GEORG-AUGUST-UNIVERSITÄT
GÖTTINGEN

On LA-Courant algebroids and Poisson Lie 2-algebroids

A geometrisation of [2]-manifolds.

by

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Roytenberg/Severa

tangent prolongations

Symplectic Lie 2-algebroids $\leftrightarrow$ Courant algebroids.

Lie 2-algebroids $\leftrightarrow$ VB-Courant algebroids.

Li-Bland

"vector bundle"

$\boxed{\text{"Geometrisation of [2]-manifolds"}}$

this

1. definitions
2. double vector bundles and
3. add. geometric structures.

Graded manifolds and Lie n-algebroids

Positively graded manifolds

"[n]-manifold"

An $\mathbb{N}$**-graded manifold** $\mathcal{M}$ of degree n and dimension $(p; r_1, \dots, r_n)$ is a smooth p-dimensional manifold M endowed with a locally free and finitely generated sheaf $C^\infty(\mathcal{M})$ of $\mathbb{N}$-graded commutative associative unital $\mathbb{R}$-algebras, which can locally be written as

$$C^\infty(\mathcal{M})_U = C^\infty(U)\left[\xi_1^1, \dots, \xi_1^{r_1}, \xi_2^1, \dots, \xi_2^{r_2}, \dots, \xi_n^1, \dots, \xi_n^{r_n}\right] \qquad \star$$

with $r_1 + \dots + r_n$ graded commutative generators ξ_i^j of degree i for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, r_i\}$.

- NB: an [n]-manifold is always noncanonically split over M.
  $\mathcal{M} \simeq E_1[1] \oplus \dots \oplus E_n[n]$
  sections of E_1^* as gen. of deg 1.
  E_2^* deg 2
- a [1]-mfd is always $\mathcal{M} = E[1]$
  i.e. $C^\infty(\mathcal{M}) = \Gamma(\Lambda^\bullet E^*) = \Omega^\bullet(E)$

Lie algebroids

A Lie algebroid over a manifold M is a vector bundle $A \to M$ with a Lie algebra bracket $[\cdot\,,\cdot]$ on $\Gamma(A)$ and an anchor $\rho\colon A \to TM$, such that

$$[a_1, fa_2] = f[a_1, a_2] + \mathcal{L}_{\rho(a_1)}(f)a_2$$

for all $a_1, a_2 \in \Gamma(A)$ and $f \in C^\infty(M)$.

$(A[1], d_A)$

vector field
of degree 1
with $d_A^2 = 0$

"homological vector field"

d_A "de Rham like"

$d_A : \Omega^\bullet(A) \to \Omega^{\bullet+1}(A)$

$d_A^2 = 0$
$\|$
$\frac{1}{2}[d_A, d_A]$

~~Example~~: Lie 1-algebroid

Take an $\mathbb{N}$-graded manifold of degree 1, i.e. $C^\infty(\mathcal{M}) = \Gamma(\bigwedge^\bullet E)$ for a vector bundle E over M and take a trivialising chart $U \subseteq M$ for E. Any homological vector field $\mathcal{Q}$ on $\mathcal{M}$ can locally be written as

$$\mathcal{Q}_U = \sum_{ij} \rho(\varepsilon_j)(x_i) e_i \partial_{x_j} - \sum_{ijk} \langle [\varepsilon_i, \varepsilon_j], e_k \rangle e_i e_j \partial_{e_k},$$

defining locally a Lie algebroid structure on $E^*|_U$. This structure is in fact global, and Lie 1-algebroids are equivalent to Lie algebroids. (This is due to Arkady Vaintrob.)

Lie algebroids $\longleftrightarrow$ [1]-mfd + hom. VF "Lie 1-algebroids".

$\longrightarrow$ Lie n-algebroid : $(\mathcal{M}, \mathcal{Q})$, $\mathcal{M}$ [n]-mfd, $\mathcal{Q}$ hom. VF degree 1 $\mathcal{Q}^2 = 0$.

How about Lie 2-algebroids vs Courant algebroids?

