

Beyond Toric Geometry and

Piecewise Linear Combinatorics

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Joint with

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Polyptych Varieties

- more general than toric varieties
- admit combinatorial description
- admit many toric degenerations

T

$$N = \text{Hom}(\mathbb{C}^*, T)$$

$$M = \text{Hom}(T, \mathbb{C}^*)$$

$$N \times M \rightarrow \mathbb{Z}$$

pairing

$$\Sigma \subseteq N_{\mathbb{R}}$$

fans

where do limits go?

$$\Delta \subseteq M_{\mathbb{R}}$$

polytopes

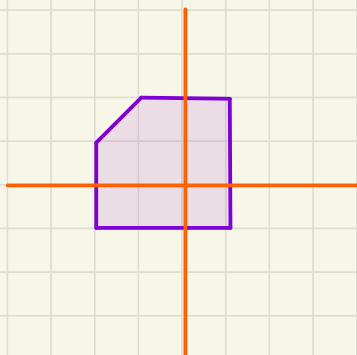
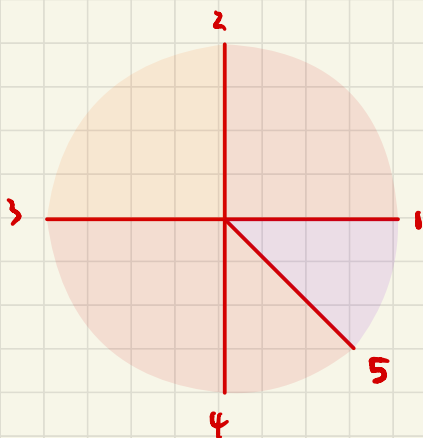
which functions extend?

$$\Sigma(1) = \{v_1, \dots, v_n\}$$

$$X_\Sigma \setminus T = D_1 \cup \dots \cup D_n$$

$$f = \sum_{m \in M} c_m t^m$$

$$\text{ord}_{D_i}(f) = \text{MIN} \{ \langle v_i, m \rangle \mid c_m \neq 0 \}$$



$$2D_1 + D_2 + D_3 + 2D_4 + 3D_5$$

X_Σ proper

smooth

projective

Σ complete

"smooth"

inner normal fan
of a polytope

- class group, Picard group

N, M are the same kind of object

$\Downarrow$
Duality

Beyond Toric Varieties

combinatorial varieties

Grassmannians, Spherical Varieties

T -Varieties, $\dim T < \dim X$

Toric Vector Bundles

Mori Dream Spaces

Beyond Toric Varieties

(M-side)

- Combinatorial model of the coordinate ring

(N-side)

- Combinatorial collection of boundary divisors
(divisorial valuations)

M-side

Toric Degeneration

X - a projective variety

$$\begin{array}{c} E \\ \downarrow \pi \hookrightarrow \mathbb{C}^* \\ \mathbb{C} \end{array}$$

$$\pi^{-1}(1) = X$$

$$\pi^{-1}(0) = X_{\Delta}$$

a toric variety
with polytope Δ

M-side

Toric Degeneration

$$X \Rightarrow X_{\Delta}$$

$\Delta \subseteq M_{\mathbb{Q}}$ combinatorial Model
of the coordinates
on X

M-side

Mutation

Theorem (Escobar, Harada)

Suppose $X \Rightarrow X_{\Delta_1}, X_{\Delta_2}$

are "close" toric degenerations,

then there is a piece-wise linear

bijection $\mu: M_{1,\mathbb{Q}} \rightarrow M_{2,\mathbb{Q}}$

with $\mu(\Delta_1) = \Delta_2$.

