Homological Quantum Mechanics

Allison Pinto
HU Berlin

Joint Work with Christoph Chiaffrino & Olaf Hohm
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Outline

Key points: the cohomology of the BV differential computes quantum expectation values, and we give a recipe on how to compute them.

1. Batalin-Vilkovisky Algebras
2. Finite-dimensional case ([Gwilliam])
3. Homotopy Retract
4. Perturbation Lemma
5. Applications

BV Algebras

A BV algebra is a graded commutative algebra $(A, \cdot)$ equipped with a graded Lie bracket $\{- , -\}$ of degree $+1$ and operator $\Delta : A \to A$ such that:

- for any $a \in A$, $\{a , -\}$ acts as a derivation on A,

$$\{a, b \cdot c\} = \{a, b\} \cdot c + (-1)^{(a+1)b} b \cdot \{a, c\} ;$$

- Δ acts as a derivation for $\{- , -\}$,

$$\Delta\{a, b\} = \{\Delta a, b\} + (-1)^{a+1}\{a, \Delta b\} ;$$

- $\{- , -\}$ is the failure of Δ being a derivation for $\cdot$,

$$(-1)^a\{a, b\} = \Delta(a \cdot b) - \Delta a \cdot b - (-1)^a a \cdot \Delta b .$$

Finite-dimensional case: toy model

[Gwilliam]
Consider $S = \frac{1}{2}ax^2$, and we want to compute

$$\langle f \rangle = \frac{1}{N} \int dx\, e^{iS/\hbar} f(x)\,, \quad \text{where } f \text{ is a polynomial}\,.$$

Want to show that the cohomology of the BV differential computes this.
Graded vector space given by space of functions:

$$F(x, x^*) = f(x) + x^* g(x)\,,$$

$$\{F, G\} := \frac{\partial_r F}{\partial x^*}\frac{\partial G}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial_l G}{\partial x^*}\,, \qquad \Delta = -\frac{\partial^2}{\partial x^* \partial x}\,.$$

One can define BV differential δ,

$$\delta := \{S, \cdot\} - i\hbar\Delta = -ax\frac{\partial}{\partial x^*} + i\hbar\frac{\partial^2}{\partial x \partial x^*}\,, \qquad \delta^2 = 0\,.$$

where S satisfies

$$\frac{1}{2}\{S, S\} - i\hbar\Delta S = 0\,.$$

Finite-dimensional case: compute δ-cohomology

Apply δ on $F(x, x^*)$:

$$\delta F(x, x^*) = -axg(x) + i\hbar g'(x)\,,$$

This is zero if $g(x) = e^{-iax^2/2\hbar}$, but this is not a polynomial, so $\ker\delta$ is given by $g(x) = 0$: $\ker\delta = \{F(x, x^*) \equiv f(x)\}$.
We want to find the cohomology classes of x^n for all n.

$$\delta(x^* x^n) = -ax^{n+1} + i\hbar n x^{n-1}$$

In cohomology we see that $x^{n+1} \sim \frac{i\hbar n}{a} x^{n-1}$.
For instance, with $n = 0$, $x \sim 0$, and $n = 1$, $x^2 \sim i\hbar/a$.
By recursion,

$$x^n \;\sim\; \begin{cases} 0 & \text{for } n \text{ odd} \\ \left(\frac{i\hbar}{a}\right)^{\frac{n}{2}} (n-1)(n-3)\cdot \dots \cdot 1 & \text{for } n \text{ even} \end{cases}$$

This is the expectation value $\langle x^n \rangle$.

Setup

Now we want quantum mechanics described by an action,

$$S[\phi] = \int_{t_i}^{t_f} \mathrm{d}t\, L(\phi(t), \dot{\phi}(t), t)\,, \quad \phi \in C^\infty([t_i, t_f])\,.$$

Enlarge space of dynamical variables, $V = V^0 \oplus V^1$,

$$V^0 \equiv C^\infty([t_i, t_f])\,, \quad V^1 \equiv \Pi C^\infty([t_i, t_f])$$

where fields in V^0 denoted by ϕ, anti-fields in V^1 denoted by ϕ^*.
Free theory is described by the chain complex (V, ∂):

$$0 \longrightarrow V^0 \xrightarrow{\partial} V^1 \longrightarrow 0$$

BV Structure

To define BV structure, we work with functionals $F[\phi, \phi^*]$,
Graded vector space given by

$$\mathcal{F}(V) = \cdots \oplus \mathcal{F}(V)^{-2} \oplus \mathcal{F}(V)^{-1} \oplus \mathcal{F}(V)^{0}$$

where the grading is -[number of ϕ^*] in the functional.
Build the BV complex: $(\mathcal{F}(V), \delta)$.

$$\delta := Q - i\hbar\Delta\,,$$

$$Q = -\int_{t_i}^{t_f} \mathrm{d}t\, EL(\phi(t)) \frac{\delta}{\delta\phi^*(t)}\,, \quad \Delta = -\int_{t_i}^{t_f} \mathrm{d}t\, \frac{\delta}{\delta\phi(t)\delta\phi^*(t)}\,,$$

where $EL(\phi(t)) = 0$ are the Euler-Lagrange equations.

