Dirac fields on Kerr spacetime and the Hawking radiation I

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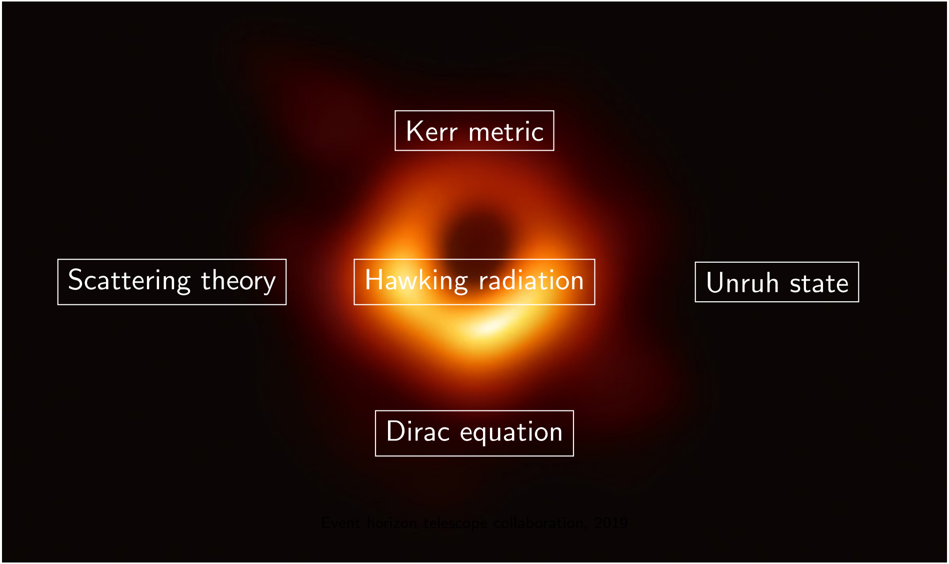


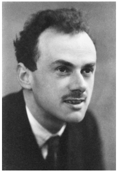
Event horizon telescope collaboration, 2019

QFT on curved spacetimes

Second quantized solutions of linear $\not{D}\hat{\phi} = 0$ on (fixed) blackhole spacetime (M, g). What is the physical solution $\hat{\phi}$?

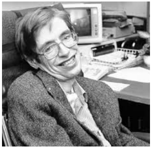
1. Locally, $\hat{\phi}$ has to look the same as vacuum solution on Minkowski. (high frequency singularities in the sense of microlocal analysis)
2. But black hole spacetimes have asymptotic symmetries, therefore global conditions (scattering theory, low frequency analysis)
3. In realistic situations, the spacetime outside a star has enough symmetries to define a vacuum state. One observes then the emergence of a thermal state when the star collapses to a black hole.

$\Rightarrow$ Quantum effects on curved spacetimes!


Paul Dirac


Roy Kerr

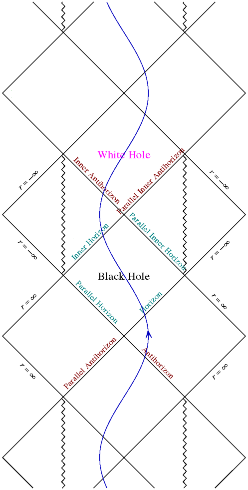

William Unruh


Stephen Hawking

Plan

1. The Kerr metric.
2. Scattering theory for Dirac fields (with Jean-Philippe Nicolas).
3. The Unruh state (with Christian Gérard, Michał Wrochna).
4. The Hawking effect.

Part I : The Kerr metric



Subrahmanyan Chandrasekhar:

"In my entire scientific life, extending over forty-five years, the most shattering experience has been the realization that an exact solution of Einstein's equations of general relativity, discovered by the New Zealand mathematician, Roy Kerr, provides the absolutely exact representation of untold numbers of massive black holes that populate the universe. This shuddering before the beautiful, this incredible fact that a discovery motivated by a search after the beautiful in mathematics should find its exact replica in Nature, persuades me to say that beauty is that to which the human mind responds at its deepest and most profound."

Einstein vacuum equation (1915)

Einstein vacuum solutions

$$\mathrm{Ric}(g) = 0,$$

where g is a Lorentzian metric $(+---)$ on a 4-manifold M.

