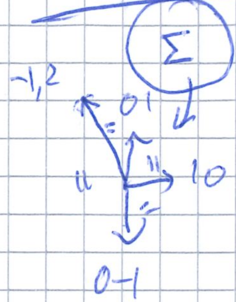
Lecture 2 (First 2 Pages New)



# Preamble: How to find Nef + Mori Cones.

$\Sigma$




$$0 \to \mathbb{Z}^2 \xrightarrow{M} \mathbb{Z}^4 \to \mathbb{Z}^2 \to 0$$

$$M = \begin{array}{c} 1 \\ \\ \\ 4 \end{array}\begin{pmatrix} -1 & 2 \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\mathbb{Z}^2 = Cl(X_\Sigma)$ ← divisors, up to rational equivalence.

$F_2$ Antonella

$$B = \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

← the degree matrix

Construct: $\mathbb{C}^{|\Sigma(1)|} \setminus V(B)/G$, $G = \mathrm{Hom}_{\mathbb{Z}}(Cl(X_\Sigma), \mathbb{C}^*)$

$B$ = monomial ideal, generated by complements of max faces $\Sigma$

$\cap$

$S = \mathbb{C}[x_1 \ldots x_d]$, $d = |\Sigma(1)|$; $\deg x_i$ = column $i$ of $B$.

In our example, $B = \langle \overline{12}, \overline{23}, \overline{34}, \overline{41} \rangle = \langle x_3x_4, \ldots x_2x_3 \rangle$

$= \underbrace{\langle x_1, x_3 \rangle \cap \langle x_2, x_4 \rangle}$

$\{v_1 \ldots v_k\}$ not face, all subsets are ← primitive collections

~~Batyrev~~ ($\Sigma$ simplicial, $X_\Sigma$ projective)

**Def** $D = \sum_{i \in \Sigma(1)} a_i D_i$, $\phi_D$ a PL cont fn "support function"

* $\phi_D(v_i) = -a_i$

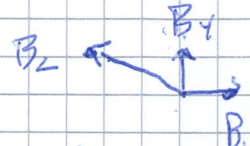
**Thm:** $D$ ample iff $\phi_D$ strictly convex, nef iff convex.

**Thm** Batyrev nef $\iff \phi_D(\sum v_i) \geq \sum \phi_D(v_i)$ over all prim. collections.

(-1)

$C_B$ = cone of Effective divisors

= cone of multidegees




So, prim collectors = $\{1,3\}$ $\{2,4\}$

Nef $\iff$ $\phi_D(1+3) \geq \phi_D(1) + \phi_D(3)$

$\phi_D(2+4) \geq \phi_D(2) + \phi_D(4)$

Simplify ~~first~~ D first, we'll use $D_3 + D_4$

(look @ dyree matrix)

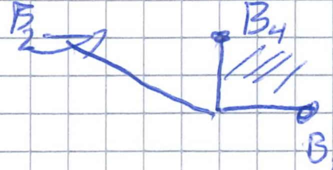
So: $D = 0D_1 + 0D_2 + xD_3 + yD_4$

$v_1 + v_3 = (0,2) = 2 \cdot v_2$, $\phi_D(v_2) = 0 \geq \underset{0}{\underset{\parallel}{\phi_D(v_1)}} + \underset{-x}{\underset{\parallel}{\phi_D(v_3)}}$

$0 \geq -x \Rightarrow x \geq 0$

same thing for

$\underset{0}{\underset{\parallel}{(v_2 + v_4)}}$ $\underset{0}{\underset{\parallel}{\phi_D(0)}} \geq \underset{0}{\underset{\parallel}{\phi_D(v_2)}} + \underset{-b}{\underset{\parallel}{\phi_D(v_4)}} \Rightarrow b \geq 0$




Nef cone ☺

Last lecture - mentioned similar for Mori.

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$H^0(D) = \{ m \mid \langle m, v_i \rangle \geq -a_i \}$ "slide to hyperplanes along rays"

$\underset{\sum a_i D_i}{\parallel}$

# LECTURE 2: Secondary Fan $\Sigma_{GKZ}$ [toric case], Gale Duality
## Polyhedron of character, empty facet ideal, GIT

N.B. We will go "backwards" wrt toric book = GIT (general $\boxed{\text{at end}}$
Basic? ~~idea~~: vary choice of character $\longleftrightarrow$ what happens?

