



GEORG-AUGUST-UNIVERSITÄT
GÖTTINGEN

On LA-Courant algebroids and Poisson Lie 2-algebroids

A geometrisation of [2]-manifolds.

by

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Poytenbers/
Severa

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"vector bundle"

Li-Blaad

"Geometrisation of [2] - manifolds"

- ① definitions
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Graded manifolds and Lie n -algebroids

Positively graded manifolds

An \mathbb{N} -graded manifold \mathcal{M} of degree n and dimension $(p; r_1, \dots, r_n)$ is a smooth p -dimensional manifold M endowed with a locally free and finitely generated sheaf $C^\infty(\mathcal{M})$ of \mathbb{N} -graded commutative associative unital \mathbb{R} -algebras, which can locally be written as

$$C^\infty(\mathcal{M})_U = \underbrace{C^\infty(U)}_{\text{smooth}} \underbrace{[\xi_1^1, \dots, \xi_1^{r_1}, \xi_2^1, \dots, \xi_2^{r_2}, \dots, \xi_n^1, \dots, \xi_n^{r_n}]}_{\text{generators}} \quad \star$$

with $r_1 + \dots + r_n$ graded commutative generators ξ_i^j of degree i for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, r_i\}$.

• NB: An $[n]$ -manifold is always non-canonically split

$$\mathcal{O}_{\mathcal{M}} \simeq E_1[1] \oplus \dots \oplus E_n[n] \quad \text{over } M.$$

sections of E_1^* as gen. of deg 1.
 E_2^* deg 2

• a $[1]$ -mfld is always

$$\mathcal{O}_{\mathcal{M}} = E[1] \quad \text{i.e.} \quad C^\infty(\mathcal{O}_{\mathcal{M}}) = \Gamma(\wedge^* E^*) = \Omega^*(M)$$

Lie algebroids

A Lie algebroid over a manifold M is a vector bundle $A \rightarrow M$ with a Lie algebra bracket $[\cdot, \cdot]$ on $\Gamma(A)$ and an anchor $\rho: A \rightarrow TM$, such that

$$[a_1, fa_2] = f[a_1, a_2] + \mathcal{L}_{\rho(a_1)}(f)a_2 \quad]$$

for all $a_1, a_2 \in \Gamma(A)$ and $f \in C^\infty(M)$.

$(A[1], d_A)$

↑
vector field
of degree 1
with $d_A^2 = 0$

"homological vector field"

d_A "de Rham like"

$$d_A: \Omega^k(A) \rightarrow \Omega^{k+1}(A)$$

$$d_A^2 = 0$$

$$\parallel$$

$$\frac{1}{2}[d_A, d_A]$$

Example: Lie 1-algebroid

Take an \mathbb{N} -graded manifold of degree 1, i.e. $C^\infty(\mathcal{M}) = \Gamma(\wedge^\bullet E)$ for a vector bundle E over M and take a trivialising chart $U \subseteq M$ for E . Any homological vector field Q on \mathcal{M} can locally be written as

$$Q_U = \sum_{ij} \rho(\varepsilon_j)(x_i) e_i \partial_{x_j} - \sum_{ijk} \langle [\varepsilon_i, \varepsilon_j], e_k \rangle e_i e_j \partial_{e_k},$$

defining locally a Lie algebroid structure on $E^*|_U$. This structure is in fact global, and Lie 1-algebroids are equivalent to Lie algebroids. (This is due to Arkady Vaintrob.)

Lie algebroids \leftrightarrow [1]-nfd + hom. VF
"Lie 1-algebroids".

\rightarrow Lie n -algebroid : $(\mathcal{G}, \mathcal{Q})$
[n]-nfd, \uparrow hom. VF
degree 1
 $\mathcal{Q}^2 = 0$

How about Lie 2-algebroids vs Courant algebroids?