Special solution : Minkowski space.

$$M = \mathbb{R}_t \times \mathbb{R}^3_x,$$
$$g_{(0,0)} = dt^2 - dx^2 = dt^2 - dr^2 - r^2 g_{\mathbb{S}^2}.$$

Remark

We discuss only the case when $\Lambda = 0$. All solutions will be asymptotically Minkowskian. In the case $\Lambda > 0$ the corresponding solutions are De Sitter, De Sitter Schwarzschild and De Sitter Kerr. These solutions are asymptotically De Sitter.

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Schwarzschild solution (1916)

Schwarzschild black holes (mass $\mathfrak{m} > 0$). Boyer Lindquist coordinates.

$$\begin{aligned} M &= \mathbb{R}_t \times (0,\infty)_r \times \mathbb{S}^2, \\ g_{(\mathfrak{m},0)} &= \left(1 - \frac{2\mathfrak{m}}{r}\right) dt^2 - \left(1 - \frac{2\mathfrak{m}}{r}\right)^{-1} dr^2 - r^2 g_{\mathbb{S}^2} \\ &= g_{(0,0)} + \mathcal{O}(r^{-1}). \end{aligned}$$

Theorem (Birkhoff, 1923)

If a given spacetime is spherically symmetric and satisfies the Einstein vacuum equations (with $\Lambda = 0$), then it is a part of Schwarzschild's spacetime.

Singularities

$$g_{(\mathfrak{m},0)} = f(r)dt^2 - f(r)^{-1}dr^2 - r^2 g_{\mathbb{S}^2},\ f(r) = \left(1 - \frac{2\mathfrak{m}}{r}\right)$$

1. $r = 0$. Curvature singularity. Kretschmann invariant $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = 48\frac{\mathfrak{m}^2}{r^6}$.
2. $r = 2\mathfrak{m}$. Consider spherically symmetric null curves:

$$\left(1 - \frac{2\mathfrak{m}}{r}\right)\dot{t}^2 - \left(1 - \frac{2\mathfrak{m}}{r}\right)^{-1}\dot{r}^2 = 0.$$

Then $\frac{dt}{dr} = \pm\left(1 - \frac{2\mathfrak{m}}{r}\right)^{-1}$ and thus $t \pm r_* = const.$, where $r_* = r + 2\mathfrak{m}\ln(r - 2\mathfrak{m})$. Let $t^* = t + r_*$. In the coordinates (t^*, r, ω) we have

$$g_{(\mathfrak{m},0)} = \left(1 - \frac{2\mathfrak{m}}{r}\right)dt^{*2} - 2dt^*dr - r^2 g_{\mathbb{S}^2}$$

This is smooth up to $r = 2\mathfrak{m}$!

The event horizon and null infinty

1. Event horizon. The hypersurface $\mathcal{H}^+ = \mathbb{R}_{t^*} \times \{2\mathfrak{m}\}_r \times S^2_\omega$ is called the future event horizon. In a similar way we can introduce ${}^*t = t - r_*$. The surface $\mathcal{H}^- = \mathbb{R}_{{}^*t} \times \{2\mathfrak{m}\}_r \times S^2_\omega$ is called the past event horizon. $\mathcal{H}^\pm$ are null hypersurfaces.
2. Null infinity. Conformally rescaled metric : $\hat{g}_{(\mathfrak{m},0)} = x^2 g_{(\mathfrak{m},0)}$, $x = \frac{1}{r}$. Coordinates $({}^*t, x, \omega)$.

$$\hat{g}_{(\mathfrak{m},0)} = x^2(1 - 2\mathfrak{m}x)d^*t^2 - 2d^*tdx - d\omega^2.$$

The surface $\mathscr{I}^+ = \mathbb{R}_{{}^*t} \times \{0\}_x \times S^2_\omega$ is called future null infinity. Past null infinity is defined as $\mathscr{I}^- = \mathbb{R}_{t^*} \times \{0\}_x \times S^2_\omega$. The surfaces $\mathscr{I}^\pm$ are null hypersurfaces.