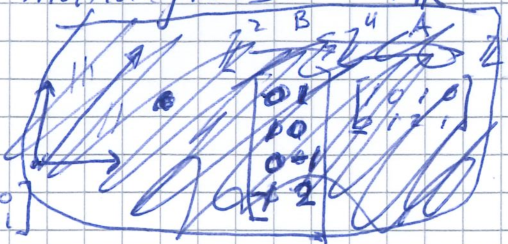
## § Toric GIT, Gale duality

14.3

**Def**: $\Sigma$ a fan, $C_\nu$ = cone gen'd by $\rho \in \Sigma(1) \subseteq N_{\mathbb{R}}$
$C_\beta$ = cone of multidegrees. $\subseteq \hat{G}_{\mathbb{R}}$

**Example** $\mathcal{H}_2$ $C_\nu = \mathbb{R}^2$, $C_\beta =$

do as in book $\begin{pmatrix} -1 & 2 \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\mathbb{Z}^4$ $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

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**Gale Duality** (just dual vector spaces)

$$0 \to V \xrightarrow{\delta} \mathbb{R}^r \xrightarrow{\gamma} W \to 0 \Rightarrow 0 \to W^* \to \mathbb{R}^r \to V^* \to 0$$

$V \to \mathbb{R}^r$: rays = rows; $\mathbb{R}^r \to W$: degs = cols; $W^* \to \mathbb{R}^r$: degs = rows; $\mathbb{R}^r \to V^*$: rays = cols

Think this way, but obviously works in general.

**Exercise**:

- $\{B_i\}$, $i \in I$ basis of $W \leftrightarrow v_j \mid j \in [r] \setminus I$ basis $V^*$
- $C_B = W \Rightarrow C_\nu$ strongly convex in $V^*$
  $\Leftarrow$ with additional condition all $v_i \neq 0$

## § Def (Key): Polyhedron of a character, virtual facet

Players

Choose a character, $a \in \hat{G}_{\mathbb{R}}$ (n.b. this works more generally, for now stay toric)

$(0 \to G \to (\mathbb{C}^*)^r \to T_N \to 0 \Leftrightarrow 0 \to M \to \mathbb{Z}^r \to Cl(X_\Sigma) \to 0)$

So this corresponds to $\bar{a} \in \mathbb{R}^r$

$$\boxed{\begin{aligned} P_{\bar{a}} &= \{ m \in M_{\mathbb{R}} \mid \langle m, v_i \rangle \geq -a_i,\ i = 1, \dots, r \} \\ F_{i\bar{a}} &= \{ m \in P_{\bar{a}} \mid \langle m, v_i \rangle = -a_i \end{aligned}}$$

$\leftarrow$ Polyhedron $\bar{a}$

$\leftarrow$ virtual facet

$\Sigma(1)$

$$\begin{bmatrix} -1 & 2 \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$$

14.4.17

☆ $G = \langle t, t^2 u, t, u \rangle$ rows of $A$ give exponent for $G$

$\Delta$ basis $\tilde{u} = t^2 u$

But using

5.1.1 $G = \prod_\rho t_\rho^{\langle m, u_\rho \rangle}$ so going with $m = e_1, e_2$

$m = e_1 \Rightarrow t_1^{-1} t_3 = 1 \Rightarrow t_1 = t_3 = t$

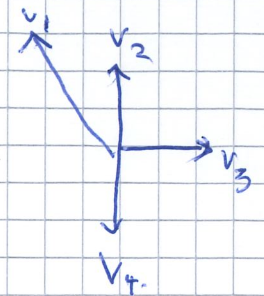
$m = e_2 \Rightarrow t_1^2 t_2 t_4^{-1} = 1 = t_4 = t_1^2 t_2$ $u = t_2$

$\Rightarrow \langle t, u, t, t^2 u \rangle$ $t, \tilde{u} t^{-2}, t, \tilde{u} \rangle =$ ☆

<u>Example</u>: $\mathbb{Z}^2 \xrightarrow{B} \mathbb{Z}^4 \xrightarrow{A} \mathbb{Z}^2$