Courant algebroids

A Courant algebroid over a manifold M is a vector bundle $E \rightarrow M$ with a fibrewise nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, a bracket $[[\cdot, \cdot]]$ on the smooth sections $\Gamma(E)$, and an anchor $\rho: E \rightarrow TM$, which satisfy the following conditions

1. $[[e_1, e_2]] + [[e_2, e_1]] = \rho^* \mathbf{d}\langle e_1, e_2 \rangle$, ←
2. $[[e_1, [[e_2, e_3]]]] = [[[[e_1, e_2]], e_3]] + [[e_2, [[e_1, e_3]]]]$, ← Jacobi
3. $\rho(e_1)\langle e_2, e_3 \rangle = \langle [[e_1, e_2]], e_3 \rangle + \langle e_2, [[e_1, e_3]] \rangle$ ←

for all $e_1, e_2, e_3 \in \Gamma(E)$.

problem: $TM \oplus T^*M \rightarrow M$

Split Lie 2-algebroids

A split Lie 2-algebroid is a sequence

$$\mathcal{H} \cong E_1[1] \oplus E_2[2]$$

$$\begin{array}{ccc} B^* & \xrightarrow{l} & Q \xrightarrow{\rho} TM \\ E_2 & & E_1 \end{array}$$

of vector bundles over M with a skew-symmetric dull bracket

$\rightarrow \llbracket \cdot, \cdot \rrbracket: \Gamma(Q) \times \Gamma(Q) \rightarrow \Gamma(Q)$, a linear connection

$\nabla: \Gamma(Q) \times \Gamma(B) \rightarrow \Gamma(B)$ and a 3-form $\omega \in \Omega^3(Q, B^*)$, such that

(i) $\nabla_{l(\beta_1)}^* \beta_2 + \nabla_{l(\beta_2)}^* \beta_1 = 0$ for all $\beta_1, \beta_2 \in \Gamma(B^*)$,

(ii) $\llbracket q, l(\beta) \rrbracket = l(\nabla_q^* \beta)$ for $q \in \Gamma(Q)$ and $\beta \in \Gamma(B^*)$,

(iii) $\text{Jac}_{\llbracket \cdot, \cdot \rrbracket} = l \circ \omega \in \Omega^3(Q, Q)$,

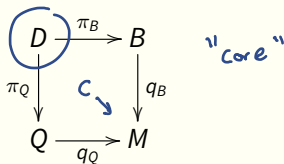
(iv) $R_{\nabla}(q_1, q_2)b = l^* \langle \mathbf{i}_{q_2} \mathbf{i}_{q_1} \omega, b \rangle$ for $q_1, q_2 \in \Gamma(Q)$ and $b \in \Gamma(B)$,
and

(v) $\mathbf{d}_{\nabla^*} \omega = 0$.

Geometrisation of [2]-manifolds

Double vector bundles

A double vector bundle is a commutative square



of vector bundles such that the structure maps of the vertical bundles define morphisms of the horizontal bundles.

Prototype :

$$\begin{array}{ccc}
 TE & \xrightarrow{\ast} & TM \\
 \ast \downarrow & E \searrow & \downarrow \\
 \underline{E} & \longrightarrow & M
 \end{array}
 \quad \underline{E} \rightarrow M \quad \text{VB}$$

NBI D is always noncanonically isomorphic to a decomposed

$$\begin{array}{ccc}
 q_B^{-1}(\mathbb{Q} \oplus \mathbb{C}) & \simeq & B \times_M \mathbb{Q} \times_M \mathbb{C} \longrightarrow B \\
 \downarrow \alpha & & \downarrow \\
 \mathbb{Q} & \longrightarrow & M
 \end{array}$$

Metric double vector bundles

$$\begin{array}{ccc}
 \mathbb{E} & \xrightarrow{\langle \cdot, \cdot \rangle} & B \leftarrow \text{metric} \\
 \downarrow \mathcal{Q} & & \downarrow \mathcal{Q}' \\
 \mathcal{Q} & \rightarrow & M
 \end{array}
 \quad \exists \text{ dec.}$$

$$\mathbb{E} \simeq B \times_{\mathbb{R}} \mathcal{Q} \times_{\mathbb{R}} \mathcal{Q}'$$

s.t.