Kruskal coordinates

$$U = -e^{-{}^*t/4\mathfrak{m}},\ V = e^{t^*/4\mathfrak{m}},\ T = (U+V)/2,\ X = (U-V)/2.$$

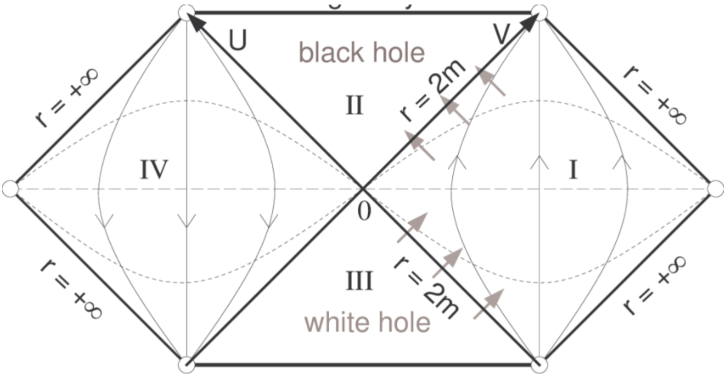
$$g_{(\mathfrak{m},0)} = \frac{32\mathfrak{m}^3 e^{-r/2\mathfrak{m}}}{r}(dT^2 - dX^2) - r^2 d\omega^2.$$

Relation between (t, r) and (T, X) :

$$\left(\frac{r}{2\mathfrak{m}} - 1\right) e^{r/2\mathfrak{m}} = X^2 - T^2,$$
$$\frac{t}{2\mathfrak{m}} = \ln\left(\frac{T+X}{X-T}\right) = 2\tanh^{-1}(T/X).$$
$$r > 0 \Leftrightarrow X^2 - T^2 > -1.$$

$\kappa = \frac{1}{4\mathfrak{m}}$ is called the surface gravity of the horizon.

The Penrose diagram



Null geodesics

γ null geodesic in the equatorial plane. Constants of motion :

1. The "mass" of the geodesic:

$$0 = g(\dot\gamma, \dot\gamma) = (1 - \frac{2\mathfrak{m}}{r})\dot t^2 - (1 - \frac{2\mathfrak{m}}{r})^{-1}\dot r^2 - r^2\dot\varphi^2. \qquad (1)$$

2. Energy linked to the Killing field ∂_t.

$$E = g(\dot\gamma, \partial_t) = (1 - \frac{2M}{r})\dot t. \qquad (2)$$

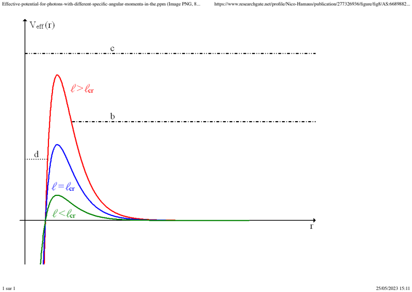
3. Angular momentum linked to the Killing field ∂_φ:

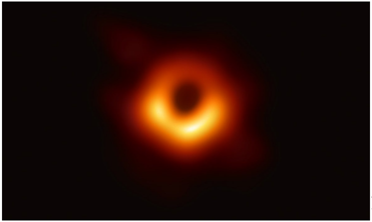
$$L = -g(\dot\gamma, \partial_\varphi) = r^2\dot\varphi. \qquad (3)$$

Substituting (3) and (2) into (1) we obtain:
$\frac{1}{2}\dot r^2 + V_L(r) = \frac{1}{2}E^2$.

Unit mass particle of energy $E^2/2$ in ordinary one-dimensional, nonrelativistic mechanics moving in an effective potential $V_L(r) = \frac{1}{2}\left(1 - \frac{2\mathfrak{m}}{r}\right)\frac{L^2}{r^2}$. V_L has a unique maximum at $r = 3\mathfrak{m}$ and $r = 3\mathfrak{m}$ is solution for suitable E.