14.2.17
14.2.20

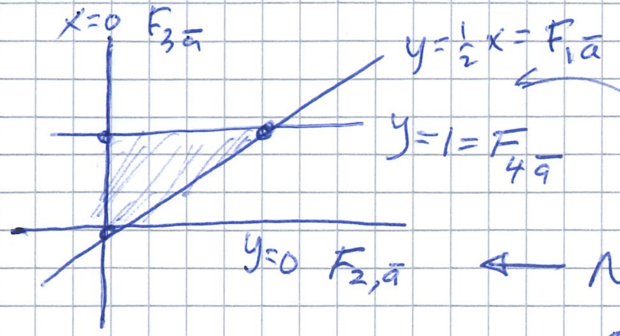
$$\begin{bmatrix} -1 & 2 \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$




facet "name"

Let $\bar{a} = (0,0,0,1)$ then $P_a =$

| | | |
|---|---|---|
| $(x,y)\cdot(-1,2) \geq 0$ | $y \geq \frac{1}{2}x$ | $F_1$ |
| $(0,1) \geq 0$ | $y \geq 0$ | $F_2$ |
| $(1,0) \geq 0$ | $x \geq 0$ | $F_3$ |
| $(0,-1) \geq -1$ | $y \leq 1$ | $F_4$ |




$\leftarrow$ Notice $F_{2\bar{a}}$ is not really a facet <u>and</u> if we vary $\bar{a}$ we "slide" facets.

<u>Def</u> $P_\chi$, $\chi \in \hat{G}$ $= \{ (\langle m, v_1\rangle + a_1, \dots, \langle m, v_r\rangle + a_r) \mid m \in P_{\bar{a}} \} = \delta_R(P_{\bar a}) + \bar{a}$

$\chi = \bar{a}$

Hence, $F_{i\bar{a}} \subseteq P_a \leftrightarrow$ elts of $P_\chi$ with $i^{\text{th}}$ coord <u>zero</u>.

Skip

<u>Lemma</u> $\bar{p} \in \mathbb{C}^r$, $I(\bar{p}) = \{ i \mid p_i = 0 \}$. $p \in (\mathbb{C}^r)^{ss}_\chi \Leftrightarrow \bigcap_{i \in I(p)} F_{i,\chi} \neq \emptyset$

(we'll need when we do "general" GIT.

<u>Def:</u> $B(\chi) = \langle \prod_{i \notin I} x_i \mid \bigcap_{i \in I} F_{i,\chi} \neq \emptyset \rangle \subseteq S \leftarrow$ cox ring

like toric irrel. ideal!

<u>Thm</u> $\mathbb{C}^r /\!/_\chi G \simeq \mathbb{C}^r \setminus V(B(\chi)) /\!/ G$

<u>What is this</u>? Idea is for closed subgroup $G \subseteq (\mathbb{C}^\times)^r$ to take a <u>character</u> $\chi$ of $G$, build a line bundle $\mathcal{L}_\chi$, take ring $R_\chi = \bigoplus_d \Gamma(\mathbb{C}^r, \mathcal{L}_\chi^{d})^G$, then $\mathbb{C}^r /\!/_\chi G = \operatorname{Proj} R_\chi \leftarrow$ ok because f'gen'd algebra ($G$ reductive)

# Example (to keep us grounded) $\mathcal{H}_2$.

14.4.6 $\quad \overline{a} = (a_1, a_2, a_3, a_4) \iff D = \sum_1^4 a_i D_i \iff$ degree in $S$

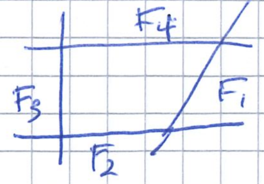
gives a polyhedron defined by $\quad (x,y)\cdot(-1,2) \geq -a_1 \quad F_1$