$$\langle (b, q, z), (b, q', z') \rangle = z'(q) + z(q')$$

$$\begin{array}{l}
 b \in \mathcal{R}(B) : \quad \check{b} \in \mathcal{R}_{\mathcal{Q}}^{\ell}(\mathbb{E}) \quad \check{b}(q_m) = (b(m), q_m, 0_{\mathcal{Q}'}) \\
 z \in \mathcal{R}(\mathcal{Q}^{\vee}) : \quad z^+ \in \mathcal{R}_{\mathcal{Q}}^c(\mathbb{E}) \quad z^+(q_m) = (0_{\mathcal{Q}}, q_m, z(z))
 \end{array}$$

$$\mathcal{C}(\mathbb{E}) = \left\{ \chi \in \mathcal{R}_{\mathcal{Q}}^p(\mathbb{E}) \mid \chi \text{ isotropic} \right\}$$

$$\langle \chi(q), \chi(q') \rangle = 0 \quad (q, q') \in \mathcal{Q} \times_{\mathbb{R}} \mathcal{Q}$$

Involutive double vector bundles

$$\begin{array}{ccc} \mathbb{E} \times \mathbb{Q} & \rightarrow & \mathbb{Q} \\ \downarrow & \mathbb{B}^x \rightarrow & \downarrow \\ \mathbb{Q} & \rightarrow & \mathbb{M} \end{array}$$

$$\mathbb{I}: \mathbb{E} \times \mathbb{Q} \rightarrow \mathbb{E} \times \mathbb{Q} \quad \mathbb{I}^2 \stackrel{\text{def}}{=} \text{id}.$$

$$\exists \text{ dec. } \mathbb{E} \times \mathbb{Q} = \mathbb{Q} \times_n \mathbb{Q} \times_n \mathbb{B}^x$$

$$\text{s.t. } \mathbb{I}(q, q', \beta) = (q', q, \beta).$$

Metric double vector bundle charts

Let M be a smooth manifold and D a set with a map $\Pi: D \rightarrow M$. A **metric double vector bundle chart** is a quadruple $c = (U, \Theta, V_1, V_2)$, where U is an open set in M , V_1, V_2 are two vector spaces and $\Theta: \Pi^{-1}(U) \rightarrow U \times V_1 \times V_2 \times V_2^*$ is a bijection such that $\Pi = \text{pr}_1 \circ \Theta$.

Two double vector bundle charts c and c' are **compatible** if the “change of chart” $\Theta' \circ \Theta^{-1}$ over $U \cap U'$ sends (x, v_1, v_2, l) to

$$(x, \underline{A(x)v_1}, \underline{B(x)v_2}, \underline{(B(x)^{-1})^*l + \omega(x)(v_1)(B(x)v_2)})$$

with $x \in U \cap U'$, $v_i \in V_i$, $\underline{A} \in C^\infty(U \cap U', \text{Gl}(V_1))$, $\underline{B} \in C^\infty(U \cap U', \text{Gl}(V_2))$ and $\underline{\omega} \in C^\infty(U \cap U', V_1^* \otimes V_2^* \wedge V_2^*)$.

Cocycle conditions: the standard ones for A 's and B 's, and
$$\left[\omega^{\gamma\beta}(x)(v_1) = B^{\alpha\gamma}(x)^t \cdot \omega^{\alpha\beta}(x)(v_1) \cdot B^{\alpha\gamma}(x) + \omega^{\gamma\alpha}(x)(A^{\alpha\beta}(x)v_1). \right]$$

Geometrisation of [2]-manifolds

Theorem (JL 2018)

The category of positively graded manifolds of degree 2 is equivalent to the category of involutive double vector bundles.

Degree 1:

$$\xi_i^\beta = \sum_{j=1}^{r_1} A_{ji}^{\alpha\beta} \xi_j^\alpha.$$

[2]-mfd's

↓

m DVBS's.

