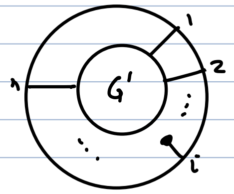
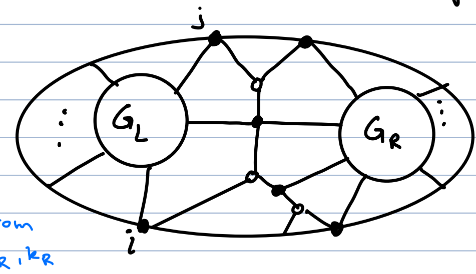
§0: Motivation

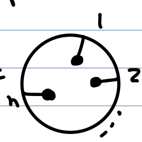
BCFW recursion for $\mathcal{N}=4$ SYM tree-level amplitudes (in momentum twistors): pick i

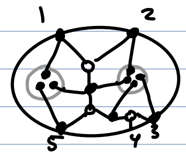
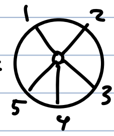
$A_{n,k} = \sum_{G'}$  $+ \sum_{j, G_L, G_R}$ 

G' from $A_{n-1,k}$; G_L from A_{n_L,k_L}; G_R from A_{n_R,k_R}; $k_L + k_R = k-1$

- Ea graph is a rational function, sums are over graphs/fcns appearing for smaller n, k.

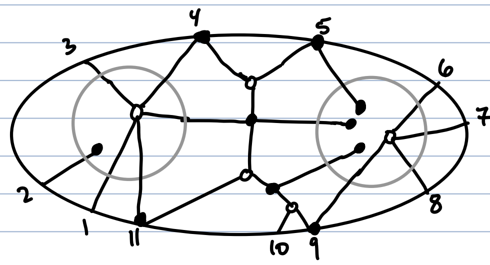
Few steps in recursion:

$A_{n,0} =$  $\rightsquigarrow 1$

$A_{5,1} =$  $=$  $\rightsquigarrow \frac{1}{\langle 1234\rangle\langle 1235\rangle\langle 1245\rangle\langle 1345\rangle\langle 2345\rangle}$

$i=5$

Pictures here are plabic graphs of type (k,n)

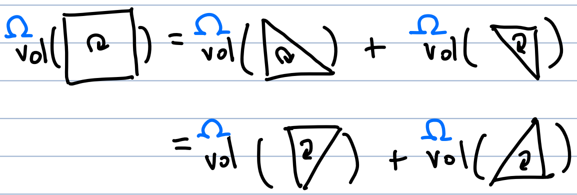
$A_{11,3} = \quad \cdots \quad +$  $+ \cdots$

$i=11$

Many many formulas for amplitude! $n=8, k=2$, 2624 formulas
$n=9, k=2$, $> 10^7$

Q: Why so many formulas? Explanation for why they're equal? Why do "spurious" poles cancel?

Hodges '09: can understand $k=1$ BCFW by looking at a polytope in $\mathbb{R}^4$.

Schematic: 

Ω: top degree form, log. singularities on bdry.

Spurious poles cancel bc that piece of bdry gives to other triangle w/ correct orientation.

Problem: Find geometric object which decomposes into pieces according to BCFW recursion, whose "canonical form" is scattering amplitude.
top degree, logarithmic singularities along bdry, extra cond. on residues.

Answer by Arkani-Hamed & Trnka '14: amplituhedron $\mathcal{A}^{Z}_{n,k,m}$.
↳ Builds on work of Postnikov on $Gr^{\geq 0}_{k,n}$, which gives a geom. space for ea. graph.

§1: Totally non-negative Grassmannian & positroids

Defn: $k \leq n$. The Grassmannian is $Gr_{k,n} := \{V \subseteq \mathbb{R}^n : \dim V = k\} \simeq GL_k \backslash Mat^{*}_{k,n}$

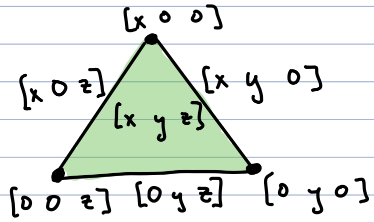
$\text{rowsp}(A) \longleftarrow [A]$

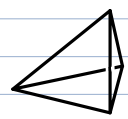
For $I \in \binom{[n]}{k}$ (k-elt subsets of $[n] := \{1, \dots, n\}$), the Plücker coordinate $p_I(V) :=$ max'l minor of A using cols I.