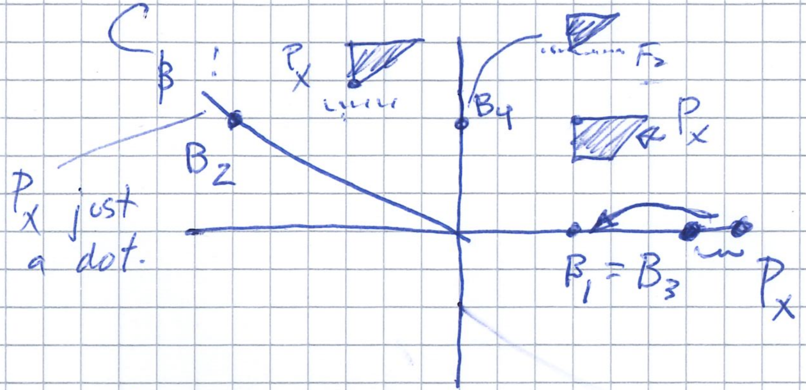
$(0,1) \geq -a_2 \quad F_2$

$(1,0) \geq -a_3 \quad F_3 \qquad \circledast$

$(0,-1) \geq -a_4 \quad F_4$

$\beta = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$



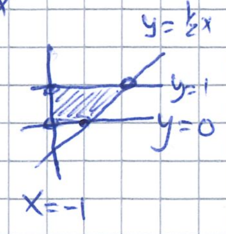

Examples: • Using $B_3, B_4$ as basis for $Cl(X_\Sigma)$, the point $(1,1)$ in drawing above $\iff$ divisor $D_3 + D_4$

so $\overline{a} = (0,0,1,1) \Rightarrow$ in $\circledast$ $\quad y \geq \frac{1}{2}x$

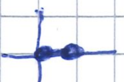
$y \geq 0$

$x \geq -1$

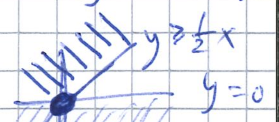
$y \leq 1$




• while $B_3$ by itself $= D_3$

we replace $y \leq 1$ with $y \leq 0 \Rightarrow$  (slide $y=1$ down)

• How about $B_2 = D_2$ ; $\left.\begin{matrix} y \geq \frac{1}{2}x & x \geq 0 \\ y \geq -1 & y \leq 0 \end{matrix}\right\}$




We see above two types of changes

- Fan of $P_a$ changes "empty" virtual facets.
- $F_{i\overline{a}} \cap P_a$ changes. Define $I_\phi = \{ i \mid F_{i\overline{a}} \cap P_a = \phi \}$

# § Constructing $\Sigma_{GKZ}$.

Key idea: We get a fan in $C_\beta$, s.t. $C_\beta = |\Sigma_{GKZ}|$ by studying $\mathbb{C}^r \setminus V(B(\chi)) /\!/ G$ as $\chi$ varies in $C_\beta$.

In particular, as we saw for $\mathcal{H}_2$, the data $(P_a, I_\phi)$ partitions $C_\beta$ into cones, a general phenomenon.

$\boxed{\text{Thm}}$ $\Sigma \subseteq N_{\mathbb{R}}$ a (generalized) fan, $I_\phi \subseteq \{1 \ldots r\}$, $r = |\Sigma(1)|$

14.4.1 Then $(\Sigma, I_\phi) \longleftrightarrow \bar{a} \in \mathbb{Z}^r \longleftrightarrow$
- $|\Sigma| = C_\nu$
- $X_\Sigma$ semiproj (= full dim lattice polytope)
- every $\sigma \in \Sigma \longleftrightarrow$ $\text{Cone}\{v_i \mid v_i \in \sigma, i \notin I_\phi\}$

"Build our cones"

**Def:** $\Sigma, I_\phi$ as here. The GKZ cone (Thm that it is a cone!)

$$\widetilde{\Gamma}_{\Sigma, I_\phi} = \{ \bar{a} \in \mathbb{R}^r \mid \exists\, \phi \text{ convex support fn on } \Sigma$$
- such that $\phi(v_i) = -a_i,\ i \notin I_\phi$
- $\phi(v_i) \geq -a_i,\ i \in I_\phi \}$

nef!

This says exactly that $\bar{a}$ is nef, and $i \notin I_\phi \Rightarrow F_i \cap P_a \neq \phi$

14.4.3 + following $\boxed{\boxed{\text{Thm}}}$ $\widetilde{\Gamma}_{\Sigma, I_\phi}$ is a RPC in $\mathbb{R}^r$, w/ minimal face $M_{\mathbb{R}}$.

Now $0 \to M_{\mathbb{R}} \xrightarrow{\delta_{\mathbb{R}}} \mathbb{R}^r \xrightarrow{\gamma_{\mathbb{R}}} \hat{G}_{\mathbb{R}} \to 0$ has $C_\beta$ cone in $\hat{G}_{\mathbb{R}}$.

and $\bigcup_{\Sigma, I_\phi} \widetilde{\Gamma}_{\Sigma, I_\phi} = \gamma_{\mathbb{R}}^{-1}(C_\beta)$. Every face of a GKZ cone is again a GKZ cone;