N -side

Tropical Skeleton

Suppose $\mathcal{U}$ is birational
to many tri T_i $i \in \mathcal{I}$,

Then there are **piecewise linear**
bijections

$$i, j \in \mathcal{I} \quad \mu_{ij} : N_i \rightarrow N_j$$

Polyptych Varieties

$\mathcal{U}$ - affine variety with a
combinatorial coordinate ring,
and a large collection
of divisorial valuations

$X \supseteq \mathcal{U}$ Divisorial compactification

$$X \setminus \mathcal{U} = D_1 \cup \dots \cup D_n$$

pairing

$$f, D \longrightarrow \text{ord}_D(f)$$

Both Functions and Divisors
are parametrized by the same
sort of object.

Polyptych Lattices

$$\mathcal{M} = \{M_i, \mu_{ij} \mid i, j \in \mathbb{I}\}$$

$$M_i \approx \mathbb{Z}^r \quad \text{rank}(\mathcal{M}) = r$$

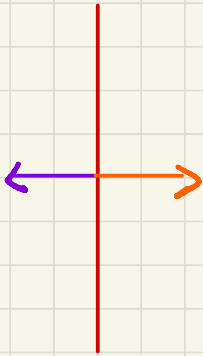
$$M_i \xrightarrow{\mu_{ij}} M_j$$

piecewise linear bijection

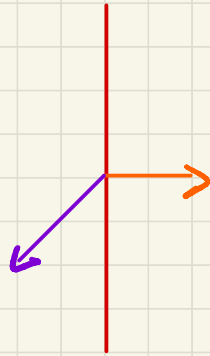
$$\mu_{ij} \circ \mu_{jk} = \mu_{ik}$$

$$\mu_{12}(a, b) = (a, b - \min\{a, 0\})$$

$$\mu_{21}(x, y) = (x, y + \min\{x, 0\})$$



μ_1



μ_2

Elements

$$\mathcal{M} = \bigcup_{i \in \gamma} M_i / \mu_{ij} (m_j) \sim m_i$$

$$\mathcal{M}_{\mathbb{R}} \text{ real elements } (M_i \otimes \mathbb{R})$$

Chart maps

$$\phi_i: \mathcal{M} \rightarrow M_i$$

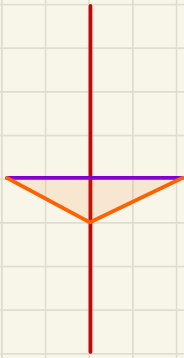
Chart Addition

$$m_1 +_i m_2 = \phi_i^{-1}(\phi_i m_1 + \phi_i m_2)$$

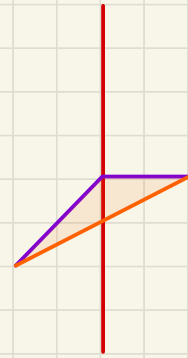
Polytypic Polyhedra

$C \subseteq \mathcal{M}_{\mathbb{R}}$ is convex

if $\phi_i C$ is convex $\forall i \in \mathcal{I}$



M_1



M_2

Canonical Semi algebra

$$S_M = \langle [m] \mid m \in M \rangle /$$

$$\bigoplus_{m \in S} m \sim \bigoplus_{n \in T} n \quad \text{if } \text{conv}(S) = \text{conv}(T)$$

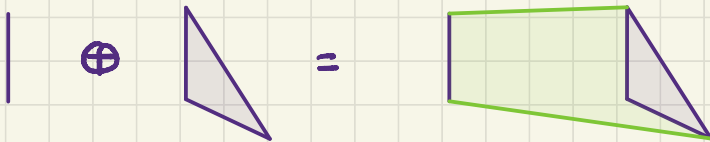
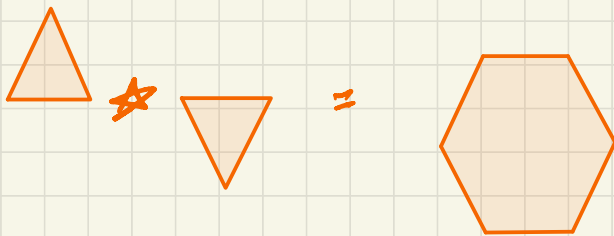
$$[m_1] \star [m_2] = \bigoplus_{i \in I} [m_1 +_i m_2]$$