Photon sphere



[1]

[1] Event horizon telescope collaboration, 2019

Hamiltonian flow

1. Wave equation :

$$(\partial_t^2 - \frac{1}{r^2} f(r)\partial_r r^2 f(r)\partial_r - V(r)\Delta_{S^2_\omega})\psi = 0,\ V(r) = \frac{f(r)}{r^2}.$$

2. Symbol : $p(z) = f(r)^2\xi^2 + V(r)(\alpha^2 + \frac{\beta^2}{\sin^2\theta}) - z^2.$
3. Bicharacteristics $p(z) = 0$. Conserved quantities along the bicharacteristic flow : $p(z) = 0,\ \mathcal{Q} = (\alpha^2 + \frac{\beta^2}{\sin^2\theta}),\ z,\ L = -\beta.$ Remaining equations.

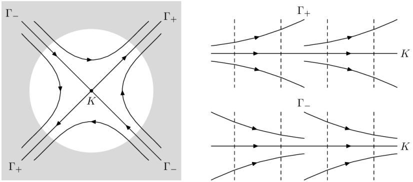
$$\dot{\xi} = -\frac{\partial V}{\partial r}\mathcal{Q} - 2f(r)f'(r)\xi^2,$$
$$\dot{r} = 2f^2(r)\xi.$$

Trapped set $\{r = 3\mathfrak{m},\ \xi = 0\}$.

Nature of the flow near the trapped set

Linearization around $\{r = 3\mathfrak{m}, \xi = 0\}$:

$$\frac{d}{ds}\begin{pmatrix} r - 3\mathfrak{m} \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & \frac{2}{9} \\ -\frac{\partial^2 V}{\partial r^2}(3\mathfrak{m})\mathcal{Q} & 0 \end{pmatrix}\begin{pmatrix} r - 3\mathfrak{m} \\ \xi \end{pmatrix}.$$



The trapping r-normally hyperbolic for each r, a property which is stable with respect to perturbations.

The Kerr solution

(mass $\mathfrak{m} > 0$, angular momentum $\mathfrak{a} \in \mathbb{R}^3$, $a = |\mathfrak{a}|$, $a < \mathfrak{m}$).

$$
\begin{aligned}
g_{(\mathfrak{m},\mathfrak{a})} &= \frac{\Delta - a^2 \sin^2\theta}{\varrho^2} dt^2 + \frac{4a\mathfrak{m}r\sin^2\theta}{\varrho^2} dt d\varphi \\
&- \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\varrho^2} \sin^2\theta d\varphi^2 - \varrho^2\left(\frac{dr^2}{\Delta} + d\theta^2\right) \\
&= g_{(\mathfrak{m},0)} + \mathcal{O}(r^{-2}), \\
&\Delta = r^2 - 2\mathfrak{m}r + a^2, \quad \varrho^2 = r^2 + a^2\cos^2\theta.
\end{aligned}
$$

1. The Kerr metric is asymptotically Minkowskian.
2. ∂_t, ∂_φ are Killing vector fields.
3. $\varrho^2 = 0$ is a curvature singularity. Kretschmann invariant :

$$
\begin{aligned}
R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} &= \frac{\mathfrak{m}^2}{\varrho^{12}}(r^2(r^2 - 3a^2\cos\theta^2)^2 \\
&- a^2\cos^2\theta(3r^2 - a^2\cos^2\theta)^2).
\end{aligned}
$$

4. Note that $\Delta = (r - r_+)(r - r_-)$, $r_\pm = \mathfrak{m} \pm \sqrt{\mathfrak{m}^2 - a^2}$.

Boyer-Lindquist blocks

$$I : r > r_+,\ II : r_- < r < r_+,\ III : r < r_-.$$

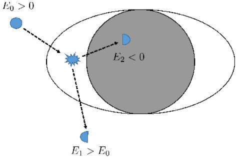
The corresponding spacetimes are denoted M_I, M_{II}, M_{III}.

1. M_I and M_{II} are causal, meaning that there doesn't exist any closed non spacelike curve.
2. The coefficient $-\frac{(r^2+a^2)^2-\Delta a^2\sin^2\theta}{\varrho^2}\sin^2\theta$ in front of $d\varphi^2$ becomes positive in a region near the ring singularity $r^2+a^2\cos^2\theta=0$, thus the closed integral curves of ∂_φ become timelike in this region. This is called the Carter time machine. Block III is vicious, for any $p, q \in III$, there exists a timelike future-pointing curve in III from p to q.
3. We will only be interested in blocks I and II.