☆ $\widetilde{\Gamma}_{\Sigma', I_\phi'} \prec \widetilde{\Gamma}_{\Sigma, I_\phi} \Leftrightarrow$ $\Sigma$ refines $\Sigma'$, $I_\phi' \subseteq I_\phi$

Flashback to $\mathcal{H}_2$ example, but first 

To make life easier, since $M_{\mathbb{R}}$ minimal face of all $\widetilde{\Pi}_{\Sigma, I_\phi}$, work in smaller space:

<u>Def:</u> $\Pi_{\Sigma,I_\phi} = \widetilde{\Pi}_{\Sigma,I_\phi} / \mathrm{Ker}\,\gamma_{\mathbb{R}} = M_{\mathbb{R}} \subseteq \mathbb{R}^r / \mathrm{Ker}\,\gamma_{\mathbb{R}} = \hat{G}_{\mathbb{R}}$

<u>THM</u>:
- $\Sigma_{GKZ} = \bigcup$ is a fan in $\hat{G}_{\mathbb{R}}$, $|\Sigma_{GKZ}| = C_B$
- $\Pi_{\Sigma',I'_\phi} < \Pi_{\Sigma,I_\phi} \iff \circledast$
- $\mathbb{C}^r /\!\!/_X G \simeq X_\Sigma$ if $X \in \mathrm{Relint}(\Pi_{\Sigma,I_\phi})$ NOT ON A BOUNDARY

<u>Exercise</u>: Fill out rest of $\mathcal{H}_2$ picture.

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## § Irrelevant ideals and $\Sigma_{GKZ}$

<u>Def (again)</u> $B(X) = \langle \prod_{i\in I} x_i \mid I \subseteq \{1..r\}, \bigcap_{i\in I} F_{i,X} \neq \emptyset \rangle$

This is a "parametric" version of our

☆ good old toric irrelevant ideal, <u>tuned</u> to be the ideal for the fan of $P_a = P_X$

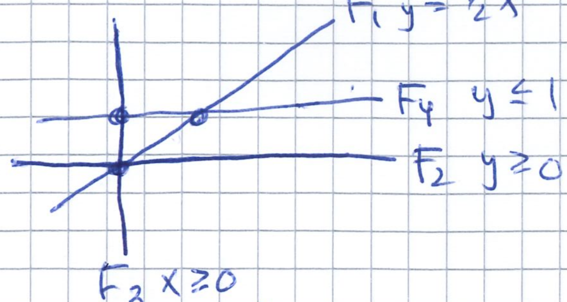
<u>Careful</u> → don't forget the <u>complement</u>: last lecture for a <u>fixed fan</u> ($\Sigma$), we took the monomials $\iff$ <u>complement</u> of max faces.

**Example** For Hirzebruch, when we took

$B_4 = D_4$, $\overset{0001}{=}$ we get the picture

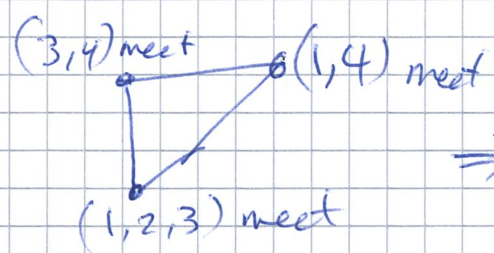
| | | | | |
|---|---|---|---|---|
| $(x,y)$ | $(-1,2) \geq 0$ | $F_1$ | | $y \geq \frac{1}{2}x$ |
| | $(0\ 1) \geq 0$ | $F_2$ | | $y \geq 0$ |
| | $(1\ 0) \geq 0$ | $F_3$ | $\rightsquigarrow$ | $x \geq 0$ |
| | $(0\ {-1}) \geq -1$ | $F_4$ | | $1 \geq y$ |

$\rightsquigarrow$

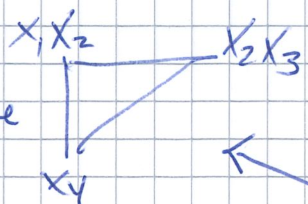



$I_\phi$ is **empty** (all facets meet $P_x$)

What virtual facets have nontrivial intersection?