" ∞ " - additive identity

$$[m] = \{ [m] \mid m \in M \} \subseteq S_M$$

If $\mathcal{M}$ has one chart: $\mathbb{Z}^r$

- $S_{\mathcal{M}}$ is the semi algebra of integral polytopes in $\mathbb{R}^r$ w/r/t Minkowski sum / convex hull of union



Semi algebra of Convex Functions

$$\mathcal{P}_M = \{ f: M \rightarrow \mathbb{R} \mid$$

$f \circ \phi_i^{-1}$ is convex,

piecewise linear

for each

$$\phi_i: M \rightarrow M_i \}$$

$$\Pi_M \subseteq \mathcal{P}_M$$

$$= \{ p: M \rightarrow \mathbb{R} \mid$$

$p \circ \phi_i^{-1}$ is linear

for some chart $\}$

Duality

$\mathcal{M}, \mathcal{N}$ are a dual pair if

$$S_{\mathcal{M}} \cong \rho_{\mathcal{N}} \quad [\mathcal{M}] \xrightarrow{\cong} \pi_{\mathcal{N}}$$

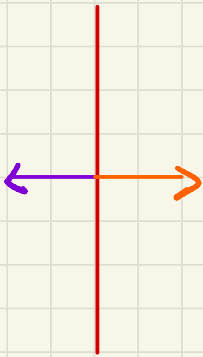
$$\rho_{\mathcal{M}} \cong S_{\mathcal{N}} \quad \pi_{\mathcal{M}} \xleftarrow{\cong} [\mathcal{N}]$$

Inducing a pairing

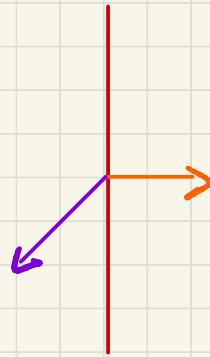
$$\mathcal{N} \times \mathcal{M} \rightarrow \mathbb{Z}$$

$$\mu_{12}(a, b) = (a, b - \min\{a, 0\})$$

$$\mu_{21}(x, y) = (x, y + \min\{x, 0\})$$



M_1



M_2

M is Self-dual

Example (Escobar, Frías-Medina, Hecada, Magee, M)

Let A be a skew-symmetric,
non-degenerate cluster algebra
with $U = \text{Spec}(A)$

$\text{Trop}(U)$ is a self-dual polytych lattice

- $\text{Gr}_m(n)$
- Del Pezzo Surfaces
- Moduli of local systems

$$M_{r,d} \quad r, d \geq 2$$

$$M_{r,d}(i) = \mathbb{Z}^{r+d-1} = \{(a_1, \dots, a_r, b_1, \dots, \hat{b}_i, \dots, b_d)\}$$

$$(\bar{a}, b_1, \dots, \hat{b}_i, \dots, b_d)$$

$$\downarrow \mu_{j,i}$$

$$(\bar{a}, b_1, \dots, b_{i-1}, \min(\bar{a}) - \sum b_j, b_{i+1}, \dots, \hat{b}_i, \dots, b_d)$$

$M_{r,d}$ and $M_{d,r}$ are dual.

$\mathcal{M}$ is finite type if $|\mathcal{I}| < \infty$.

For $\mathcal{M}$ finite type let $\Sigma \mathcal{M}$ denote the common refinement of the subdivisions defined by μ_{ij} into domains of linearity.

Theorem If $\mathcal{M}$ is finite type then $\mathcal{M}$ has a dual $\mathcal{N}$ iff

- ① Each μ_{ij} is convex.
- ② Each $\gamma \in \Sigma \mathcal{M}$ is convex.
- ③ $\Pi(\mathcal{M})_{\mathbb{R}} / \mathbb{R}_{\geq 0}$ is a polyhedral sphere.