The ergosphere

$$\mathcal{E} := \{(r, \theta, \phi);\ r^2 + a^2 \cos^2 \theta - 2\mathfrak{m}r < 0\}.$$

The Killing field $v_{\mathscr{I}^+} = \partial_t$ is spacelike in $\mathcal{E}$.



The equivalent phenomenon for the wave equation is called superradiance. Different linear combinations $\partial_t + C\partial_\varphi$ are timelike in different regions, but there is no timelike Killing field in the whole exterior region. A particular role is played by the Killing vector field $v_{\mathcal{H}^+} = \partial_t + \Omega_{\mathcal{H}}\partial_\varphi$. The constant $\Omega_{\mathcal{H}} = \frac{a}{r_+^2 + a^2}$ is the local velocity of the horizon. $v_{\mathcal{H}^+}$ is timelike close to the horizon and spacelike at infinity.

Principal null directions

1. Suppose $\mathrm{Ric}(g) = 0$. Weyl tensor

$$W(X, Y, Z, T) = g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, T).$$

2. A null vector $l \in T_x\mathcal{M}$ is called principal of multiplicity at least 2 if for all $a, b \in T_x\mathcal{M}$ such that $g(l, a) = g(l, b) = 0$ $W(l, a, l, b) = 0$.
3. Principal null directions in Kerr $V^{\pm} = \frac{r^2+a^2}{\Delta}\partial_t \pm \partial_r + \frac{a}{\Delta}\partial_\varphi$.
4. Killing tensor $K_{ab} = \Delta V^{+}_{(a} V^{-}_{b)} + r^2 g_{ab}$. $\nabla_{(a} K_{bc)} = 0$. If γ is a geodesic with tangent u^a, then $K_{ab} u^a u^b$ is conserved.

Horizons, null infinity

- Kerr-star coordinates :

$$t^* = t + r_*, r, \theta, \varphi^* = \varphi + \Lambda(r), \ \frac{dr_*}{dr} = \frac{r^2+a^2}{\Delta}, \ \frac{d\Lambda(r)}{dr} = \frac{a}{\Delta}.$$

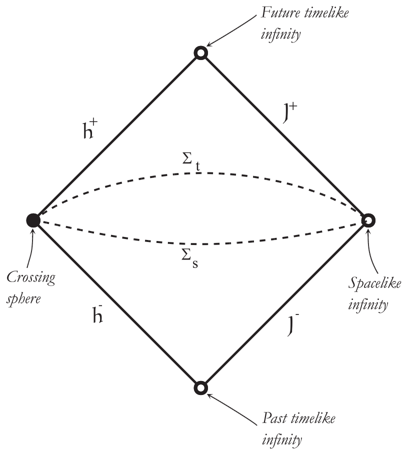
  Along incoming principal null geodesics :
  $\dot{t}^* = \dot{\theta} = \dot{\varphi}^* = 0, \ \dot{r} = -1.$

- Form of the metric in Kerr-star coordinates :
  $g = g_{tt}dt^{*2} + 2g_{t\varphi}dt^*d\varphi^* + g_{\varphi\varphi}d\varphi^{*2} + g_{\theta\theta}d\theta^2 - 2dt^*dr + 2a\sin^2 d\varphi^* dr.$

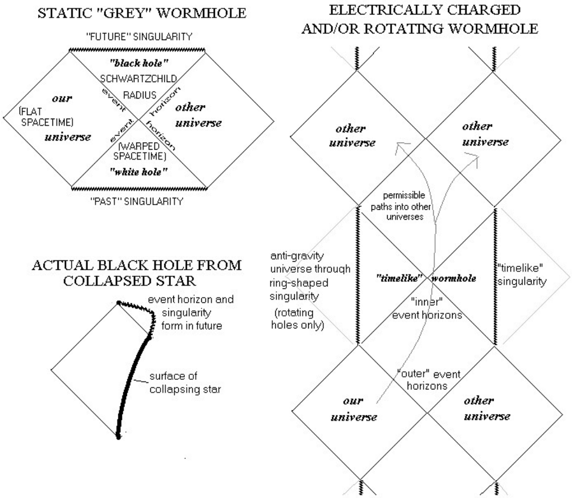
- Future event horizon : $\mathcal{H}^+ := \mathbb{R}_{t^*} \times \{r = r_+\} \times S^2_{\theta,\phi^*}$.