Roadmap

Goal is to compute cohomology of δ.

- First, we establish a homotopy retract from (V, ∂) to the phase space $(\mathbb{R}^2, 0)$.
- This homotopy retract gives rise to a homotopy retract $(\mathcal{F}(V), Q_0) \to (\mathcal{F}(\mathbb{R}^2), 0)$, where Q_0 is the free, classical part of δ.
- Then we apply the perturbation lemma and we derive the cohomology of δ.

Homotopy Retract

Given $(x_i, x_f) \in \mathbb{R}^2$, let ϕ_{x_i,x_f} be unique solution with these bcs:

$$(\phi_{x_i,x_f}(t_i), \phi_{x_i,x_f}(t_f)) = (x_i, x_f)$$

We can define the maps

$$i : \mathbb{R}^2 \longrightarrow V\,, \qquad (x_i, x_f) \longmapsto (\phi, \phi^*) = (\phi_{x_i,x_f}, 0)\,.$$

$$p : V \longrightarrow \mathbb{R}^2\,, \qquad (\phi, \phi^*) \longmapsto (\phi(t_i), \phi(t_f))\,,$$

Diagramatically,

$$\begin{array}{ccccccc} 0 & \longrightarrow & V^0 & \xrightarrow{\partial} & V^1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & \mathbb{R}^2 & \xrightarrow{0} & 0 & \longrightarrow & 0 \end{array}$$

We have a homotopy retract if there is a map $h : V^1 \to V^0$,

$$p \circ i = \mathrm{id}\,, \quad i \circ p = \mathrm{id} - h \circ \partial\,.$$

Homotopy Retract

Homotopy retract from $(V, \partial) \to (\mathbb{R}^2, 0)$ gives rise to a homotopy retract from $(\mathcal{F}(V), Q_0) \to (\mathcal{F}(\mathbb{R}^2), 0)$.
Lifting to space of functionals

$$i^* : \mathcal{F}(V) \longrightarrow \mathcal{F}(\mathbb{R}^2) \,, \qquad i^*(F) := F \circ i \,.$$

$$p^* : \mathcal{F}(\mathbb{R}^2) \longrightarrow \mathcal{F}(V) \,, \qquad p^*(f) = f \circ p \,.$$

Any function f defines a functional via

$$F[\phi, \phi^*] = f(p(\phi, \phi^*)) = f(\phi(t_i), \phi(t_f)) \,.$$

We can show that

$$Q_0 H(\phi(t)) = \phi(t) - p^* i^* \phi(t) \,, \quad HQ_0(\phi^*(t)) = \phi^*(t)$$

where $\phi(t)$ is now viewed as functional, H is the homotopy map on $\mathcal{F}(V)$, and Q_0 is the free part of Q, e.g. for harmonic oscillator,

$$Q_0 = \int_{t_i}^{t_f} \mathrm{d}t (\ddot{\phi}(t) + \omega^2 \phi(t)) \frac{\delta}{\delta \phi^*(t))}$$

The Recipe

Want to compute the normalised expectation value of $F[\phi, \phi^*]$:
Find a member of the cohomology class of F, i.e. $F' = F + \delta G$, such that

$$F' = I(f) = p^*(f) = f \circ p$$

where f is a function of the boundary conditions.

Then f is the normalised expectation value of F.

In operator language,

$$f(x, y) = \frac{\langle y; t_f | T(F) | x; t_i \rangle}{\langle y; t_f | x; t_i \rangle}$$

$|x; t_i\rangle$, $|y; t_f\rangle$ are eigenstates of $\hat{\phi}(t)$, i.e. $\hat{\phi}(t)\,|x; t\rangle = x\,|x; t\rangle$,
T: time ordering.

Perturbation Lemma

Given the homotopy retract $(\mathcal{F}(V), Q_0, H) \to (\mathcal{F}(\mathbb{R}^2), 0)$, for the free theory, we want to extend to quantum mechanics on the interacting theory.