$\mathcal{N}$ is finite and unique up to isomorphism.

Geometry

N, M

$$T = N \otimes \mathbb{C}^*$$

N, m

?

$A/\mathbb{C}$ a domain

A valuation

$$v: A \rightarrow S_m$$

satisfies

$$v(fg) = v(f) \star v(g)$$

$$v(f+g) \geq v(f) \oplus v(g)$$

$$v(cf) = v(f) \quad c \in \mathbb{C}^*$$

$$v(0) = \infty$$

$$\nu: A \rightarrow S_m$$

is a **Detropicalization**
of $\mathcal{M}$ if

① $\text{krull}(A) = \text{rank}(\mathcal{M})$

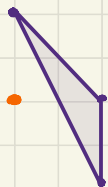
② ν is onto

Like choosing coefficients
for S_m

If $\mathcal{M}$ has one chart: $\mathbb{Z}^r$

- Any deropicalization is a Laurent Polynomial ring in r variables.
- $S_{\mathcal{M}}$ is the semialgebra of integral polytopes in $\mathbb{R}^r$ w/r/t Minkowski sum / convex hull of union
- $v(f)$ is the Newton polytope of f .

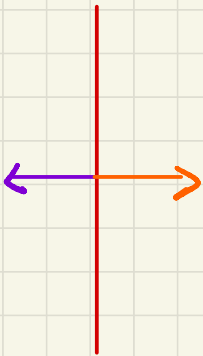
$$x + y + \frac{x}{y}$$



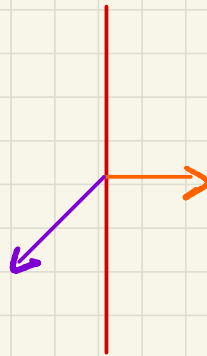
$$\mathbb{C}[x, y, t^{\pm}] / xy - t - 1$$

$$\mu_{1,2}(a, b) = (a, \min\{a, 0\} - b)$$

$$\mu_{2,1}(x, y) = (x, y + \min\{x, 0\})$$



M_1



M_2

$$M_{r,d} \quad r, d \geq 1$$

$$M_{r,d}(i) = \mathbb{Z}^{r+d-1} = \{(a_1, \dots, a_r, b_1, \dots, \hat{b}_i, \dots, b_d)\}$$

$$(\bar{a}, b_1, \dots, \hat{b}_i, \dots, b_d)$$

$$\downarrow \mu_{j,i}$$

$$(\bar{a}, b_1, \dots, b_{i-1}, \min(\bar{a}) - \sum b_j, b_{i+1}, \dots, \hat{b}_i, \dots, b_d)$$

$$\mathbb{C}[x_1, \dots, x_r; y_1^+ \dots y_d^+]/$$

$$\langle x_1 \dots x_r - y_1 \dots y_d \rangle$$

$\mathcal{M}, \mathcal{N}$ dual

In general There are
Moduli W_m of
Detropicalizations of $\mathcal{M}$

$\mathcal{M}, \mathcal{N}$ dual

$A/\mathbb{C}$ a deprojectalization of $\mathcal{M}$

$$\mathcal{U} = \text{Spec}(A)$$

$\mathcal{U}$ is a normal variety

each $n \in \mathcal{N}$ defines an
integral, divisorial valuation

$$v_n: \mathbb{C}(\mathcal{U}) \setminus 0 \rightarrow \mathbb{Z}$$

Polytych Varieties

$\mathcal{M}, \mathcal{N}$ dual pair

$\mathcal{U} = \text{Spec}(A)$ for a

detropicalization $\sigma: A \rightarrow S_m$

Definition (Polytych Variety)

