The Fuzzy Onion

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The goal

We want(ed) to make a matrix formulation of three-dimensional quantum space. Sort of like the fuzzy sphere but with one more (radial) dimension.

The Fuzzy Sphere

A finite-dimensional representation of $su(2)$ expressed in terms of $N \times N$ Hermitian matrices with a natural cut-off, $l \leq N-1$.

$$[L_i^{(N)}, L_j^{(N)}] = i\varepsilon_{ijk} L_k^{(N)},$$

$$[L_i^{(N)}, [L_i^{(N)}, Y_{lm}^{(N)}]] = l(l+1) Y_{lm}^{(N)}, \quad [L_3^{(N)}, Y_{lm}^{(N)}] = m Y_{lm}^{(N)},$$

$$\Phi^{(N)} = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} c_{lm}^{(N)} Y_{lm}^{(N)}.$$

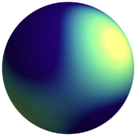
This comes without the mildly annoying $^{(N)}$ and usually with physical scales $x_i = \lambda L_i$, $x^2 = r^2$. Note that $N \sim r/\lambda + \dots$

The Fuzzy Sphere

$$\Phi^{(N)} = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} c_{lm}^{(N)} Y_{lm}^{(N)}$$

$$\Phi^{(\infty)}(\theta, \phi) = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} c_{lm}^{(\infty)} Y_{lm}^{(\infty)}$$

$$\begin{pmatrix} -0.182241 & -0.356949 + 0.0169752\, i & 0.0260558 + 0.055678\, i & -0.0418167 - 0.358403\, i \\ -0.356949 - 0.0169752\, i & 0.723061 & -0.266625 - 0.323709\, i & -0.209613 - 0.250825\, i \\ 0.0260558 - 0.055678\, i & -0.266625 + 0.323709\, i & 0.93628 & 0.115833 + 0.0969497\, i \\ -0.0418167 + 0.358403\, i & -0.209613 + 0.250825\, i & 0.115833 - 0.0969497\, i & 0.30945 \end{pmatrix}$$



The Fuzzy Sphere

Also usually one is interested in some kind of field theory on the fuzzy sphere

$$S_N[\Phi^{(N)}] = \frac{4\pi}{N}\mathrm{tr}_N \left(a\ \Phi^{(N)}\mathcal{K}^{(N)}\Phi^{(N)} + b\ (\Phi^{(N)})^2 + c\ (\Phi^{(N)})^4 \right) ,$$

where

$$\mathcal{K}^{(N)}\Phi^{(N)} = [L_i^{(N)}, [L_i^{(N)}, \Phi^{(N)}]] \ .$$

With this, one can compute mean values of observables:

$$\langle \mathcal{O}(\Psi) \rangle = \frac{1}{Z}\int d\Psi e^{-S(\Psi)}\mathcal{O}(\Psi) \ , \ d\Psi = \prod_{N=1}^{M} d\Phi^{(N)} \ .$$

This is nice!

Bosonic contruction

A similar starting point

$$[x_i, x_j] = 2\lambda i \varepsilon_{ijk} x_k .$$

We can now invoke an auxiliary Fock space and two sets of c/a bosonic operators:

$$[\mathrm{a}_\alpha, \mathrm{a}_\beta^\dagger] = \delta_{\alpha\beta}, \quad [\mathrm{a}_\alpha, \mathrm{a}_\beta] = [\mathrm{a}_\alpha^\dagger, \mathrm{a}_\beta^\dagger] = 0 \, ,$$

$$\frac{(\mathrm{a}_1^\dagger)^{n_1} \, (\mathrm{a}_2^\dagger)^{n_2}}{\sqrt{n_1! \, n_2!}} \, |0\rangle = |n_1, n_2\rangle \quad .$$

Now one can take

$$x_i = \lambda a^\dagger \sigma_i a$$

to satisfy the commutation relation and

$$r = \lambda \left(\mathrm{a}^\dagger \mathrm{a} + 1 \right) .$$

Bosonic contruction

One can define physical content using this construction; for example take $\Psi = \Psi(x)$ and

$$H_0\Psi = \frac{1}{2\lambda r}[\mathrm{a}^\dagger_\alpha, [\mathrm{a}_\alpha, \Psi]].$$

This was done thoroughly for the Coulomb problem and the spectrum was found (exactly)

$$E^l_{\lambda\, n} = \frac{\hbar}{m_e \lambda^2}\left(1 - \sqrt{1 + \frac{m_e q \lambda}{\hbar^2 n}^2}\right)$$

This is also nice but differently.

