Harmonic maps and random walks on countable groups

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Geometry beyond Riemann: Curvature and Rigidity
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A consequence of Main Theorem

Γ: a countable group
Y: a CAT(0) space loc. cpt or of finite telescopic dim. (introduced by Caprace-Lytchak '10)

Theorem (I. '23)

$\rho\colon \Gamma \to \mathrm{Isom}(Y)$: a homomorphism
μ: irreducible symmetric probability measure on Γ with finite 2nd moment w.r.t. ρ.
If $\rho(\Gamma)$ does not fix a point in ∂Y, then either

- $\exists F \subset Y$ a flat subspace with $\rho(\Gamma)(F) = F$, or
- $\exists \varphi\colon \partial_P \Gamma \to \partial Y$: a canonical ρ-equivariant map.

$\partial_P \Gamma$: Poisson boundary of (Γ, μ), ∂Y: boundary at ∞ of Y.
$\varphi\colon \partial_P \Gamma \to \partial Y$: ρ-equiv. $\Leftrightarrow$ $\rho(\gamma)\varphi(\xi) = \varphi(\gamma\xi)$ for $\gamma \in \Gamma$, $\xi \in \partial_P \Gamma$.

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This refines a theorem due to Bader-Duchesne-Lécureux ('16).
Our φ is canonical: φ is obtained as an extension of an orbit map $\gamma \mapsto \rho(\gamma)p$ to $\partial_P \Gamma$ for $p \in Y$.

CAT(0) spaces

$Y = (Y, d)$: a complete metric space

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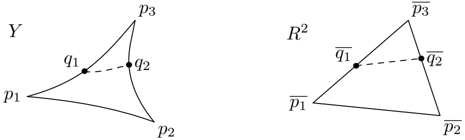

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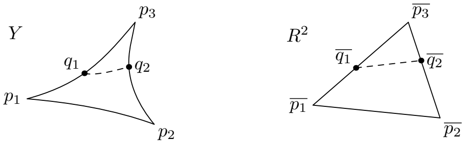

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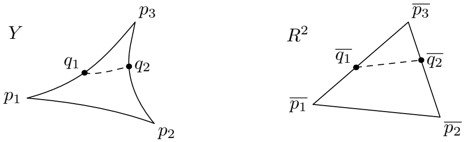

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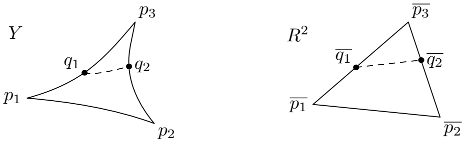

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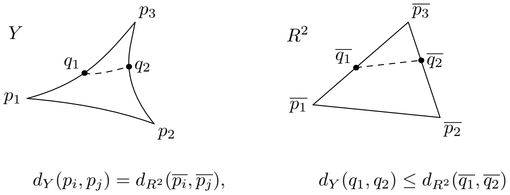

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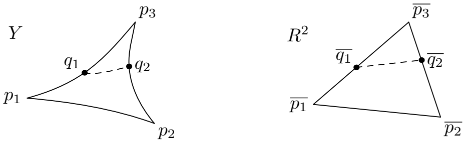

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• u has the "weakest convexity" "⇒" Y is flat.

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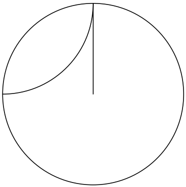

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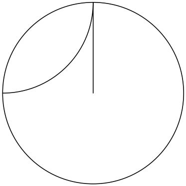

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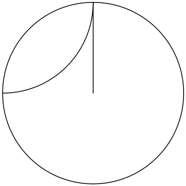

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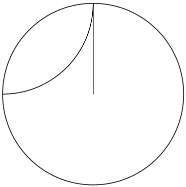

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Note. If Y is locally compact, then $Y \cup \partial Y$ becomes compact.