$X \supseteq \mathcal{U}$ open

$X \setminus \mathcal{U} = D_1 \cup \dots \cup D_\ell$ divisorial
boundary

where $\text{ord}_{D_i} = n_i$

for some $n_1, \dots, n_\ell \in \mathcal{N}$

Let $X \supseteq \mathcal{U}$

For every chart M_i of $\mathcal{M}$

There is a degeneration

$$\mathcal{U} \Rightarrow T^{M_i} = \text{Hom}(M_i, \mathbb{C}^*)$$

$\cap$

$\cap$

$$X \Rightarrow X_i \text{ normal toric variety}$$

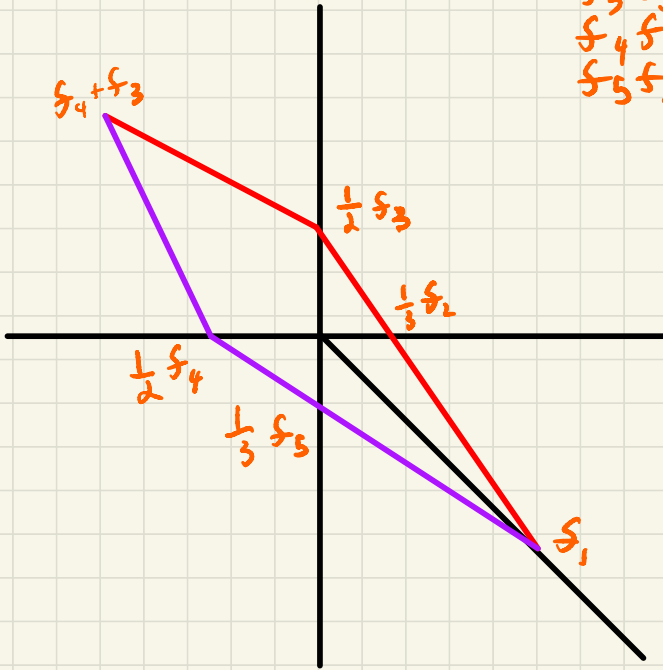
For every chart N_j of $\mathcal{N}$

There is an open Torus

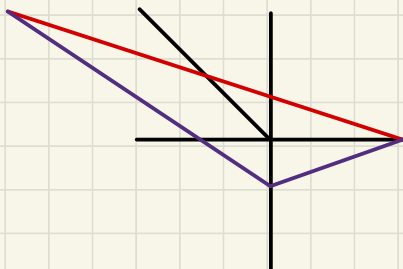
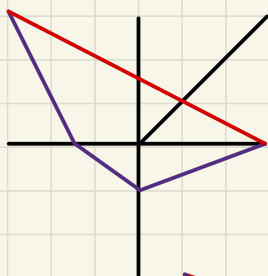
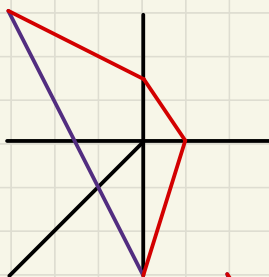
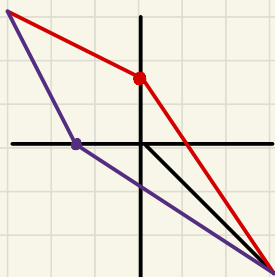
$$N_j \otimes \mathbb{C}^* = T_{N_j} \hookrightarrow \mathcal{U} \subseteq X$$

Bigon

$$A = \mathbb{C}[f_1, f_2, f_3, f_4, f_5] / \begin{array}{l} f_1 f_3 - f_2 - 1 \\ f_2 f_4 - f_3 - 1 \\ f_3 f_5 - f_4 - 1 \\ f_4 f_1 - f_5 - 1 \\ f_5 f_2 - f_1 - 1 \end{array}$$



$$X \setminus \mathcal{U} = D_1 \cup D_2$$



Classification

There is a notion of fan

$$\Sigma = \left\{ \begin{array}{l} \sigma_{\alpha, \beta} \in \mathcal{N}_{\mathbb{R}} \\ \text{"faces"} \end{array} \right\}$$