Fact: $\begin{array}{ccc} Gr_{k,n} & \longrightarrow & \mathbb{P}^{\binom{n}{k}-1} \\ V & \longmapsto & [p_I(V)] \end{array}$ is an embedding, image is cut out by quadratic <u>Plücker relns</u> such as $p_{13}p_{24} = p_{12}p_{34} + p_{14}p_{23}$

Defn: The <u>totally nonnegative (TNN) Grassmannian</u> is $Gr_{k,n}^{\geq 0} := \{V \in Gr_{k,n} : p_I(V) \geq 0 \ \forall I \in \binom{[n]}{k}\}$
[Postnikov, Lusztig]

e.g.

$Gr_{1,3}^{\geq 0} \approx$  $\subseteq Gr_{1,3} = \mathbb{P}^2$

$Gr_{1,4}^{\geq 0} \approx$  $\subseteq Gr_{1,4}$. In general, curvy!

Def: For $\mathcal{M} \subseteq \binom{[n]}{k}$, $C_{\mathcal{M}} := \{V \in Gr_{k,n}^{\geq 0} : p_I > 0 \iff I \in \mathcal{M}\}$. ($= Gr_{k,n}^{\geq 0} \cap$ matroid stratum)

If $C_{\mathcal{M}} \neq \emptyset$, $\mathcal{M}$ is a <u>positroid</u> of type (k,n) & $C_{\mathcal{M}}$ is a <u>positroid cell</u>.

e.g. $C_{\binom{[4]}{2} \setminus \{12\}} = \left\{ \text{rowsp} \begin{bmatrix} 1 & a & 0 & -b \\ 0 & 0 & 1 & c \end{bmatrix} : a,b,c \in \mathbb{R}_{>0} \right\}$

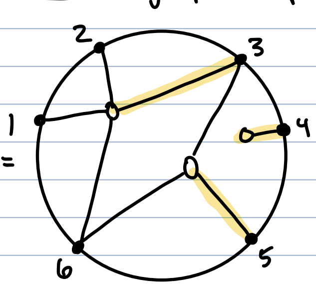
Thm: $Gr_{k,n}^{\geq 0} = \bigsqcup_{\substack{\mathcal{P} \text{ type } (k,n) \\ \text{positroid}}} C_{\mathcal{P}}$ is a <u>regular CW-clx homeo. to closed ball</u>
[Galashin-Karp-Lam], using [Williams]

"next-best-thing to a polytope" - L. Williams

- In particular, $C_{\mathcal{P}} \cong (\mathbb{R}_{>0})^{\dim C_{\mathcal{P}}}$ [Rietsch, Postnikov] & $\overline{C_{\mathcal{P}}} = \bigsqcup_{\mathcal{P}' \leq \mathcal{P}} C_{\mathcal{P}'}$ [Rietsch]

Postnikov: Many different indexing sets for positroid cells, including plabic graphs ("up to moves").

<u>Plabic graph</u>: bipartite, embedded in disk w/ black bdry vt $1, \ldots, n$ clockwise.
(Every leaf must be adj. to a bdry vt. Every conn. comp. must include a bdry vt. Bdry of disk b/w $i, i+1$ <u>NOT</u> an edge).

$G =$ 

$\partial M = 345$

Defn: An <u>almost-perfect matching</u> M of G is a matching (i.e. subset of edges, s.t. no vt is in >1 edge in M) which covers all <u>internal</u> vertices.

$\partial M = \{i \in [n] : i \text{ covered by an edge of } M\}$.

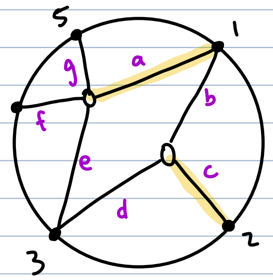
Thm: G plabic graph. Then $\{\partial M : M$ a.p.m. of $G\} = \mathcal{P}(G)$ is a positroid of type ($k = \#\text{white} - \#\text{internal black}$, n). All positroids arise this way.
[Postnikov]
w/ an a.p.m.

- Will write C_G for $C_{\mathcal{P}(G)}$.

