Dirac fields on Kerr spacetime and the Hawking radiation I

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Kerr metric

Scattering theory

Hawking radiation

Unruh state

Dirac equation

Event horizon telescope collaboration, 2019
QFT on curved spacetimes

**Second quantized solutions** of linear $\mathcal{D}\hat{\phi} = 0$ on (fixed) blackhole spacetime $(M, g)$. What is the physical solution $\hat{\phi}$?

1. **Locally**, $\hat{\phi}$ has to look the same as vacuum solution on Minkowski. (high frequency singularities in the sense of **microlocal analysis**)

2. But black hole spacetimes have **asymptotic symmetries**, therefore **global** conditions (scattering theory, low frequency analysis)

3. In realistic situations, the spacetime outside a star has enough symmetries to define a vacuum state. One observes then the **emergence** of a thermal state when the star collapses to a black hole.

⇒ **Quantum effects** on curved spacetimes!

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Paul Dirac  
Roy Kerr  
William Unruh  
Stephen Hawking

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Dirac fields on Kerr spacetime and the Hawking radiation I
1. The Kerr metric.
2. Scattering theory for Dirac fields (with Jean-Philippe Nicolas).
3. The Unruh state (with Christian Gérard, Michał Wrochna).
4. The Hawking effect.
Part I: The Kerr metric
"In my entire scientific life, extending over forty-five years, the most shattering experience has been the realization that an exact solution of Einstein’s equations of general relativity, discovered by the New Zealand mathematician, Roy Kerr, provides the absolutely exact representation of untold numbers of massive black holes that populate the universe. This shuddering before the beautiful, this incredible fact that a discovery motivated by a search after the beautiful in mathematics should find its exact replica in Nature, persuades me to say that beauty is that to which the human mind responds at its deepest and most profound.”
Einstein vacuum equation (1915)

Einstein vacuum solutions

\[ \text{Ric}(g) = 0, \]

where \( g \) is a Lorentzian metric \((+---)\) on a 4-manifold \( M \).

Special solution: Minkowski space.

\[ M = \mathbb{R}_t \times \mathbb{R}^3_x, \]
\[ g_{(0,0)} = dt^2 - dx^2 = dt^2 - dr^2 - r^2 g_{S^2}. \]

Remark

We discuss only the case when \( \Lambda = 0 \). All solutions will be asymptotically Minkowskian. In the case \( \Lambda > 0 \) the corresponding solutions are De Sitter, De Sitter Schwarzschild and De Sitter Kerr. These solutions are asymptotically De Sitter.
Schwarzschild solution (1916)

Schwarzschild black holes (mass $m > 0$). Boyer Lindquist coordinates.

$$M = \mathbb{R}_t \times (0, \infty)_r \times S^2,$$

$$g(m,0) = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 g_{S^2}$$

$$= g(0,0) + O(r^{-1}).$$

Theorem (Birkhoff, 1923)

*If a given spacetime is spherically symmetric and satisfies the Einstein vacuum equations (with $\Lambda = 0$), then it is a part of Schwarzschild’s spacetime.*
Singularities

\[ g(m,0) = f(r) dt^2 - f(r)^{-1} dr^2 - r^2 g_{S^2}, \quad f(r) = \left(1 - \frac{2m}{r}\right) \]

1. \( r = 0 \). Curvature singularity. Kretschmann invariant \( R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} = 48 \frac{m^2}{r^6} \).

2. \( r = 2m \). Consider spherically symmetric null curves:

\[ \left(1 - \frac{2m}{r}\right) \dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2 = 0. \]

Then \( \frac{dt}{dr} = \pm \left(1 - \frac{2m}{r}\right)^{-1} \) and thus \( t \pm r_* = \text{const.} \), where \( r_* = r + 2m \ln(r - 2m) \). Let \( t^* = t + r_* \). In the coordinates \( (t^*, r, \omega) \) we have

\[ g(m,0) = \left(1 - \frac{2m}{r}\right) dt^*{}^2 - 2dt^* dr - r^2 g_{S^2} \]

This is smooth up to \( r = 2m \)!
Event horizon. The hypersurface $\mathcal{H}^+ = \mathbb{R}_t^* \times \{2m\}_r \times S_\omega^2$ is called the future event horizon. In a similar way we can introduce $^*t = t - r_*$. The surface $\mathcal{H}^- = \mathbb{R}_r^* \times \{2m\}_r \times S_\omega^2$ is called the past event horizon. $\mathcal{H}^\pm$ are null hypersurfaces.