Classification

$A/\mathbb{C}$ a detropicalization of $\mathcal{M}$

$$\mathcal{N} = \text{Spec}(A)$$

A has a distinguished^{*}
generating set B

$$\text{Let } \mathcal{P} = \{n_1, \dots, n_e\} \subseteq \mathcal{N}$$

be a collection of primitive elements

$$\alpha \in \mathcal{P}$$

$$\beta \in B$$



$$\mathcal{U}_{\alpha, \beta} \cong \mathcal{U}$$

an affine variety

$$\sigma_{\alpha, \beta} \subseteq \mathcal{N}$$

Theorem

Let N, m be a finite dual pair,

Let $\mathcal{U}$ be a factorial detropicalization of m

① Let Σ be a fan

there is a polytypic variety

$$\mathcal{U} \subseteq X_{\Sigma}$$

Theorem (continued)

② If $X \ni \mathcal{U}$ satisfies

$$X \setminus \mathcal{U} = D_1 \cup \dots \cup D_\ell$$

with $\text{ord } \rho_i = r_{n_i}$, then

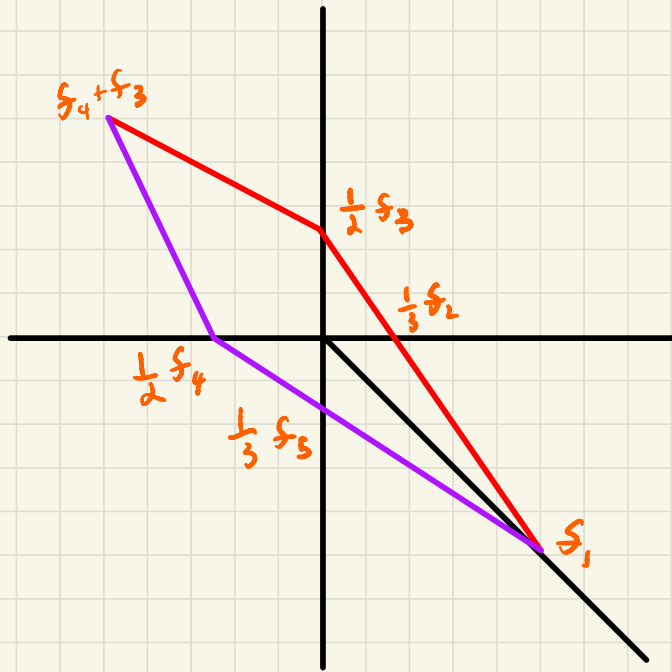
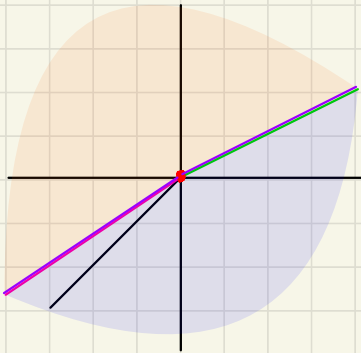
there is a unique fan Σ
on rays $\rho = \{ \rho_1, \dots, \rho_\ell \}$

with $X = X_\Sigma$

③ X is a Mori Dream Space

Bigon

$$\mathbb{C}[f_1, f_2, f_3, f_4, f_5] / \begin{aligned} &f_1 f_3 - f_2 - 1 \\ &f_2 f_4 - f_3 - 1 \\ &f_3 f_5 - f_4 - 1 \\ &f_4 f_1 - f_5 - 1 \\ &f_5 f_2 - f_1 - 1 \end{aligned}$$



$$\mathcal{CL} \simeq \mathbb{Z} \oplus \mathbb{Z}$$

$$\text{Pic} \simeq \mathbb{Z}(-7, 8) + \mathbb{Z}(8, -7)$$

Cox Ring!

