on E, M.

$$\nabla_i \Theta^\alpha = \Theta^\alpha_i - \Gamma^\alpha_{\beta i} \Theta^\beta.$$

$$\nabla_j \nabla_i \Theta^\alpha = D_j(\nabla_i \Theta^\alpha) + \Gamma^k_{ji} \nabla_k \Theta^\alpha - \Gamma^\alpha_{j\beta} \nabla_i \Theta^\beta.$$

Build a global homotopy operator.

* Sheaf theory

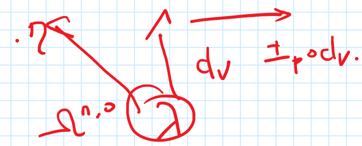
First Variational Formula

$$* \to \Omega^{n-1,s} \xrightarrow{d_H} \Omega^{n,s} \xrightarrow{I_p} \Omega^{n,s}$$

$\omega \in \Omega^{n,s}$, $I_p(\omega) = 0$

$I_p(\omega) = 0 \Rightarrow \omega = d_H \eta$, $\eta \in \Omega^{n-1,s}$ globally.

$*$ $\lambda \in \Omega^{n,0}$, $\lambda = L(x^i, u, \partial u, \partial^2 u \ldots)\, dx^1 \ldots dx^n$



$\alpha = d_V \lambda - I_p\, d_V(\lambda) = (1 - I)(d_V \lambda)$,

$I_p(\alpha) = 0$ (because $I_p^2 = I_p$).

$$\Rightarrow \boxed{d_V \lambda = E(\lambda) + d_H \eta}$$ globally, over all of $J^\infty(E)$.

Global First Variational Formula.

Note: Is there a mapping $\Omega^{n,0} \to \Omega^{n-1,1}$, $\lambda \to \Theta(\lambda)$

such that

① $\mathcal{L}_{prX}\, \Theta(\lambda) = \Theta(pr X(\lambda))$ [Natural, symmetry pres.]

② $d_V \lambda = E(\lambda) + d_H(\Theta(\lambda))$

In general no.

[exceptions, $n = 1$
order $\lambda = 2$]

Theorem. Every global infinitesimal symmetry of $\lambda \in \Omega^{n,0}$ generates a global conservation law.

X on E $\quad \mathcal{L}_X \lambda = 0$

$\exists \xi$ $\quad E(\lambda) = 0$

$d_H \xi = 0$ on sol^n,

$\Omega^{n-1,0}$.

Proof.

$$\begin{cases} \mathrm{pr}\, X = \mathrm{tot}\, X + \mathrm{pr}\, X_{ev} \\ d\lambda = d_V \lambda = E(\lambda) + d_H \eta \end{cases}$$

$$\mathcal{L}_{\mathrm{pr}X} \lambda = \mathrm{pr}X \lrcorner d\lambda + d(\mathrm{pr}X \lrcorner \lambda)$$
$$= \dots$$
$$= X_{ev} \lrcorner E(\lambda) + d_H \xi$$

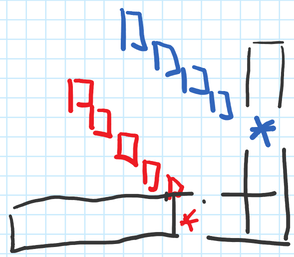
$$\boxed{\xi = \mathrm{pr}\, X_{ev} \lrcorner \eta + \mathrm{tot}\, X \lrcorner \lambda} \quad \text{global.}$$

$\ast$ Local calculations do not give this global result.

Cohomology

Interior rows of a double complex exact
$\Rightarrow$
edge cohomology = total cohomology

Explicit isomorphism. Π



$\Pi(\omega) = *$

$\Pi(\omega) = *$

* $H^p(\Omega^*(J^\infty(E))) = H^p(\mathcal{E}^*(J^\infty(E)))$

de Rham

* If $H^n_{deR}(\Omega^*(\underline{E})) = 0$ every null Lagrangian is a tot. der $\lambda = d_H \xi$

$\Omega^{n,1}$

* Let $\Delta = \Delta_\alpha(x, u, \partial u \ldots)\,\theta^\alpha \wedge \nu$ is a source form with $\mathcal{H}(T) = I_p \circ d_V(T) = 0$
If $H^{n+1}(\Omega^*(E)) = 0$ then $\underline{\Delta}$ is global variational, $\Delta = EL(\lambda)$.

$H^p(E)$

Question from yesterday.

Wednesday, August 25, 2021 9:46 AM

Question about total derivative vectorfields.

Classical Frobenius.

Let $X_1 \cdots X_r$ be pointwise indep. vectorfields on M^n.

Suppose they form a "closed system" under Lie brackets

$$[X_i, X_j] = \lambda^k_{ij} X_k$$

Then there are $n-r$ first integrals. $X_i(f_1) = X_i(f_2) = \cdots X_i(f_r) = 0.$

$$D_x = \partial_x + u_x \partial_u + u_{xx} \partial_{u_x} + u_{xy} \frac{\partial}{\partial u_y} + \cdots$$

$$D_y = \partial_y + u_y \partial_u + u_{xy} \partial_{u_x} + u_{yy} \partial_{u_y} + \cdots$$

$[D_x, D_y] = 0$ but $D_x f = 0$, $D_y f = 0 \Rightarrow f =$ constant.