Null infinity. Conformally rescaled metric:

$$\hat{g}(m,0) = x^2 g(m,0), \quad x = \frac{1}{r}. \quad \text{Coordinates} \ (^*t, x, \omega).$$

$$\hat{g}(m,0) = x^2 (1 - 2mx) d^*t^2 - 2d^*tdx - d\omega^2.$$ 

The surface $\mathcal{I}^+ = \mathbb{R}_r^* \times \{0\}_x \times S_\omega^2$ is called future null infinity. Past null infinity is defined as $\mathcal{I}^- = \mathbb{R}_t^* \times \{0\}_x \times S_\omega^2$. The surfaces $\mathcal{I}^\pm$ are null hypersurfaces.
Kruskal coordinates

\[ U = -e^{-*t/4m}, \quad V = e^{t^*/4m}, \quad T = (U + V)/2, \quad X = (U - V)/2. \]

\[ g(m,0) = \frac{32m^3e^{-r/2m}}{r}(dT^2 - dX^2) - r^2d\omega^2. \]

Relation between \((t,r)\) and \((T,X)\):

\[ \left(\frac{r}{2m} - 1\right)e^{r/2m} = X^2 - T^2, \]

\[ \frac{t}{2m} = \ln\left(\frac{T + X}{X - T}\right) = 2\tanh^{-1}(T/X). \]

\[ r > 0 \iff X^2 - T^2 > -1. \]

\[ \kappa = \frac{1}{4m} \] is called the surface gravity of the horizon.
The Penrose diagram

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Null geodesics

$\gamma$ null geodesic in the equatorial plane. Constants of motion:

1. The "mass" of the geodesic:

$$0 = g(\dot{\gamma}, \dot{\gamma}) = (1 - \frac{2m}{r})\dot{t}^2 - (1 - \frac{2m}{r})^{-1}\dot{r}^2 - r^2 \dot{\phi}^2. \tag{1}$$

2. Energy linked to the Killing field $\partial_t$.

$$E = g(\dot{\gamma}, \partial_t) = (1 - \frac{2M}{r})\dot{t}. \tag{2}$$

3. Angular momentum linked to the Killing field $\partial_\varphi$:

$$L = -g(\dot{\gamma}, \partial_\varphi) = r^2 \dot{\phi}. \tag{3}$$

Substituting (3) and (2) into (1) we obtain:

$$\frac{1}{2}\dot{r}^2 + V_L(r) = \frac{1}{2}E^2.$$  

Unit mass particle of energy $E^2/2$ in ordinary one-dimensional, nonrelativistic mechanics moving in an effective potential $V_L(r) = \frac{1}{2} \left(1 - \frac{2m}{r}\right) \frac{L^2}{r^2}$. $V_L$ has a unique maximum at $r = 3m$ and $r = 3m$ is solution for suitable $E$. 

Photon sphere

Event horizon telescope collaboration, 2019

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1 Wave equation:

\[
(\partial_t^2 - \frac{1}{r^2} f(r) \partial_r r^2 f(r) \partial_r - V(r) \Delta_{S^2}) \psi = 0, \quad V(r) = \frac{f(r)}{r^2}.
\]

2 Symbol: \( p(z) = f(r)^2 \xi^2 + V(r)\left(\alpha^2 + \frac{\beta^2}{\sin^2 \theta}\right) - z^2. \)

3 Bicharacteristics \( p(z) = 0. \) Conserved quantities along the bicharacteristic flow: \( p(z) = 0, \ Q = \left(\alpha^2 + \frac{\beta^2}{\sin^2 \theta}\right), \ z, \ L = -\beta. \)

Remaining equations.

\[
\dot{\xi} = -\frac{\partial V}{\partial r} Q - 2f(r)f'(r)\xi^2,
\]

\[
\dot{r} = 2f^2(r)\xi.
\]

Trapped set \( \{r = 3m, \ \xi = 0\}. \)
Nature of the flow near the trapped set

Linearization around \( \{r = 3m, \xi = 0\} \):

\[
\frac{d}{ds} \begin{pmatrix} r - 3m \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & 2/9 \\ -\frac{\partial^2 V}{\partial r^2}(3m)Q & 0 \end{pmatrix} \begin{pmatrix} r - 3m \\ \xi \end{pmatrix}.
\]

The trapping \( r \)-normally hyperbolic for each \( r \), a property which is stable with respect to perturbations.
The Kerr solution

(mass \( m > 0 \), angular momentum \( a \in \mathbb{R}^3 \), \( a = |a| \), \( a < m \)).