G also gives a parametrization of C_G! Weight edges of G w/ x_e.

The <u>boundary measurements</u> of G are, for $I \in \binom{[n]}{k}$

$$\Delta_I^G := \sum_{\substack{M: \\ \partial M = I}} \prod_{e \in M} x_e$$

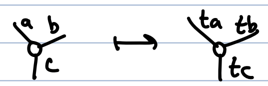
e.g. $G =$ 

$\Delta_{12}^G = ac$ $\quad\quad$ $\Delta_{13}^G = ad + be$

$\mathcal{P}(G) = \binom{[5]}{2} \setminus \{12\}$

<u>Thm:</u> [Postnikov, Lam, see also Speyer] The boundary meas. map $\Delta^G: \mathbb{R}_{>0}^E \to \mathbb{P}^{\binom{n}{k}-1}$ is a surjection onto C_G.
$(x_e) \mapsto (\Delta_I^G)$

Moreover, if G is reduced & edge wts are considered modulo guage transform,
Δ^G is a homeo $\mathbb{R}_{>0}^{\#\text{faces}-1} \xrightarrow{\sim} C_G$.

- <u>Gauge transformation</u>:  rescales all Δ_I^G by t, so no effect projectively

Edge wts modulo guage: $\mathbb{R}_{>0}^E / \mathbb{R}_{>0}^{V_{int}} \simeq \mathbb{R}_{>0}^{\#\text{faces}-1}$.

(In practice, fix some edge wts to 1 to "use up" gauge)

- <u>Reducedness</u>: G has as few faces as possible among all G' w/ $\mathcal{P}(G) = \mathcal{P}(G')$.

- Now have geom. space C_G for ea. plabic G. Does this help us understand BCFW recursion geometrically?

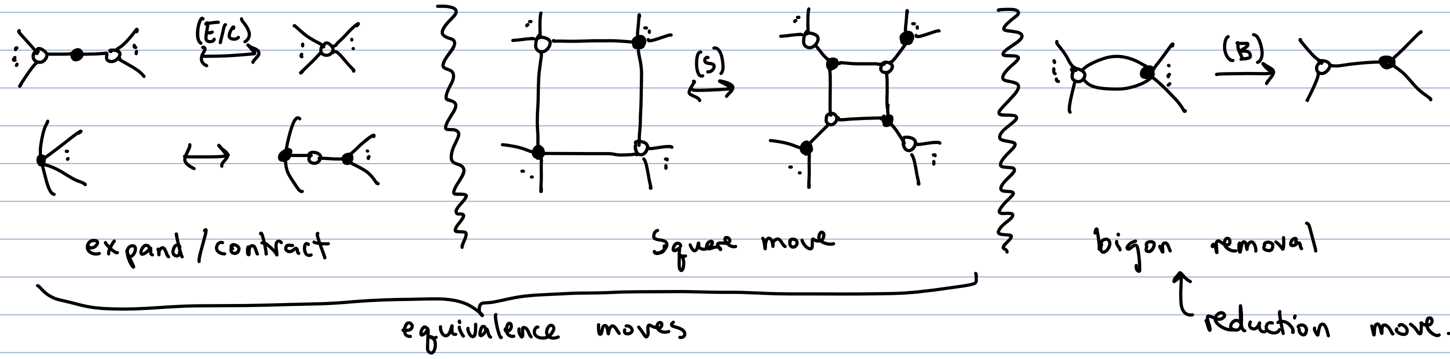
<u>No</u>: BCFW graphs index cells C_G of $\dim = 4k \overset{\text{usually}}{\ll} \dim Gr_{k,n}^{\geq 0} = k(n-k)$

↳ These cells are somewhere in bdry of $Gr_{k,n}^{\geq 0}$, & BCFW expr. corresp. to disjoint clumps of the bdry.

↳ "Natural" form assoc. to C_G is $\bigwedge_{e \text{ edge}} \frac{dx_e}{x_e}$; doesn't have much to do w/ BCFW rat'l fcns.

<u>Idea</u>: To bring these bdry cells closer together, should "squish" $Gr_{k,n}^{\geq 0}$.

Some moves don't change $\mathcal{P}(G)$:



Defn: G is <u>reduced</u> if no graph move equiv. to G has a bigon face.