Two similar models of quantum space

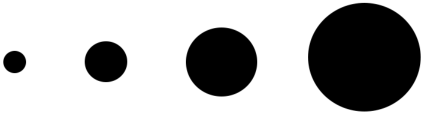
The fuzzy sphere: Convenient to work with, but it is a two-dimensional space.

Bosonic construction of $\mathbb{R}^3_\lambda$: Is three-dimensional (like our space). Working with it is slightly cumbersome.

Here we introduce **the fuzzy onion**: a three-dimensional quantum space with matrix realisation that mimics the bosonic construction.

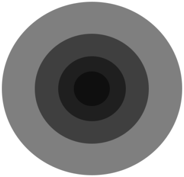
The fuzzy onion

$$\begin{pmatrix} \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$



The fuzzy onion

$$\Psi = \begin{pmatrix} (\cdot) & & & \\ & \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} & & \\ & & \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} & \\ & & & \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \end{pmatrix}$$



The kinetic term

The angular part of the kinetic term is easy to define

$$\mathcal{K}_L \Psi = r^{-2} \begin{pmatrix} \mathcal{K}^{(1)}\Phi^{(1)} & & & & \\ & \mathcal{K}^{(2)}\Phi^{(2)} & & & \\ & & \mathcal{K}^{(3)}\Phi^{(3)} & & \\ & & & \ddots & \\ & & & & \mathcal{K}^{(M)}\Phi^{(M)} \end{pmatrix}$$

Considered before by Vitale, Jurić, Wallet, Poulain and others.

The kinetic term

The radial part of the kinetic term is harder to define. We want to compare objects with different numbers of d.o.f.

$$\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \leftrightarrow \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

$$\Phi^{(N)} = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} c_{lm}^{(N)} Y_{lm}^{(N)}$$

All expansion coefficients can be matched but the ones corresponding to the highest momentum. Solution: ignore them.

The kinetic term

$$\mathcal{D} : \Phi^{(N+1)} \to \Phi^{(N)} = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} c_{lm}^{(N)} Y_{lm}^{(N)} \ , \ c_{lm}^{(N)} = c_{lm}^{(N+1)}$$

$$\mathcal{U} : \Phi^{(N)} \to \Phi^{(N+1)} = \sum_{l=0}^{N} \sum_{m=-l}^{l} c_{lm}^{(N+1)} Y_{lm}^{(N+1)} \ , \ \begin{cases} c_{lm}^{(N+1)} = c_{lm}^{(N)} \\ c_{Nm}^{(N+1)} = 0 \end{cases}$$

$$\partial_r^{(N)} \Phi^{(N)} = \frac{\mathcal{D}\Phi^{(N+1)} - \mathcal{U}\Phi^{(N-1)}}{2\lambda}$$

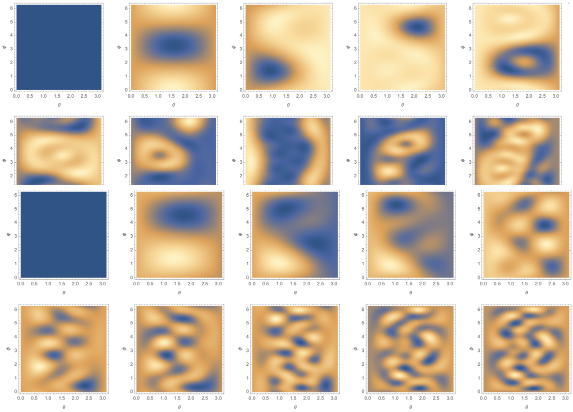
$$\mathcal{K}_R \Psi = \sum_{N,l,m} \frac{(N+1)c_{lm}^{(N+1)} + (N-1)c_{lm}^{(N-1)} - 2Nc_{lm}^{(N)}}{N\lambda^2} Y_{lm}^{(N)}$$

The fuzzy onion

We have the field content Ψ and the kinetic term $\mathcal{K} = \mathcal{K}_L + \mathcal{K}_R$.
Now we can do physics!