Courant algebroids

A Courant algebroid over a manifold M is a vector bundle $\mathsf{E} \to M$ with a fibrewise nondegenerate symmetric bilinear form $\langle \cdot\,,\cdot \rangle$, a bracket $[\![\cdot\,,\cdot]\!]$ on the smooth sections $\Gamma(\mathsf{E})$, and an anchor $\rho\colon \mathsf{E} \to TM$, which satisfy the following conditions

1. $[\![e_1, e_2]\!] + [\![e_2, e_1]\!] = \rho^* \mathbf{d}\langle e_1, e_2\rangle$,
2. $[\![e_1, [\![e_2, e_3]\!]]\!] = [\![[\![e_1, e_2]\!], e_3]\!] + [\![e_2, [\![e_1, e_3]\!]]\!]$, ← Jacobi
3. $\rho(e_1)\langle e_2, e_3\rangle = \langle [\![e_1, e_2]\!], e_3\rangle + \langle e_2, [\![e_1, e_3]\!]\rangle$

for all $e_1, e_2, e_3 \in \Gamma(\mathsf{E})$.

prototype: $TM \oplus T^*M \longrightarrow M$

Split Lie 2-algebroids

A split Lie 2-algebroid is a sequence $\mathcal{M} \cong E_1[1] \oplus E_2[2]$

$$\underset{E_2}{B^*} \xrightarrow{l} \underset{E_1}{Q} \xrightarrow{\rho} TM$$

of vector bundles over M with a skew-symmetric dull bracket $[\![\cdot\,,\cdot]\!] \colon \Gamma(Q) \times \Gamma(Q) \to \Gamma(Q)$, a linear connection $\nabla \colon \Gamma(Q) \times \Gamma(B) \to \Gamma(B)$ and a 3-form $\underline{\omega \in \Omega^3(Q, B^*)}$, such that

(i) $\nabla^*_{l(\beta_1)}\beta_2 + \nabla^*_{l(\beta_2)}\beta_1 = 0$ for all $\beta_1, \beta_2 \in \Gamma(B^*)$,

(ii) $[\![q, l(\beta)]\!] = l(\nabla^*_q\beta)$ for $q \in \Gamma(Q)$ and $\beta \in \Gamma(B^*)$,

(iii) $\mathrm{Jac}_{[\![\cdot,\cdot]\!]} = l \circ \omega \in \Omega^3(Q, Q)$,

(iv) $R_\nabla(q_1, q_2)b = l^*\langle \mathbf{i}_{q_2}\mathbf{i}_{q_1}\omega, b\rangle$ for $q_1, q_2 \in \Gamma(Q)$ and $b \in \Gamma(B)$, and

(v) $\mathbf{d}_{\nabla^*}\omega = 0$.

Geometrisation of $[2]$-manifolds

Double vector bundles

A double vector bundle is a commutative square

$$\begin{tikzcd} D \arrow[r, "\pi_B"] \arrow[d, "\pi_Q"'] \arrow[dr, phantom, "C"] & B \arrow[d, "q_B"] \\ Q \arrow[r, "q_Q"'] & M \end{tikzcd}$$

"core"

of vector bundles such that the structure maps of the vertical bundles define morphisms of the horizontal bundles.

Prototype:

$$\begin{tikzcd} TE \arrow[r, "*"] \arrow[d, "*"'] \arrow[dr, phantom, "E"] & TM \arrow[d] \\ E \arrow[r] & M \end{tikzcd}$$

$E \to M$ VB

NB! D is always noncanonically isomorphic to a decomposed DVB:

$$\begin{tikzcd} q_B^!(Q \oplus C) \simeq B \times_M Q \times_M C \arrow[r] \arrow[d] \arrow[dr, phantom, "C"] & B \arrow[d] \\ Q \arrow[r] & M \end{tikzcd}$$

Metric double vector bundles

$$\begin{tikzcd} \mathbb{E} \arrow[r, "\langle\cdot,\cdot\rangle"] \arrow[d] & B \arrow[d] \\ Q \arrow[u, bend left, "\tilde{b}"] \arrow[r] & M \arrow[u, bend right, "b"'] \end{tikzcd} \qquad Q^*$$