$$\mathbb{C}[\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, t_1, t_2]$$

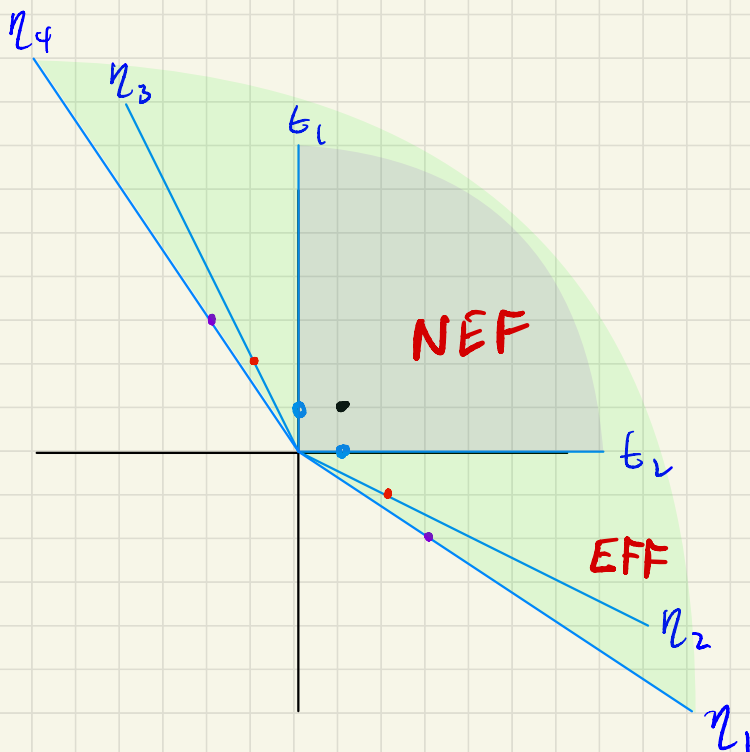
$$\eta_1 \eta_4 - \eta_0 - t_1 t_2$$

$$\eta_0 \eta_2 - \eta_1 t_2^2 - t_1^3$$

$$\eta_1 \eta_3 - t_2 \eta_2 - t_1^2$$

$$\eta_2 \eta_4 - t_1 \eta_3 - t_2^2$$

$$\eta_3 \eta_0 - t_1^2 \eta_4 - t_2^3$$



Polytopes

$\mathcal{M}, \mathcal{N}$ dual pair $\mathcal{U}$ factorial

$\Delta \in \mathcal{M}_{\mathbb{R}}$ a Polytopical Polytope

there is an inner normal fan Σ_{Δ}

X_{Σ} is projective
iff

$$\Sigma = \Sigma_{\Delta}$$

for a polytope Δ

Polytopes

$\mathcal{M}, \mathcal{N}$ dual pair $\mathcal{U}$ factorial

For any divisor D on X_Σ

there is a polytope $\Delta_D \subseteq \mathcal{M}_{\mathbb{R}}$

such that $h^0(X, D) = |\Delta_D \cap \mathcal{M}|$

Polytopes

$\mathcal{M}, \mathcal{N}$ dual pair $\mathcal{U}$ factorial

Δ "Gorenstein - Fano" makes sense

In this case, $X_{\Sigma_{\Delta}}$ is a Fano variety.

If $\mathcal{N}$ has desingularization $\mathcal{V}$

There is a dual Fano $Y_{\Sigma_{\Delta^{\vee}}} \geq \mathcal{V}$

What can be said about the hypersurfaces in these varieties?

Polytopes from Combinatorics

Theorem (Higashitani, Fujita
arXiv 2603.21588)

For a fixed marked, graded
poset, (Π, Π^*, λ) , the marked
chain order polytopes

$$\Delta_{C, \mathcal{O}}(\Pi, \Pi^*, \lambda)$$

for various partitions $\Pi \setminus \Pi^* = C \cup \mathcal{O}$
form a Polytopych Polytope
in a certain dualizable polytopych
lattice $\mathcal{M}_{\Pi, \Pi^*, \lambda}$