First expansive returns and cross sections

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April 24, 2025

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1 Families of generalized α-cf of Calta-Kraaikamp-S

2 2-D extensions

3 Cross section conjecture

4 Cross section and entropy

5 Sketch of proof

Group

Fix $n \geq 3$. Let $\nu = \nu_n = 2\cos\pi/n$ and $t = 1 + \nu$.
Let G_n of index $(3, n, \infty)$ be generated by

$$A = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},\ B = \begin{pmatrix} \nu & 1 \\ -1 & 0 \end{pmatrix},\ C = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix},$$

and note that $C = AB$.

The maps: Rotate until exit, translate back

Fix $\alpha \in [0,1]$ and define

$$\mathbb{I}_\alpha := \mathbb{I}_{n,\alpha} = [\,(\alpha-1)t, \alpha t\,)\,.$$

Let

$$T_\alpha = T_{n,\alpha} : x \mapsto A^k C^\ell \cdot x,$$

- $\ell > 0$ is *minimal* such that $C^\ell \cdot x \notin \mathbb{I}_\alpha$ Rotate until exit $\mathbb{I}_\alpha$.

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- $\ell > 0$ is *minimal* such that $C^\ell \cdot x \notin \mathbb{I}_\alpha$ Rotate until exit $\mathbb{I}_\alpha$.
- $k = -\lfloor (C^\ell \cdot x)/t + 1 - \alpha \rfloor$. Translate back into $\mathbb{I}_\alpha$.

Graphs

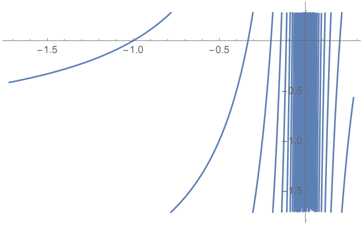

Figure: The graph of the function $x \mapsto T_{3,3,0.14}(x)$. Each branch is given by some $x \mapsto A^k C \cdot x$.

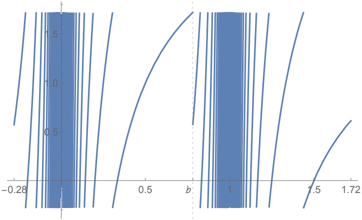

Figure: The graph of the function $x \mapsto T_{3,3,0.86}(x)$. Branches agree with $x \mapsto A^k C^2 \cdot x$ for various values of k when $x \geq b = \mathfrak{b}_\alpha$.

Calta-Kraaikamp-S: Ergodic measures, Matching intervals, continuity of entropy.

2-D set up

- Let $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. For $M \in \mathrm{SL}_2(\mathbb{R})$ and an interval $\mathbb{I}_M$, let

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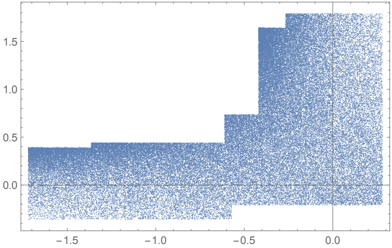
- The measure μ on $\mathbb{R}^2$ given by

$$d\mu = \frac{dx\, dy}{(1 + xy)^2}$$

is (locally) $\mathcal{T}_M$-invariant.

Measures for measure



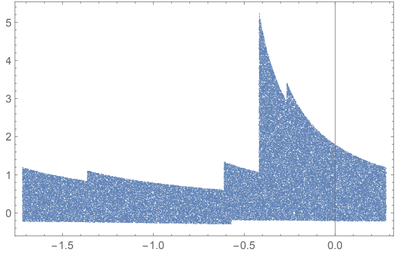


Figure: The domain $\Omega_{3,0.14}$ plotting 100,000 points. Left: $d\mu = (1+xy)^{-2} dx\, dy$ is invariant. Right: Lebesgue measure is invariant for system conjugated by $\mathcal{Z}(x,y) = (x, y/(1+xy))$.

For each T_α-system, gave an explicit planar domain on which 2-D map is bijective.

Conjecture

- Here, only have 'eventual expansiveness'. So, define $U(x) = T^k(x)$ with k minimal for $|(T^k)'(x)| > 1$.

Conjecture (CKS)

For all $n \geq 3$ and for all $\alpha \in (0,1)$ we conjecture that the first pointwise expansive power of $T_{n,\alpha}$ has its natural extension given by the first return of the geodesic flow to a cross section in the unit tangent bundle of the hyperbolic orbifold uniformized by G_n.

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- For each n, proved that this holds if and only if for any α

$$h(T_{n,\alpha})\,\mu(\Omega_{n,\alpha}) = \mathrm{vol}(T^1\, G_n \backslash \mathbb{H})\,.$$

Showed for $n = 3$.