$\leftarrow$ metric

$\exists$ dec. $\mathbb{E} \simeq B \times_M Q \times_M Q^*$

s.t.

$$\langle (b, q, \tau), (b, q', \tau') \rangle = \tau'(q) + \tau(q')$$

$b \in \Gamma(B)$: $\quad \tilde{b} \in \Gamma^{\ell}_{Q}(\mathbb{E}) \quad \tilde{b}(q_m) = (b(m), q_m, 0^{Q^*}_m)$

$\tau \in \Gamma(Q^*)$: $\quad \tau^+ \in \Gamma^{c}_{Q}(\mathbb{E}) \quad \tau^+(q_m) = (0^{Q^*}_m, q_m, \tau(m))$

$$C(\mathbb{E}) = \{ \chi \in \Gamma^{\ell}_{Q}(\mathbb{E}) \mid \chi \text{ isotropic} \}$$

$$\langle \chi(q), \chi(q') \rangle = 0 \qquad (q, q') \in Q \times_M Q$$

Involutive double vector bundles

$$\begin{tikzcd} \mathbb{E}^*Q \arrow[r] \arrow[d] & Q \arrow[d] \\ Q \arrow[r] & M \end{tikzcd} \qquad B^* \to M$$

$\mathcal{I} : \mathbb{E}^*Q \to \mathbb{E}^*Q \qquad \mathcal{I}^2 = \mathrm{id}.$

$\exists$ dec. $\mathbb{E}^*Q \simeq Q \times_M Q \times_M B^*$

s.t. $\mathcal{I}(q, q', \beta) = (q', q, \beta).$

Metric double vector bundle charts

Let M be a smooth manifold and D a set with a map $\Pi\colon D \to M$. A **metric double vector bundle chart** is a quadruple $c = (U, \Theta, V_1, V_2)$, where U is an open set in M, V_1, V_2 are two vector spaces and $\Theta\colon \Pi^{-1}(U) \to U \times V_1 \times V_2 \times V_2^*$ is a bijection such that $\Pi = \mathrm{pr}_1 \circ \Theta$.

Two double vector bundle charts c and c' are **compatible** if the "change of chart" $\Theta' \circ \Theta^{-1}$ over $U \cap U'$ sends (x, v_1, v_2, l) to

$$(x, A(x)v_1, B(x)v_2, (B(x)^{-1})^* l + \omega(x)(v_1)(B(x)v_2))$$

with $x \in U \cap U'$, $v_i \in V_i$, $A \in C^\infty(U \cap U', \mathrm{Gl}(V_1))$, $B \in C^\infty(U \cap U', \mathrm{Gl}(V_2))$ and $\omega \in C^\infty(U \cap U', V_1^* \otimes V_2^* \wedge V_2^*)$.

Cocycle conditions: the standard ones for A's and B's, and

$$\omega^{\gamma\beta}(x)(v_1) = B^{\alpha\gamma}(x)^t \cdot \omega^{\alpha\beta}(x)(v_1) \cdot B^{\alpha\gamma}(x) + \omega^{\gamma\alpha}(x)(A^{\alpha\beta}(x)v_1).$$

Geometrisation of [2]-manifolds

Theorem (JL 2018)

The category of positively graded manifolds of degree 2 is equivalent to the category of involutive double vector bundles.

[2]-mfds
$\downarrow$
m DVB's.

Degree 1:

$$\xi_i^\beta = \sum_{j=1}^{r_1} A_{ji}^{\alpha\beta} \xi_j^\alpha .$$

Degree 2:

$$\eta_i^\beta = \sum_{j=1}^{r_2} B_{ji}^{\alpha\beta} \eta_j^\alpha + \sum_{1 \le k < l \le r_1} \underline{\omega_{kli}^{\alpha\beta}} \xi_k^\alpha \wedge \xi_l^\alpha .$$

$$\begin{tikzcd} \mathbb{E} \arrow[r] \arrow[d] \arrow[dr, phantom, "Q^*"] & B \arrow[d] \\ Q \arrow[r] & M \end{tikzcd}$$

$\to C^\infty(\mathcal{G}(\mathbb{E}))$

$C(\mathbb{E})$ gen. of dg 2
$\Gamma(Q^*)$ gen. of dg 1
$C^\infty(M)$ gen of dg 0.