We view the BV-differential as a perturbed Q_0.

$$\delta = Q_0 + \eta \,, \qquad \eta = Q_I - i\hbar\Delta$$

Perturbation lemma gives us the homotopy retract $(\mathcal{F}(V), \delta, H') \to (\mathcal{F}(\mathbb{R}^2), 0)$ with new projection & inclusion maps:

$$P' = P \sum_{n\geq 0} (-\eta H)^n \,, \quad I' = I \,, \text{ where } P = i^* \,, \;\; I = p^* \,.$$

Perturbation Lemma

We now have the homotopy retract $(\mathcal{F}(V), \delta, H') \to (\mathcal{F}(\mathbb{R}^2), 0)$ with $P' : \mathcal{F}(V) \to \mathcal{F}(\mathbb{R}^2)$, and $I' : \mathcal{F}(\mathbb{R}^2) \to \mathcal{F}(V)$.
Given a functional $F \in \mathcal{F}(V)$, let us define

$$f := P'(F)\,, \qquad F' := I'P'(F)\,.$$

The homotopy retract tells us that

$$F - F' = (1 - I'P')(F) = \delta(H'(F))\,,$$

so F and F' are in the same cohomology class wrt δ.

For a functional $F \in \mathcal{F}(V)$, we can find a member of its δ-cohomology class that can be written as $F' = I'(f)$ for a function f on phase space $\mathbb{R}^2$.

Computing Expectation Values

Now we want to show that by using $P' : \mathcal{F}(V) \to \mathcal{F}(\mathbb{R}^2)$, $f := P'(F)$ computes the expectation value of F.

We first perturb Q_0 by the quantum part of δ so the perturbed projection is:

$$P_1 = P \sum_{n \geq 0} (i\hbar \Delta H)^n$$

We can show that

$$P_1 = P \sum_{n \geq 0} (i\hbar \Delta H)^n = P \exp\left(\frac{i\hbar}{2} C\right),$$

where

$$C = \int \mathrm{d}t \mathrm{d}s \, K(t,s) \frac{\delta^2}{\delta\phi(t)\delta\phi(s))},$$

and $K(t,s)$ is the Green's function.

Relation to path integral

Now we apply the perturbation lemma a second time, with the interacting part of the differential so that

$$P_2 = P_1 \sum_{n \geq 0} (-Q_I H)^n$$

We find that P' (as defined by [Doubek, Jurčo, Pulmann, 2019])

$$P'(F) = \frac{P_1(F \exp(iS_I/\hbar))}{Z}, \quad Z = P_1 \exp(iS_I/\hbar).$$

is equal to $P_2(F)$.

$$\frac{\langle y; t_f | F[\phi] | x; t_i \rangle}{\langle y; t_f | x; t_i \rangle} = \frac{P_1(F \exp(iS_I/\hbar))}{Z} = P'(F) = f$$

Application: Harmonic Oscillator

BV-differential for harmonic oscillator

$$\delta = \int_{t_i}^{t_f} \mathrm{d}t \left[\left(\ddot{\phi}(t) + \omega^2 \phi(t) \right) \frac{\delta}{\delta \phi^*(t)} + i\hbar \frac{\delta^2}{\delta \phi^*(t) \delta \phi(t)} \right] .$$

Define projection: $p : \phi \to \left(\phi(t_i), \phi(t_f) \right)$ (Dirichlet bcs).
We have the homotopy map (Dirichlet propagator):

$$h(f) = \int_{t_i}^{t} \mathrm{d}s\, f(s) \frac{\sin \omega (t - s)}{\omega} - \frac{\sin \omega (t - t_i)}{\sin \omega (t_f - t_i)} \int_{t_i}^{t_f} \mathrm{d}s\, f(s) \frac{\sin \omega (t_f - s)}{\omega} .$$

We want to compute a two-point function for $F = \phi(t)\phi(s)$.
Applying the perturbation lemma,

$$f(x, y) = \prod_{r = t, s} \left\{ \frac{\sin \omega (r - t_i)}{\sin \omega (t_f - t_i)} y + \frac{\sin \omega (t_f - r)}{\sin \omega (t_f - t_i)} x \right\} - i\hbar K_{DD}(t, s) .$$

Computation with Coherent States

The most general projections we consider are:

$$
\begin{aligned}
p : V &\longrightarrow \mathbb{R}^2\,, \\
\phi &\longmapsto (a_i\phi(t_i) + b_i\dot{\phi}(t_i), a_f\phi(t_f) + b_f\dot{\phi}(t_f))\,.
\end{aligned}
$$

In this case, the boundary states change.