Matrix geodesic flow

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- Identify the unit tangent bundle of $\mathbb{H}$ with $\mathrm{PSL}(2,\mathbb{R})$
- Geodesic flow now

$$\Phi_t(A) = A \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$$

Arnoux's transversal

- Any $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$ with $\gamma > 0$ is uniquely of the form

$$A(x, y)\, g_t = \begin{pmatrix} x & xy - 1 \\ 1 & y \end{pmatrix} \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$$

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- Can show that Liouville measure on $T^1\mathbb{H} \leftrightarrow \mathrm{PSL}_2(\mathbb{R})$ is

$$dx\; dy\; dt$$

Flow back to transversal

Suppose $T(x) = M \cdot x = \dfrac{ax+b}{cx+d}$.

Assume $cx + d > 0$ and let $t_0 = -2\log cx + d$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & xy-1 \\ 1 & y \end{pmatrix} \begin{pmatrix} e^{t_0/2} & 0 \\ 0 & e^{-t_0/2} \end{pmatrix}$$

$$= \begin{pmatrix} M \cdot x & * \\ 1 & (cx+d)^2 y - c(cx+d) \end{pmatrix}$$

Planar extension Ω injects into $T^1(G\backslash\mathbb{H})$

$$
\begin{tikzcd}
{[A(x,y)]_G} \arrow[r, "\text{flow}"] \arrow[d] & {M\,A(x,y)\,g_{t_0}]_G} \arrow[d] \\
(x,y) \arrow[r, "\mathcal{T}"] \arrow[d] & \mathcal{T}_M(x,y) \arrow[d] \\
x \arrow[r, "T"] & M \cdot x
\end{tikzcd}
$$

But, is our return the first return of the flow?

Entropy inequality (expansive setting), Arnoux-S '14

$$h(T) = \int_{\mathbb{I}} \log |T'(x)| \, d\nu$$ Rohlin's formula

$$= \int_{\mathbb{I}} -2 \log |cx + d| \, d\nu$$

$$= \frac{\int_\Omega -2 \log |cx + d| \, d\mu}{\mu(\Omega)}$$ replace marginal measure

$$\geq \frac{\mathrm{vol}(T^1(G\backslash\mathbb{H}))}{\mu(\Omega)}\,.$$ Arnoux transv., Hopf

In summary of last

$$h(T)\mu(\Omega) = \int_\Omega -2\log|cx+d|\,d\mu$$

and

$$\int_\Omega -2\log|cx+d|\,d\mu \geq \mathrm{vol}(T^1(\Gamma\backslash\mathbb{H}))\,.$$

First return if and only if equality holds.

Constancy of entropy $\times$ mass allows use of $\alpha = 1$

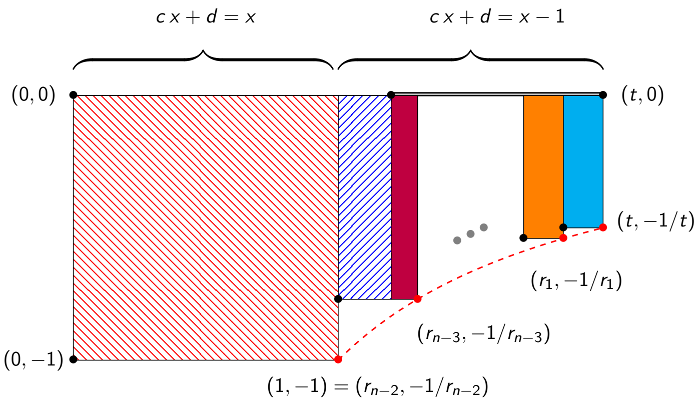

Figure: Schematic of $\Omega_{n,1}$.

Integral over square is $\pi^2/3$, enter Hecke groups

- Nakada's $\alpha = 0$ cf have as planar extension the square $\Omega_{2,3,\infty,\alpha=0} = [-1, 0] \times [0, 1]$. Entropy formula gives

$$\int_{\Omega_{2,3,\infty,\alpha=0}} -2 \log |x| \, d\mu = \pi^2/3 \,.$$

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- Now,

$$\mathrm{vol}(T^1\, G_n\backslash\mathbb{H}) - \frac{\pi^2}{3} = \frac{2(2n-3)\pi^2}{3n} - \frac{\pi^2}{3} = \frac{(n-2)\pi^2}{n}$$

$$= 2\pi(1 - \frac{1}{2} - \frac{1}{n})\pi = \mathrm{vol}(T^1(\Delta(2,n,\infty)\backslash\mathbb{H})).$$