Geometrisation of [n]-manifolds

(in preparation, with Malte Heuer)

$[n]$-manifolds $\longleftrightarrow$ n-fold vector bundles with S_n-action.

VB-and LA-Courant algebroids

Theorem (JL 2018)

The category of positively graded Poisson manifolds of degree 2 is equivalent to the category of self-dual VB-algebroids.

$(\mathcal{M}, \{\cdot,\cdot\})$

$\langle\cdot,\cdot\rangle$

$$\begin{tikzcd} \mathbb{E} \arrow[r] \arrow[d, "LA"'] & B \arrow[d, "LA"] \\ A \arrow[r] \arrow[ur, phantom, "A^*"] & M \end{tikzcd}$$

VB - algebroid.

Theorem (Li-Bland 2012, JL 2019)

The category of Lie 2-algebroids is equivalent to the category of VB-Courant algebroids.

A decomposed VB-Courant algebroid defines the structure objects as on Slide 11:

$$\Theta(\tilde{q}) = \widehat{\nabla_q} \in \mathfrak{X}(B), \quad [\![\tilde{q}, \tau^\dagger]\!] = (\Delta_q \tau)^\dagger \text{ and}$$
$$[\![\widetilde{q_1}, \widetilde{q_2}]\!] = \widetilde{[\![q_1, q_2]\!]} - R_\omega(q_1, q_2)^\dagger,$$

for all $q, q_1, q_2 \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$, where $\Delta\colon \Gamma(Q) \times \Gamma(Q^*) \to \Gamma(Q^*)$ is the “Lie derivative” that is dual to the dull bracket.

Theorem (Li-Bland 2012, JL 2020)

The category of Poisson Lie 2-algebroids is equivalent to the category of LA-Courant algebroids.

$$(\mathcal{M}, \{\cdot,\cdot\}, \mathcal{Q})$$

$\mathcal{Q}$ preserves $\{\cdot,\cdot\}$

$$\begin{tikzcd} \mathbb{E} \arrow[r, "C.A"] \arrow[d, "L.A"'] \arrow[dr, phantom, "VB"] & B \arrow[d] \\ A \arrow[r] & M \end{tikzcd}$$

in <u>a splitting</u>!

Symplectic Lie 2-algebroids correspond to tangent doubles of Courant algebroids.

$$
\begin{tikzcd}
TE \arrow[r, "\text{C.A}"] \arrow[d, "\text{LA}"'] \arrow[dr, phantom, "E"] & TM \arrow[d, "\text{LA}"] \\
E \arrow[r, "\text{CA.}"'] & M
\end{tikzcd}
$$

Matched pairs of representations

Matched pairs of representations

A, B Lie algebroids on M $\qquad A \circlearrowright B \qquad B \circlearrowright A$

$A \oplus B \to M$

Lie alg.

$A, B \subseteq A \oplus B$ sub.

$[a,b] = (-\nabla_b a, \nabla_a b)$

bicrossproduct.

$$\begin{tikzcd} A \bowtie B \arrow[r, "LA"] \arrow[d, "LA"'] \arrow[dr, phantom, "0"] & A \arrow[d] \\ B \arrow[r] & M \end{tikzcd}$$

Matched pairs of representations

$$\begin{tikzcd}
D \arrow[r, "LA"] \arrow[d, "LA"'] & A \arrow[d, "LA"] \\
 & C \arrow[u, "\partial_A"'] \arrow[dl, "\partial_B"] \arrow[d] & \\
B \arrow[r, "LA"'] & M
\end{tikzcd}$$

dec. $\downarrow$

$A \circlearrowright \; C[0] \oplus B[1]$

$B \circlearrowright \; C[0] \oplus A[1]$

matched pair

$\longrightarrow$

$$\begin{tikzcd}
(D^*A, & D^*B) \\
 & C^* \arrow[from=1-1] \arrow[from=1-2]
\end{tikzcd}$$

VB-Liebialg.