- The construction of the past event horizon $\mathcal{H}^-$ is based on outgoing principal null geodesics (star-Kerr coordinates). Similar constructions for future and past null infinities $\mathscr{I}^+$ and $\mathscr{I}^-$ using the conformally rescaled metric $\hat{g} = \frac{1}{r^2}g$.

The Penrose diagram



Maximal extension



Global hyperbolicity

Definition

- Given a subset S of M, the domain of dependence of S is the set of all poins in M such that every inextendible causal curve through p intersects S.
- A subset S of M is achronal if no timelike curve intersects S more than once.
- A Cauchy surface for M is a closed achronal set whose domain of dependence is M.
- The spacetime M is globally hyperbolic if it admits a Cauchy surface.

Proposition

$(M_{I\cup II}, g)$ *is globally hyperbolic.*

Geodesics

The principal symbol of $\frac{1}{2}\Box_g$ is :

$$p(z) := \frac{1}{2\rho^2}\left(-\frac{\sigma^2}{\Delta}z^2 - \frac{4\mathfrak{m}r}{\Delta}\beta z + \frac{\Delta - a^2\sin^2\theta}{\Delta\sin^2\theta}\beta^2 + \Delta|\xi|^2 - \alpha^2\right),$$
$$\sigma^2 := (r^2+a^2)^2 - a^2\Delta\sin^2\theta.$$

Theorem

We have the following constants of motion for the hamiltonian flow : p, z, $L = -\beta$, $Q = \alpha^2 + (p - z^2)a^2\cos^2\theta + \frac{\beta^2}{\sin^2\theta}$.

Q is the so called Carter constant. Thanks to the four constants of motion the geodesic equations are completely integrable.

Remark

$\sigma(\nabla_a K^{ab}\nabla_b) = Q - \beta^2 - 2az\beta.$

Results on the geodesic motion

Proposition

There exists $a_0 > 0$ such that for $|a|\mathfrak{m}^{-1} < a_0$ any future directed null geodesic in M_I which does not start at r_+ nor at ∞ passes through a region where both Killing vector fields $v_{\mathscr{I}^+}$ and $v_{\mathcal{H}^+}$ are timelike.

Proposition (Dyatlov '15)

For all $0 < a < \mathfrak{m}$ the trapping is $r-$ normally hyperbolic for all r.

Back to the Einstein equations: Initial value problem for $\mathrm{Ric}(g) = 0$

Given on $\Sigma = t^{-1}(0) \subset M$:

- γ: Riemannian metric on Σ,
- k: symmetric 2-tensor on Σ.

Find:

- Lorentzian metric g on M, $\mathrm{Ric}(g) = 0$,
- $\tau(g) := (-g|_\Sigma, \mathrm{II}^{\mathrm{g}}_\Sigma) = (\gamma, \mathrm{k})$.

Necessary and sufficient for local existence: constraint equations on (γ, k). (Choquet-Bruhat '52.)

Example

For $(\gamma, k) = (\gamma_b, k_b) := \tau(g_b)$, the solution of the initial value problem is g_b.

Kerr black hole stability

Kerr :

Theorem (Klainerman, Szeftel, Giorgi, Shen '22)

The future globally hyperbolic development of a general, asymptotically flat, initial data set, sufficiently close (in a suitable topology) to a Kerr $(\mathfrak{a}_0, \mathfrak{m}_0) = b_0$ initial data set, for sufficiently small $\frac{|a_0|}{\mathfrak{m}_0}$, has a complete future null infinity $\mathscr{I}^+$ and converges in its causal past $J^-(\mathscr{I}^+)$ to another nearby Kerr spacetime with parameters b_f close to the initial ones b_0.

De Sitter Kerr:

Theorem (Hintz, Vasy '16)

In an equivalent situation for De Sitter Kerr there exists g such that

$$g = g_{b_f} + \tilde{g}, \quad |\tilde{g}| \lesssim e^{-\beta t_*}, \; \beta > 0.$$

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