Random walk on a group and its Poisson boundary

Γ: a countable group with a probability measure μ.
We assume that μ is

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$$\gamma_n : \Omega \to \Gamma; \ \omega \mapsto \omega_1 \ldots \omega_n.$$

Then $\mu^n := (\gamma_n)_* \mathbb{P}$ is the n-step transition probability of the random walk generated by μ:

$$\mu^n(\gamma) = (\gamma_n)_* \mathbb{P}(\gamma) = \sum_{\omega : \gamma_n(\omega) = \gamma} \mu(\omega_1) \ldots \mu(\omega_n).$$

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The Poisson boundary of Γ w.r.t. μ is the probability space describing the distribution of the position of a random walk at "time ∞":

$$\partial_P \Gamma \text{ `` } = \text{ '' } \lim_{n\to\infty} (\Gamma, \mu^n).$$

If the random walk "diverges", then $\partial_P \Gamma$ can be viewed as a boundary at ∞ of Γ, and Γ acts on $\partial_P \Gamma$.

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$$\omega \mapsto \lim_{n\to\infty} \frac{d(p, \rho(\gamma_n(\omega))p)}{n} \to \exists C \in [0, \infty) \text{ in } L^1(\Omega, \mathbb{P}).$$

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Theorem (Karlsson-Margulis '99)

If $l_\rho(\Gamma) > 0$, then, for almost every $\omega \in \Omega$, $\{\rho(\gamma_n(\omega))p\}$ converges to a point in ∂Y.

This convergence induces an equivariant map $\varphi \colon \partial_p\Gamma \to \partial Y$.

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Main Theorem (I. '23)

Suppose that $l_\rho(\Gamma) = 0$, and that

- Y is either locally compact or of teledim$< \infty$.
- μ has finite 2nd moment w.r.t. $\rho \colon \Gamma \to \mathrm{Isom}(Y)$ and $\rho(\Gamma)$ does not fix a point in ∂Y.

Then there exists a $\rho(\Gamma)$-invariant convex subset isometric to $\mathbb{R}^k$.

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Proof of Main Theorem

The proof uses a ρ-equivariant harmonic map $f\colon \Gamma \to Y$.

- ρ-equivariance: $\rho(\gamma)f(\gamma') = f(\gamma\gamma')$ for $\forall \gamma, \gamma' \in \Gamma$,
- harmonicity: f minimizes μ-energy E_μ

$$E_\mu\colon f \mapsto \frac{1}{2}\int_{\gamma\in\Gamma} d(f(e), f(\gamma))^2 d\mu(\gamma)$$

· If $\rho(\Gamma)$ does not fix a point in ∂Y, there exists a ρ-equivariant μ-harmonic map.
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Since $f\colon \Gamma \to Y$ is μ-harmonic,

$$u\colon Y \to \mathbb{R} \text{ convex } \Rightarrow \Delta_\mu f^* u \leq 0,$$

where

$$\Delta_\mu h(\gamma) = h(\gamma) - \int_{\gamma' \in \Gamma} h(\gamma\gamma') d\mu(\gamma').$$

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If $h(\gamma) = f^* d(f(e), \cdot) = d(f(e), f(\gamma))$, we see that

$$\begin{aligned}
&\int_\Gamma \Delta_\mu h(\gamma) d\mu^n(\gamma) \\
&= \int_\Gamma \underbrace{\left(d(f(e), f(\gamma)) - \int_\Gamma d(f(e), f(\gamma\gamma')) d\mu(\gamma') \right)}_{\le 0} d\mu^n(\gamma) \\
&= \int_\Gamma d(f(e), f(\gamma_n(\omega))) d\mathbb{P}(\omega) - \int_\Gamma d(f(e), f(\gamma_{n+1}(\omega))) d\mathbb{P}(\omega).
\end{aligned}$$

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If $l_\rho(\Gamma) = 0$, for a. a $\omega \in \Omega$,

$$\Delta h(\gamma_n(\omega)) \overset{n \to \infty}{\to} 0, \quad \text{where } h(\gamma) = d(f(e), f(\gamma))$$

and hence

$$\Delta h_n(e) \overset{n \to \infty}{\to} 0, \quad \text{where } h_n(\gamma') = d(f(\gamma_n(\omega)^{-1}), f(\gamma')).$$