\[
g_{(m,a)} = \frac{\Delta - a^2 \sin^2 \theta}{\varrho^2} dt^2 + \frac{4amr \sin^2 \theta}{\varrho^2} dtd\varphi
- \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\varrho^2} \sin^2 \theta d\varphi^2 - \varrho^2 \left( \frac{dr^2}{\Delta} + d\theta^2 \right)
\]

\[
= g_{(m,0)} + O(r^{-2}),
\]

\[
\Delta = r^2 - 2mr + a^2, \quad \varrho^2 = r^2 + a^2 \cos^2 \theta.
\]

1. The Kerr metric is asymptotically Minkowskian.
2. \( \partial_t, \partial_\varphi \) are Killing vector fields.
3. \( \varrho^2 = 0 \) is a curvature singularity. Kretschmann invariant:

\[
R_{\alpha \beta \mu \nu} R^{\alpha \beta \mu \nu} = \frac{m^2}{\varrho^{12}} (r^2 (r^2 - 3a^2 \cos^2 \theta))^2
- a^2 \cos^2 \theta (3r^2 - a^2 \cos^2 \theta)^2).
\]

4. Note that \( \Delta = (r - r_+)(r - r_-), \ r_{\pm} = m \pm \sqrt{m^2 - a^2} \).
Boyer-Lindquist blocks

\[ I: r > r_+, \quad II: r_- < r < r_+, \quad III: r < r_. \]

The corresponding spacetimes are denoted \( M_I, M_{II}, M_{III} \).

1. \( M_I \) and \( M_{II} \) are causal, meaning that there doesn’t exist any closed non spacelike curve.

2. The coefficient \(-\frac{(r^2+a^2)^2-\Delta a^2 \sin^2 \theta}{\rho^2} \sin^2 \theta\) in front of \( d\varphi^2 \) becomes positive in a region near the ring singularity \( r^2 + a^2 \cos^2 \theta = 0 \), thus the closed integral curves of \( \partial \varphi \) become timelike in this region. This is called the Carter time machine. Block \( III \) is vicious, for any \( p, q \in III \), there exists a timelike future-pointing curve in \( III \) from \( p \) to \( q \).

3. We will only be interested in blocks \( I \) and \( II \).
The ergosphere

\[ \mathcal{E} := \{(r, \theta, \phi); \ r^2 + a^2 \cos^2 \theta - 2mr < 0\} . \]

The Killing field \( v_{\mathcal{I}^+} = \partial_t \) is spacelike in \( \mathcal{E} \).

The equivalent phenomenon for the wave equation is called superradiance. Different linear combinations \( \partial_t + C \partial \varphi \) are timelike in different regions, but there is no timelike Killing field in the whole exterior region. A particular role is played by the Killing vector field \( v_{\mathcal{H}^+} = \partial_t + \Omega_{\mathcal{H}} \partial \varphi \). The constant \( \Omega_{\mathcal{H}} = \frac{a}{r^2 + a^2} \) is the local velocity of the horizon. \( v_{\mathcal{H}^+} \) is timelike close to the horizon and spacelike at infinity.
1. Suppose $\text{Ric}(g) = 0$. Weyl tensor

$$W(X, Y, Z, T) = g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, T).$$

2. A null vector $l \in T_x \mathcal{M}$ is called principal of multiplicity at least 2 if for all $a, b \in T_x \mathcal{M}$ such that $g(l, a) = g(l, b) = 0$ \(W(l, a, l, b) = 0\).

3. **Principal null directions** in Kerr $V^\pm = \frac{r^2 + a^2}{\Delta} \partial_t \pm \partial_r + \frac{a}{\Delta} \partial_\phi$.

4. Killing tensor $K_{ab} = \Delta V^+_{(a} V^-_{b)} + r^2 g_{ab} \cdot \nabla_{(a} K_{bc)} = 0$. If $\gamma$ is a geodesic with tangent $u^a$, then $K_{ab} u^a u^b$ is conserved.
Kerr-star coordinates:
\[ t^* = t + r_*, r, \theta, \varphi^* = \varphi + \Lambda(r), \quad \frac{dr^*}{dr} = \frac{r^2 + a^2}{\Delta}, \quad \frac{d\Lambda(r)}{dr} = \frac{a}{\Delta}. \]

Along incoming principal null geodesics:
\[ \dot{t}^* = \dot{\theta} = \dot{\varphi}^* = 0, \quad \dot{r} = -1. \]

Form of the metric in Kerr-star coordinates:
\[ g = g_{tt}dt^*^2 + 2g_{t\varphi}dt^*d\varphi^* + g_{\varphi\varphi}d\varphi^*^2 + g_{\theta\theta}d\theta^2 - 2dt^*dr + 2a \sin^2 \varphi^*dr. \]