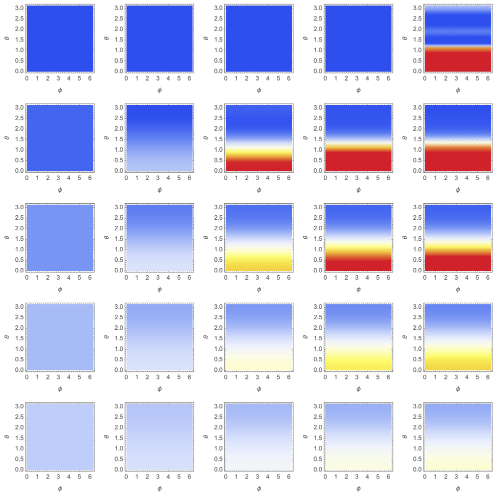
Example I: Scalar field theory

$$S[\Psi] = 4\pi\lambda^2 \mathrm{Tr}\ r\left(\ \Psi\mathcal{K}\Psi + b\ \Psi^2 + c\ \Psi^4\right)$$



Example II: Heat transfer

$$\mathcal{K}\Psi(t) = \alpha\ \partial_t \Psi(t)$$



Example III: Coulomb problem

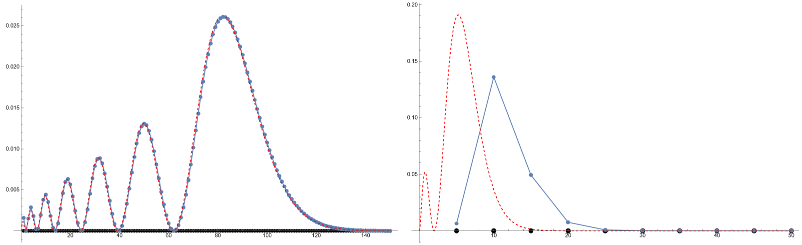
$$H = -\frac{\hbar^2}{2m_e} - \frac{q}{r}$$

$$\mathbf{H}\mathcal{C}_{lm} = E\mathcal{C}_{lm}$$

n	1	2	3	4	5	6
E_n	-0.4142	-0.1180	-0.0541	-0.0307	-0.0179	-0.0031
$E^l_{\lambda n}$	-0.4142	-0.1180	-0.0541	-0.0307	-0.0198	-0.0138
E_n^{CQM}	-0.5	-0.125	-0.0556	-0.0313	-0.02	-0.0139

$$E^l_{\lambda\, n} = \frac{\hbar}{m_e\lambda^2}\left(1 - \sqrt{1 + \frac{m_e q\lambda}{\hbar^2 n}}^{\,2}\right)$$

Example III: Coulomb problem



- For $\lambda \ll a_0$ and $\lambda N \gg a_0$ we have great agreement with QM.
- For $\lambda \sim a_0$ and $\lambda N \gg a_0$ strong quantum-space effects.
- For $\lambda N \sim a_0$ space is not large enough to capture the physics.

Outlook

Scalar field theory

- What we have done so far is a very basic preliminary study.
- Study the phase diagram.
- Correlation functions, coexistence of phases, aligning stripes.

Outlook

Classical physics

- Behavior of granular materials.
- Pixelated physics (weather forecast, planetary weather, ...)
- Neutron stars, gravitational collapse, ...

Outlook

Expansion of the space

- One can think of the radiation direction as a temporal one, $\Delta_R \to \Delta_t$.
- The model would then describe an expanding fuzzy sphere with a growing number of degrees of freedom.
- Solution to the past hypothesis? (probably not)
- Definitely not a good model of expanding universe but a nice toy model to understand some ideas.

Thank you for your attention!