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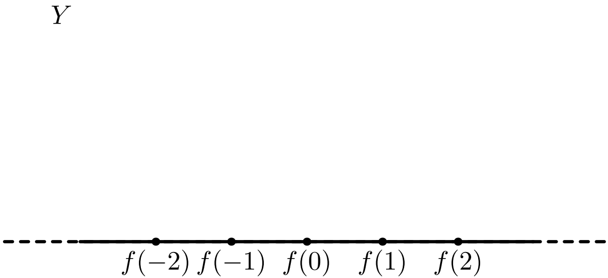
If $l_\rho(\Gamma) = 0$, $\exists \xi \in Y \cup \partial Y$ such that $\Delta f^* u \equiv 0$, where $u(p) = d(\xi, p)$ or $u(p) = b_\xi(p)$, which means that u has the weakest possible convexity around $f(\Gamma)$, which leads us to see the convex hull of $f(\Gamma)$ is flat.

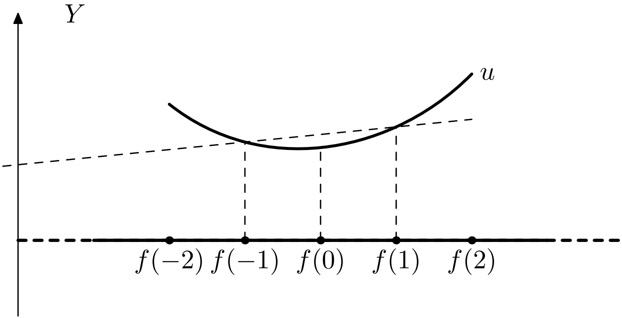
Proof of Main Theorem

Ex. $\Gamma = \mathbb{Z}$, $\mu(1) = \mu(-1) = 1/2$.
$f : \mathbb{Z} \to Y$ is μ-harmonic $\Leftrightarrow$ $f(0)$ is the midpoint of $f(\pm 1)$.
The same is true for any $n \in \mathbb{Z}$.



Proof of Main Theorem

Ex. $\Gamma = \mathbb{Z}$, $\mu(1) = \mu(-1) = 1/2$.
$f : \mathbb{Z} \to Y$ is μ-harmonic $\Leftrightarrow$ $f(0)$ is the midpoint of $f(\pm 1)$.
f^*u is subharmonic for a convex function u on Y.



Proof of Main Theorem

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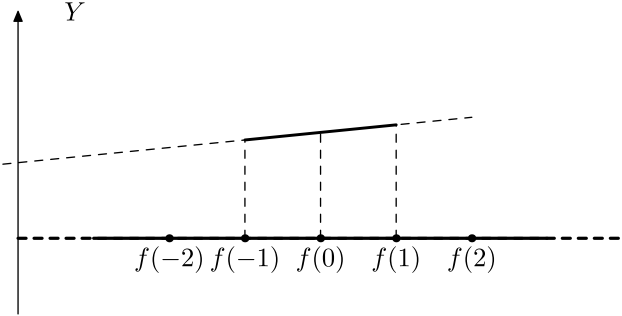
If f^*u is harmonic,



References

Application

Theorem

Γ: a countable group with property (T) with prob. meas. μ
Y: CAT(0) space which is loc. cpt or of teledim$< \infty$
ρ: $\Gamma \to \mathrm{Isom}(Y)$: a homo. with finite 2nd moment w.r.t. μ.
$\Rightarrow$ either $\rho(\Gamma)$ fixes a point in $Y \cup \partial Y$, or $\exists \varphi \colon \partial_P \Gamma \to \partial Y$.

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Theorem

Γ: acting geometrically on an irreducible Riemannian symmmetric space of noncampact type or an irreducible Euclidean building with strongly transitive full isometry.
Y: CAT(0) space which is loc. cpt or of teledim$< \infty$
$\rho\colon \Gamma \to \mathrm{Isom}(Y)$: homo. $\rho(\Gamma)$ does not fix a point in ∂Y.
$\Rightarrow \exists \mu$ s.t. any $\gamma \mapsto \rho(\gamma)p$ induces a radially affine Lipschitz map $\Phi\colon \mathrm{Cone}(\partial_{\mathrm{Tits}} X) \to \mathrm{Cone}(\partial_{\mathrm{Tits}} Y)$