Future event horizon:
\[ \mathcal{H}^+ := \mathbb{R}_t \times \{ r = r_+ \} \times S^2_{\theta, \phi^*}. \]

The construction of the past event horizon \( \mathcal{H}^- \) is based on outgoing principal null geodesics (star-Kerr coordinates). Similar constructions for future and past null infinities \( \mathcal{I}^+ \) and \( \mathcal{I}^- \) using the conformally rescaled metric \( \hat{g} = \frac{1}{r^2} g \).
Maximal extension

**STATIC "GREY" WORMHOLE**
- "future" singularity
- "black hole"
- Schwartzchild radius
- Flat spacetime universe

**ELECTRICALLY CHARGED AND/OR ROTATING WORMHOLE**
- Permeable paths into other universes
- Anti-gravity universe through ring-shaped singularity (rotating holes only)

**ACTUAL BLACK HOLE FROM COLLAPSED STAR**
- Event horizon and singularity form in future
- Surface of collapsing star

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Global hyperbolicity

Definition

- Given a subset $S$ of $M$, the **domain of dependence** of $S$ is the set of all points in $M$ such that every inextendible causal curve through $p$ intersects $S$.
- A subset $S$ of $M$ is **achronal** if no timelike curve intersects $S$ more than once.
- A **Cauchy surface** for $M$ is a closed achronal set whose domain of dependence is $M$.
- The spacetime $M$ is **globally hyperbolic** if it admits a Cauchy surface.

Proposition

$\left( M_{I\cup II} , g \right)$ is globally hyperbolic.
Geodesics

The principal symbol of $\frac{1}{2} \Box_g$ is:

$$p(z) := \frac{1}{2\rho^2} \left( -\frac{\sigma^2}{\Delta} z^2 - \frac{4mr}{\Delta} \beta z + \frac{\Delta - a^2 \sin^2 \theta}{\Delta \sin^2 \theta} \beta^2 + \Delta |\xi|^2 - \alpha^2 \right),$$

$$\sigma^2 := (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta.$$

Theorem

We have the following constants of motion for the hamiltonian flow: $p, z, L = -\beta, Q = \alpha^2 + (p - z^2)a^2 \cos^2 \theta + \frac{\beta^2}{\sin^2 \theta}.$

$Q$ is the so called Carter constant. Thanks to the four constants of motion the geodesic equations are completely integrable.

Remark

$$\sigma (\nabla_a K^{ab} \nabla_b) = Q - \beta^2 - 2az\beta.$$
Proposition

There exists $a_0 > 0$ such that for $|a| m^{-1} < a_0$ any future directed null geodesic in $M_I$ which does not start at $r_+$ nor at $\infty$ passes through a region where both Killing vector fields $v_{\mathcal{I}+}$ and $v_{\mathcal{H}+}$ are timelike.

Proposition (Dyatlov '15)

For all $0 < \alpha < m$ the trapping is $r-$ normally hyperbolic for all $r$. 

Back to the Einstein equations: Initial value problem for $\text{Ric}(g) = 0$

Given on $\Sigma = t^{-1}(0) \subset M$:
- $\gamma$: Riemannian metric on $\Sigma$,
- $k$: symmetric 2-tensor on $\Sigma$.

Find:
- Lorentzian metric $g$ on $M$, $\text{Ric}(g) = 0$,
- $\tau(g) := (-g|_{\Sigma}, \Pi_{\Sigma}^g) = (\gamma, k)$.

Necessary and sufficient for local existence: constraint equations on $(\gamma, k)$. (Choquet-Bruhat '52.)

Example
For $(\gamma, k) = (\gamma_b, k_b) := \tau(g_b)$, the solution of the initial value problem is $g_b$. 
Kerr black hole stability

Kerr:

**Theorem (Klainerman, Szeftel, Giorgi, Shen ’22)**

The future globally hyperbolic development of a general, asymptotically flat, initial data set, sufficiently close (in a suitable topology) to a Kerr \((a_0, m_0) = b_0\) initial data set, for sufficiently small \(\frac{|a_0|}{m_0}\), has a complete future null infinity \(\mathcal{I}^+\) and converges in its causal past \(\mathcal{J}^-(\mathcal{I}^+)\) to another nearby Kerr spacetime with parameters \(b_f\) close to the initial ones \(b_0\).

De Sitter Kerr:

**Theorem (Hintz, Vasy ’16)**

In an equivalent situation for De Sitter Kerr there exists \(g\) such that

\[
g = g_{b_f} + \tilde{g}, \quad |\tilde{g}| \lesssim e^{-\beta t_*}, \quad \beta > 0.
\]