Review the bidding

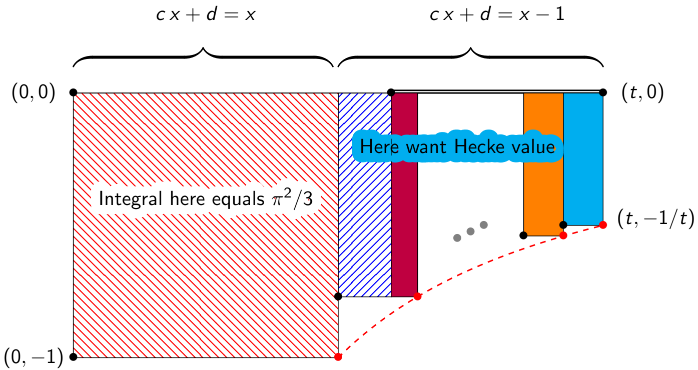

Figure: Schematic of $\Omega_{n,1}$.

Shift from Hecke to right piece

Figure: Extension of $(2, n)$ sent to $x \geq 1$ by $(x, y) \mapsto (x+1, \frac{y}{-y+1})$. [See talk.]

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Lemma

For each $n \geq 3$,

$$\int_{\Omega_{3,n,\infty,\alpha=1} \cap \{x>1\}} -2\log(x-1)\, d\mu = \int_{\Omega_{2,n,\infty,\alpha=1}} -2\log x \, d\mu.$$

Rosen, Burton-Kraaikamp-S, Nakada

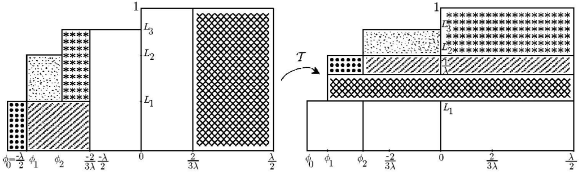

Figure: Rosen planar extension, $n = 8$

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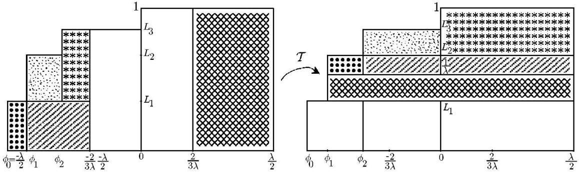

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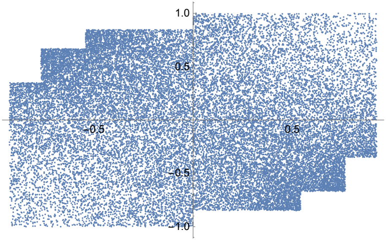
Theorem (Nakada 2010 (rephrased))

For Rosen's cf of $\Delta(2, n, \infty)$, the product of its entropy times the μ-mass of its planar extension times equals $1/2$ times the volume of the unit tangent bundle.

Symmetric Rosen



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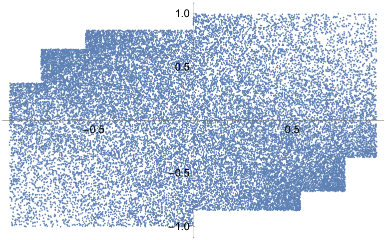

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- Corollary (Arnoux-S 2014):
  $\int_{\Omega_{2,n,\infty,\alpha=1/2}} -2\log|x|\ d\mu = \mathrm{vol}(T^1(\Delta(2,n,\infty)\backslash\mathbb{H}))$.

And ... equality holds, first return maps

Finally, as for the $m = 3$ setting, the integrals $\int_{\Omega_{2,n,\infty,\alpha}} -2\log|x|\,d\mu$ are equal. Thus, $\alpha \in (0,1)$,

$$
\begin{aligned}
h(T_{n,\alpha})\,\mu(\Omega_{n,\alpha}) &= \int_{\Omega_{3,n,\infty,\alpha}} -2\log|cx+d|\,d\mu \\
&= \int_{\Omega_{3,n,\infty,\alpha=1}} -2\log|cx+d|\,d\mu \\
&= \pi^2/3 + \mathrm{vol}(T^1(\Delta(2,n,\infty)\backslash\mathbb{H})) \\
&= \mathrm{vol}(T^1(\Delta(3,n,\infty)\backslash\mathbb{H})).
\end{aligned}
$$

Application of [Nakada, 2010]

Corollary

Suppose $f : \mathbb{I} \to \mathbb{I}$ is given piecewise by elements of a zonal G (thus determinant 1) such that

1. $\mathit{Legendre}(f) = \mathit{Lenstra}(f) \leq \frac{1}{2} \min_{c \in G} |c(g)|$,

Then, the first expansive return of f has its natural extension given by the first return system to a cross-section of $T^1(G\backslash\mathbb{H})$.

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4. $$\lim_{n\to\infty} \frac{\ln q_n}{n} = h(f).$$

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Thank you!