Speaker: Yvain Bruned
(Edinburgh)

Discussant: Tom Klose
(TU Berlin)

Discussion of

Locality for singular SPDEs

Overview:

$\xi = (\xi_1, \dots, \xi_{n_0})$
n_0-dim. noise

$$(\partial_{x_0} - \mathcal{L}^i_{x_1}) u_i = f^i(u)\,\xi + g^i(u, \partial_{x_1} u) \quad \text{on } \mathbb{R} \times \mathbb{T} \ni (x_0, x_1)$$

for $i \in [\![k_0]\!]$

$$\mathcal{L}^i_{x_1} v = a^i(\cdot)\,\partial^2_{x_1} v + b^i(\cdot)\,\partial_{x_1} v$$

2^{nd} order elliptic op. w/ smooth coeff.

2. How does one figure out the renormalised equation in greatest generality?

ξ is "rough"

⇓

u and/or $\partial_{x_1} u$ are "rough" (think: ST distributions)

⇓

$f^i(u)\xi$ and/or $g^i(u, \partial_{x_1} u)$ ill-def.

⇓

renormalised equation with "well-def." product

Conclusive answer by:

translation invariant setting i.e. a^i, b^i = const.

- [Bruned, Chandra, Chevyrev, Hairer: Renormalising SPDEs in regularity structures, JEMS '21]
- Simplified proof: [Bailleul & Bruned: Renormalised singular SPDEs]

Non-translation invariant setting

- Extension: [Bailleul & Bruned: Locality for singular SPDEs]

Philosophy, in more detail: $(\partial_t - L)\, u_\varepsilon = F(u_\varepsilon, \nabla u_\varepsilon)\, \xi_\varepsilon$ w/ $\xi_\varepsilon \hat{=}$ mollified noise (#) $[a, b = \text{const.}]$

① Formal Picard iteration

→ collection $\mathcal{T}$ of trees τ encoding ill-def. products

e.g. $\curlyvee \equiv \mathcal{I}(\Xi)^2$

$\hat{=} (P * \xi_\varepsilon)^2$

② "Extend" smooth noise ξ_ε

→ model $\Pi_x^{\xi_\varepsilon} \in \mathcal{L}(\mathcal{T}, \mathcal{S}'(\mathbb{R}^{d+1}))$

- multiplicative! (w.r.t. tree product)
- $\Pi_x^{\xi_\varepsilon}\tau \;\hat{=}$ local description of $\Pi^{\xi_\varepsilon}\tau$ at ST point x

E.g. $\Pi_x^{\xi_\varepsilon} \curlyvee = (\Pi_x^{\xi_\varepsilon} \,\Upsilon)^2 = [P * \xi_\varepsilon - P * \xi_\varepsilon(x)]^2$

③ Based on model $(\Pi_x^{\xi_\varepsilon} : x \in \mathbb{R}^{d+1})$:

→ spaces of mod. distrib. $\mathcal{D}^\gamma(\Pi)$ that contain solutions to

$\underline{u} = \mathcal{K}^Z(\bar{F}(\underline{u}, D\underline{u})\, \Xi) + [\text{I.C.}]$

$\hat{=}$ abstract version of (#)

④ One main result of [BCCH21]: $\underline{u} = \sum_\tau \frac{\tilde{F}(\tau)(\underline{u}, D\underline{u})}{S(\tau)}\, \tau$ (dep. on model Z!)

⑤ Go back to (#) via reconstruction operator $\mathcal{R}^{Z^{\xi_\varepsilon}} \underline{u} = u_\varepsilon$ → This solves (#)!

⇒ We get back what we know for $\varepsilon > 0$. Problem: $\lim_{\varepsilon \downarrow 0} u_\varepsilon$ does not exist!

($\nearrow$ ill-def. products)

⇒ Renormalisation in RS: on the level of the model in step ②.

New model: $\hat{Z}^{\xi_\varepsilon} := M_\varepsilon Z^{\xi_\varepsilon}$ gives $\hat{u}_\varepsilon := \mathcal{R}^{\hat{Z}^{\xi_\varepsilon}} \underline{u}$ ($M_\varepsilon \in \mathfrak{R} \;\hat{=}$ renormalisation group)

$\boxed{NB}$ $\varepsilon > 0$: $\hat{u}_\varepsilon(z) = \left(R^{\hat{Z}^{\xi_\varepsilon}} \underline{u}\right)(z) = \left(\hat{\Pi}_z^{\xi_\varepsilon} \underline{u}(z)\right)$ w/ $\underline{u} \overset{④}{=} \sum_\tau u_\tau \tau$ as above

One would like to have : $\hat{\Pi}^{\xi_\varepsilon} = \Pi^{\xi_\varepsilon} M_\varepsilon^*$ ⊛ to calculate eq. for $\hat{u}_\varepsilon$

!