$\hookdownarrow$

$$\begin{tikzcd}
D^*A \oplus D^*B \arrow[r] \arrow[d] & C^* \arrow[d] \\
A \oplus B \arrow[r] & M
\end{tikzcd}$$

VB-Cal

$\updownarrow$

split Lie 2.

$\xrightarrow{\text{bicrossprod.}}$

$$C \xrightarrow{(-\partial_A, \partial_B)} A \oplus B \xrightarrow{\rho_A + \rho_B} TM$$

Lie bialgebroids

(A, A^*) Lie bialgebroid

$$\begin{array}{ccc} T^*A & \to & A \\ \downarrow & T^*M & \downarrow \\ A^* & \to & M \end{array}$$

double Lie algebroid

$\updownarrow$

VB-Lie bialgebroid

(TA, TA^*)

$\downarrow$

TM

symplectic Lie 2

$T^*M \to A\oplus A^* \to TM$

$A\oplus A^* \to M$

bicrossproduct Courant algebroid

$$\begin{array}{ccc} T(A\oplus A^*) & \longrightarrow & TM \\ \mathrm{LA}\downarrow & A\oplus A^* & \downarrow \\ A\oplus A^* & \longrightarrow & M \end{array}$$

LA-Courant tangent prolongation

$(\mathfrak{g}, \langle\cdot,\cdot\rangle, [\cdot,\cdot])$ C.A over a point $\mathfrak{g}\times\mathfrak{g}^*$

$$
\begin{tikzcd}
\mathfrak{g}\times\mathfrak{g} = T\mathfrak{g} \arrow[r, "CA"] \arrow[d, "CA"'] \arrow[dr, phantom, "\mathfrak{g}"] & \{*\} \arrow[d] \\
\mathfrak{g} \arrow[r, "*"'] & \{*\}
\end{tikzcd}
\hookrightarrow
\begin{tikzcd}
T^*\mathfrak{g} \arrow[r] \arrow[d] \arrow[dr, phantom, "*"] & \mathfrak{g}^* \arrow[d] \\
\mathfrak{g} \arrow[r] & \{*\}
\end{tikzcd}
$$

Thank you for your attention!

Lie-Bialg : Lie 2-alg. $\Longleftrightarrow$ VB-Courant

$Q = 0$

$[\cdot,\cdot] = 0$

$\rho = 0$

References and tables

- *The geometrisation of $\mathbb{N}$-manifolds of degree 2*, "Journal of Geometry and Physics" (2018), Volume 133, 113-140.
- *Lie 2-algebroids and matched pairs of 2-representations – a geometric approach*, "Pacific Journal of Mathematics" (2019), Volume 301, Number 1, 143-188.
- *On LA-Courant algebroids and Poisson Lie 2-algebroids*, "Mathematical Physics, Analysis and Geometry" (2020), Volume 23, Number 31.
- *Multiple vector bundles: cores, splittings and decompositions*, with Malte Heuer, "Theory and Applications of Categories" (2020), Vol. 35, No. 19, 665-699.

Table of the supergeometric objects.

$$
\begin{tikzcd}
 & \text{[2]-manifolds} & & \\
\text{Lie 2-algebroids} \arrow[ur] & & \text{Poisson [2]-manifolds} \arrow[ul] & \\
 & \text{Poisson Lie 2-algebroids} \arrow[ul] \arrow[ur] & & \\
 & \text{Symplectic Lie 2-algebroids} \arrow[u, hook] \arrow[rr, leftrightarrow, dashed, "\text{“Core”}", "\text{Tangent functor}"'] & & \text{Courant algebroids}
\end{tikzcd}
$$

Double geometric objects.