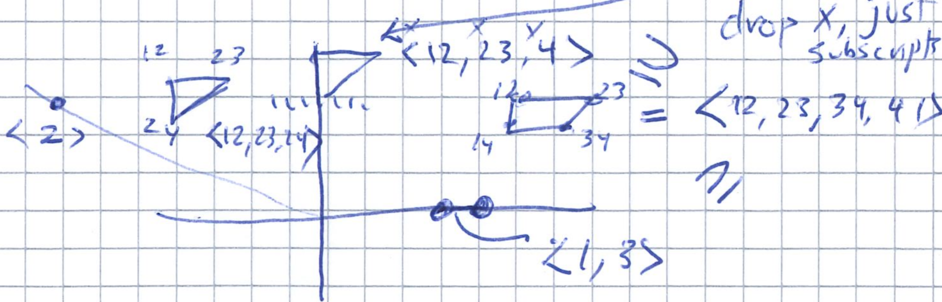

$\Rightarrow$ **Complements** are




$\Rightarrow B(X) = \langle x_4, x_1x_2, x_2x_3 \rangle$

**THM**: The containments of the $B(X)$ encode all the secondary fan data.

**Example**: $\mathcal{H}_2$




WHOLE PIX $\langle 2 \rangle \supseteq \langle 12, 23, 24 \rangle \subseteq \langle 12, 23, 4 \rangle \supseteq \langle 12, 23, 24, 14 \rangle \subseteq \langle 13 \rangle$

**THM** $\exists$ finitely many GIT quotients up to isomorphisms

**Pf**: finitely many $B(X)$ ideals.

## § GIT: general setup (sketch) 

<u>input</u>: $G$ closed subgroup of $(\mathbb{C}^*)^r$, acting on $\mathbb{C}^r$

<u>output</u> a good cat. quotient, depending on a <u>choice</u> of character, and <u>after</u> removing <u>bad points</u>, a good geometric quotient.

<u>Key steps</u> ① $\chi$ gives a trivial line bundle on $\mathbb{C}^r$

- $\chi \in \hat{G}$ acts on $\mathbb{C}^r \times \mathbb{C}$ via $g\cdot(p,t) = (gp, \chi(g)\cdot t)$ and line bundle $\mathcal{L}_\chi$, w/ sections $s(p) = (p, F(p))$, $F \in \mathbb{C}[x_1 \dots x_r]$

so $(g\cdot s)(p) = (p, \chi(g) F g^{-1} p)$ and

$$\Gamma(\mathbb{C}^r, \mathcal{L}_\chi)^G = \{F \mid F(gp) = \chi(g) F(p)\ \forall g \in G, p \in \mathbb{C}^r\}$$

<u>Example</u>: $G \leq (\mathbb{C}^*)^{|\Sigma(1)|}$, $S = \mathbb{C}[x_\rho]$ graded by $Cl(X_\Sigma)$ with $\beta \in Cl(X_\Sigma)$ then $\Gamma(\mathbb{C}^{\Sigma(1)}, \mathcal{L}_\chi)^G = S_\beta$ ☺

<u>Def</u>: • if $s \in \Gamma(\mathcal{L}_\chi)$, $\mathbb{C}^r_s := \{p \in \mathbb{C}^r \mid s(p) \neq 0\}$ is an <u>affine open</u>

• Fix $G \subseteq (\mathbb{C}^*)^r$ and $\chi \in \hat{G}$.

Semistable locus $\longrightarrow (\mathbb{C}^r_\chi)^{ss} = \{p \mid \exists d > 0,\ s \in \Gamma(\mathbb{C}^r, \mathcal{L}_{\chi^d})^G \text{ with } p \in \mathbb{C}^r_s$

stable locus $\longrightarrow (\mathbb{C}^r_\chi)^s = \{p \in (\mathbb{C}^r_\chi)^{ss}$ <u>and</u> • $G$ orbits in $\mathbb{C}^r_s$ closed
• $p \mid g(p) = p$ finite

<u>Def/Thm</u> • $R_\chi = \bigoplus_{d=0}^{\infty} \Gamma(\mathbb{C}^r, \mathcal{L}_{\chi^d})^G$ is f'gen'd $\mathbb{C}$-alg, (~~$G$~~ reductive) so $G \subseteq (\mathbb{C}^*)^r$, $\chi \in \hat{G}$, GIT quotient $\mathbb{C}^r /\!/_\chi G = \mathrm{Proj}(R_\chi)$

• $X_\Sigma$ proj toric, $\chi$ ample $\Rightarrow (\mathbb{C}^{\Sigma(1)}_\chi)^{ss} = \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)$, sim for stable, $\Sigma'$.

$\Sigma' \leq \Sigma$ simplicial subfan