Degree 2:

$$\eta_i^\beta = \sum_{j=1}^{r_2} B_{ji}^{\alpha\beta} \eta_j^\alpha + \sum_{1 \leq k < l \leq r_1} \omega_{kli}^{\alpha\beta} \xi_k^\alpha \wedge \xi_l^\alpha.$$

$$\begin{array}{ccc} E & \rightarrow & B \\ \downarrow \alpha & & \downarrow \\ Q & \rightarrow & M \end{array}$$

$$\rightarrow C^\infty(\mathcal{JG}(E))$$

$C(E)$ gen. of dg 2
 $\Gamma(Q^1)$ gen. of dg 2
 $C^\infty(M)$ gen. of dg 0.

Geometrisation of $[n]$ -manifolds

(in preparation ; with Mark Heuer)

$[n]$ -manifolds \leftrightarrow n -fold vector bundles
with S_n -action.

VB- and LA-Courant algebroids

Theorem (JL 2018)

The category of positively graded Poisson manifolds of degree 2 is equivalent to the category of self-dual VB-algebroids.

$$(\mathcal{M}, \{\cdot, \cdot\})$$

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{\langle \cdot, \cdot \rangle} & \mathbb{B} \\ \text{LA} \downarrow & \mathcal{Q}^* \rightsquigarrow & \downarrow \text{LA} \\ \mathcal{Q} & \longrightarrow & \mathbb{M} \end{array}$$

VB-algebroid.

Theorem (Li-Bland 2012, JL 2019)

The category of Lie 2-algebroids is equivalent to the category of VB-Courant algebroids.

A decomposed VB-Courant algebroid defines the structure objects as on Slide 11:

$$\Theta(\tilde{q}) = \widehat{\nabla}_q \in \mathfrak{X}(B), \quad \llbracket \tilde{q}, \tau^\dagger \rrbracket = (\Delta_q \tau)^\dagger \text{ and} \\ \llbracket \tilde{q}_1, \tilde{q}_2 \rrbracket = \widetilde{\llbracket q_1, q_2 \rrbracket} - R_\omega(q_1, q_2)^\dagger,$$

for all $q, q_1, q_2 \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$, where

$\Delta: \Gamma(Q) \times \Gamma(Q^*) \rightarrow \Gamma(Q^*)$ is the “Lie derivative” that is dual to the dull bracket.

Theorem (Li-Bland 2012, JL 2020)

The category of Poisson Lie 2-algebroids is equivalent to the category of LA-Courant algebroids.

$$(\mathcal{M}, \{ \cdot, \cdot \}, \mathcal{Q})$$

\mathcal{Q} preserves $\{ \cdot, \cdot \}$

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{\text{C.A.}} & B \\ \text{L.A.} \downarrow & \cong & \downarrow \\ \mathcal{Q} & \longrightarrow & M \end{array}$$

in a split way!

Symplectic Lie 2-algebroids correspond to tangent doubles of Courant algebroids.

$$\begin{array}{ccc} TE & \xrightarrow{\text{C.A.}} & TM \\ \downarrow \text{LA} & \text{E} \rightarrow & \downarrow \text{LA} \\ E & \xrightarrow{\text{C.A.}} & M \end{array}$$

Matched pairs of representations

Matched pairs of representations

A, B Lie algebras on M

$A \subset B$

$B \subset A$



n

$A \oplus B \rightarrow M$

lie alg.

$A, B \subseteq A \oplus B$ sub.

$[a, b] = (-\nabla_b a, \nabla_a b)$

bicrossproduct.

Matched pairs of representations

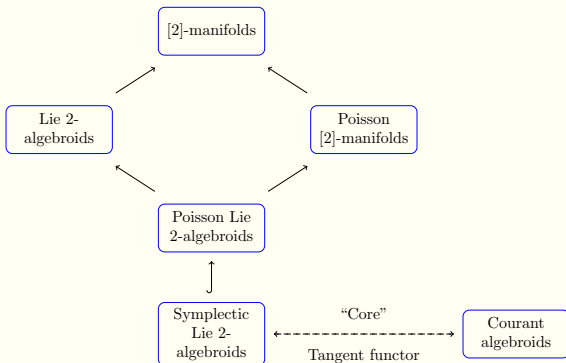
Lie bialgebroids

Thank you for your attention!

References and tables

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Table of the supergeometric objects.



Double geometric objects.