$|x, t_i\rangle$ is an eigenstate $\left(a_i\hat{\phi}(t_i) + b_i\hat{\pi}(t_i)\right)|x, t_i\rangle = x\,|x, t_i\rangle$;
$\langle y, t_f|$ is an eigenstate $\left(a_f\hat{\phi}(t_f) + b_f\hat{\pi}(t_f)\right)\langle y, t_f| = y\,\langle y, t_f|$,

where $\hat{\pi}(t)$ is the momentum operator at t.

Relevant for coherent states $|z\rangle$, which are eigenstates of the annihilation operator a,

$$
a\,|z\rangle = z\,|z\rangle\ , \quad \langle z|\,a^\dagger = \langle z|\,z\,.
$$

Computation with Coherent States

Define projection $p : \phi \to (a(t_i), a^\dagger(t_f))$, $\phi^* \to 0$, where

$$a(t) = \sqrt{\tfrac{\omega}{2\hbar}}\big(\phi(t) + \tfrac{i}{\omega}\dot{\phi}(t)\big)\,, \quad a^\dagger(t) = \sqrt{\tfrac{\omega}{2\hbar}}\big(\phi(t) - \tfrac{i}{\omega}\dot{\phi}(t)\big)\,.$$

The homotopy map is now the Feynman propagator:

$$h_F(f)(t) = i\int_{t_i}^{t} \mathrm{d}s f(s)\frac{e^{-i\omega(t-s)}}{2\omega} + i\int_{t}^{t_f} \mathrm{d}s f(s)\frac{e^{i\omega(t-s)}}{2\omega}$$

With this we compute the two-point function wrt coherent states:

$$f(w,z) = \frac{\langle w|\, T(\phi(t)\phi(s))\,|z\rangle}{\langle w|z\rangle}\,,$$

where $f(w,z) = -i\hbar K_F(t,s)$

$$+ \tfrac{\hbar}{2\omega}\big(ze^{-i\omega(t-t_i)} + we^{i\omega(t-t_f)}\big)\big(ze^{-i\omega(s-t_i)} + we^{i\omega(s-t_f)}\big)\,.$$

Setting $w = z = 0$ gives us the vacuum expectation value.

Application: Unruh Effect

Given the vacuum $|0\rangle$ annihilated by $\hat{a}_k$ in an inertial frame, we want to compute the expectation value of the number operator given by annihilation/creation operators of an accelerated frame.

$$\langle N_k \rangle \equiv \langle 0| \hat{b}_k^\dagger \hat{b}_k |0\rangle \ .$$

Fields now depend on time and space, so the theory is described by

$$0 \longrightarrow V^0 \xrightarrow{\ \partial\ } V^1 \longrightarrow 0 \ ,$$

where $V^0 = C^\infty([t_i, t_f] \times \mathbb{R})$, $V^1 = \Pi C^\infty([t_i, t_f] \times \mathbb{R})$, and $\partial(\phi) = (\partial_t^2 - \partial_x^2)\phi$.

Application: Unruh Effect

Want to compute the expectation value

$$f(c,d) = \lim_{\tilde{t}\to 0} \frac{\langle d|\, T\big(\hat{b}_k^\dagger(\tilde{t})\hat{b}_k(0)\big)\,|c\rangle}{\langle d|c\rangle}\ ,$$

where $|c\rangle\,, |d\rangle$ are coherent states wrt a_k with eigenvals $c(k), d(k)$. We can determine $F[\phi]$ by expressing $b_k, b_k^\dagger$ in terms of ϕ, e.g.

$$b_k(\tilde{t}) \ = \int \mathrm{d}\tilde{x}\, e^{-ik\tilde{x}} \sqrt{\frac{\Omega_k}{4\pi\hbar}} \left(\phi + \frac{i}{\Omega_k}\partial_{\tilde{t}}\phi\right),$$

The projector we want is $p : V \to C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R})$,

$$p(\phi,\phi^*) = (a_k(t_i), a_l^\dagger(t_f))\,.$$

We compute $f(c,d) = P_1 F(c,d)$, and with $c = d = 0$, we obtain

$$f(0,0) = (e^{2\pi k/a} - 1)^{-1} \int_{-\infty}^{\infty} \mathrm{d}l\ \frac{1}{2\pi a\omega_l} = \langle N_k\rangle\,.$$

Conclusion & Outlook

- We show that the δ-cohomology computes quantum expectation values for certain types of states.
- We found a recipe to pick the correct representative of the δ-cohomology depending on in and out states.
- Applied formulation to harmonic oscillator and QFT in curved spacetime (derivation of Unruh effect).
- Future work: gauge theories?
- Computing quantum correlation functions in cosmological perturbation theory?