- transl.-inv. case: complicated (↗ extended decorations in [BCCH21])

- NON transl.-inv. case: No hope to find $M_\varepsilon : \mathbb{R}^{d+1} \times \mathcal{T} \to \mathcal{T}$ (state space dep.) s.t. ⊛ holds (kernels on internal edges of τ dep. on another ST point than the "base point" indexing M_ε)

<u>Remedy of Yvain & Ismael:</u> Preparation maps

- <u>same</u> algebraic structure (although: state-space dep.)
- heat calculus for analytic estimates

Recap: $R : \mathbb{R}^{d+1} \times \mathcal{T} \to \mathcal{T}$ s.t. • $R(x,\cdot)|_{poly} = \mathrm{Id}$ and

- $R(x,\tau) = \tau + \sum_{i=1}^{n} \lambda_i(x) \tau_i$ for $n = n(\tau) \in \mathbb{N}_0$, $\lambda_i \in C^\infty(\mathbb{R}^{d+1}, \mathbb{R})$ bdd., $\deg(\tau_i) \geq \deg(\tau)$, $|\tau|_\Xi > |\tau_i|_\Xi$

- $\left(R(x,\cdot) \otimes \mathrm{Id}\right)\Delta = \Delta R$

renormalisation <u>fcts.</u>

(+ another crucial algebraic identity w.r.t. product $\star$ for "strong" preparation maps)

A quick run through merits and open problems w.r.t. SPM:

- strong preparation maps have nice algebraic properties:
  - right morphism property, i.e. $R^*(\sigma \star \tau) = \sigma \star R^* \tau$ for all $\sigma, \tau \in \mathcal{T}$
  - they induce good multi-pre-Lie morphisms $M_R : \mathcal{T} \to \mathcal{T}$

  open: 1-to-1 correspondence (like for BRP etc.)

- $\exists$ Canonical construction. $R = (R(x,\cdot) : x \in \mathbb{R}^{d+1})$

SPM R $\Longrightarrow$ models Z^R, $\tilde{Z}^R$ w/

- $(\Pi^R_x \tau)(y) = \tilde{\Pi}^R_x (R(y)\tau)(y)$
- $\tilde{\Pi}^R_x$ is multiplicative

$\Longrightarrow$ $v \in \mathcal{D}^\gamma$ for $\gamma > 0$:

$$\left(\mathcal{R}^{Z^R} v\right)(x) = \tilde{\Pi}^R_x (R(x)\, v(x))(x)$$

$\Downarrow$ $\exists$ smooth

$$(\partial_t - L_{x_1})\, u = \mathcal{R}^{Z^R} v$$
$$= F(R^* \Xi)\,(u(x), \partial_{x_2} u(x))\, \xi(x)$$

Ex.: BPHZ renormalisation [Bruned, Hairer, Zambotti, Inventiones '19]

$$R^*_\ell(\tau) := \sum_{\sigma \in B^-} \ell(\sigma)(\tau \star \sigma), \quad \text{e.g. } \ell = \ell_\varepsilon \text{ w/ } \ell_\varepsilon(x,\tau) = \mathbb{E}[\Pi^{R_{\ell_\varepsilon}} \tau(x)]$$

B^-: basis for $\mathcal{T}^-$

(consistent w/ transl. inv. setting where $\ell_\varepsilon(\tau) = \mathbb{E}[\Pi^{\xi_\varepsilon} \tilde{\mathcal{A}}_- \tau(0)]$, $\tilde{\mathcal{A}}_- \hat{=}$ twisted antipode)

open: Convergence $Z^{R_{\ell_\varepsilon}}$ as $\varepsilon \downarrow 0$ like in [Chandra, Hairer '16] in non-transl. inv. setting

Questions

- 1-to-1 correspondence w/ good multi-pre Lie morphisms

  Suppose all good multi-pre Lie morphisms do / do not induce SPMs.

  ▶ What would be the consequence / impact of that?

- Representation of character group A^- in $\mathcal{L}(T)$

  Last remark in the paper suggests that one cannot find a similar representation in the non-transl. inv. like in the transl.-inv. setting.

  ▶ There is some similar identity for PM. Is there hope of identifying more structure?