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GÖTTINGEN

On LA-Courant algebroids and Poisson Lie 2-algebroids

A geometrisation of [2]-manifolds.

by

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"vector bundle"

Li-Blaad

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Graded manifolds and Lie n -algebroids

Positively graded manifolds

An \mathbb{N} -graded manifold \mathcal{M} of degree n and dimension $(p; r_1, \dots, r_n)$ is a smooth p -dimensional manifold M endowed with a locally free and finitely generated sheaf $C^\infty(\mathcal{M})$ of \mathbb{N} -graded commutative associative unital \mathbb{R} -algebras, which can locally be written as

$$C^\infty(\mathcal{M})_U = \underbrace{C^\infty(U)}_{\text{smooth manifold}} [\xi_1^1, \dots, \xi_1^{r_1}, \xi_2^1, \dots, \xi_2^{r_2}, \dots, \xi_n^1, \dots, \xi_n^{r_n}]$$

with $r_1 + \dots + r_n$ graded commutative generators ξ_i^j of degree i for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, r_i\}$.

• NB: An $[n]$ -manifold is always

“ $[n]$ -manifold”

Lie algebroids

A Lie algebroid over a manifold M is a vector bundle $A \rightarrow M$ with a Lie algebra bracket $[\cdot, \cdot]$ on $\Gamma(A)$ and an anchor $\rho: A \rightarrow TM$, such that

$$[a_1, fa_2] = f[a_1, a_2] + \mathcal{L}_{\rho(a_1)}(f)a_2$$

for all $a_1, a_2 \in \Gamma(A)$ and $f \in C^\infty(M)$.

Example: Lie 1-algebroid

Take an \mathbb{N} -graded manifold of degree 1, i.e. $C^\infty(\mathcal{M}) = \Gamma(\wedge^\bullet E)$ for a vector bundle E over M and take a trivialisating chart $U \subseteq M$ for E . Any homological vector field Q on \mathcal{M} can locally be written as

$$Q_U = \sum_{ij} \rho(\varepsilon_j)(x_i) e_i \partial_{x_j} - \sum_{ijk} \langle [\varepsilon_i, \varepsilon_j], e_k \rangle e_i e_j \partial_{e_k},$$

defining locally a Lie algebroid structure on $E^*|_U$. This structure is in fact global, and Lie 1-algebroids are equivalent to Lie algebroids. (This is due to Arkady Vaintrob.)

How about Lie 2-algebroids vs Courant algebroids?

Courant algebroids

A Courant algebroid over a manifold M is a vector bundle $E \rightarrow M$ with a fibrewise nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, a bracket $[[\cdot, \cdot]]$ on the smooth sections $\Gamma(E)$, and an anchor $\rho: E \rightarrow TM$, which satisfy the following conditions

1. $[[e_1, e_2]] + [[e_2, e_1]] = \rho^* \mathbf{d}\langle e_1, e_2 \rangle$,
2. $[[e_1, [[e_2, e_3]]]] = [[[e_1, e_2]], e_3]] + [[e_2, [[e_1, e_3]]]]$,
3. $\rho(e_1)\langle e_2, e_3 \rangle = \langle [[e_1, e_2]], e_3 \rangle + \langle e_2, [[e_1, e_3]] \rangle$

for all $e_1, e_2, e_3 \in \Gamma(E)$.

Split Lie 2-algebroids

A split Lie 2-algebroid is a sequence

$$B^* \xrightarrow{l} Q \xrightarrow{\rho} TM$$

of vector bundles over M with a skew-symmetric dull bracket

$[\![\cdot, \cdot]\!] : \Gamma(Q) \times \Gamma(Q) \rightarrow \Gamma(Q)$, a linear connection

$\nabla : \Gamma(Q) \times \Gamma(B) \rightarrow \Gamma(B)$ and a 3-form $\omega \in \Omega^3(Q, B^*)$, such that

- (i) $\nabla_{l(\beta_1)}^* \beta_2 + \nabla_{l(\beta_2)}^* \beta_1 = 0$ for all $\beta_1, \beta_2 \in \Gamma(B^*)$,
- (ii) $[\![q, l(\beta)]\!] = l(\nabla_q^* \beta)$ for $q \in \Gamma(Q)$ and $\beta \in \Gamma(B^*)$,
- (iii) $\text{Jac}_{[\![\cdot, \cdot]\!]} = l \circ \omega \in \Omega^3(Q, Q)$,
- (iv) $R_{\nabla}(q_1, q_2)b = l^* \langle \mathbf{i}_{q_2} \mathbf{i}_{q_1} \omega, b \rangle$ for $q_1, q_2 \in \Gamma(Q)$ and $b \in \Gamma(B)$,
and
- (v) $\mathbf{d}_{\nabla^*} \omega = 0$.

Geometrisation of [2]-manifolds

Double vector bundles

A double vector bundle is a commutative square

$$\begin{array}{ccc} D & \xrightarrow{\pi_B} & B \\ \pi_Q \downarrow & & \downarrow q_B \\ Q & \xrightarrow{q_Q} & M \end{array}$$

of vector bundles such that the structure maps of the vertical bundles define morphisms of the horizontal bundles.

Metric double vector bundles

Involutive double vector bundles

Metric double vector bundle charts

Let M be a smooth manifold and D a set with a map $\Pi: D \rightarrow M$. A **metric double vector bundle chart** is a quadruple $c = (U, \Theta, V_1, V_2)$, where U is an open set in M , V_1, V_2 are two vector spaces and $\Theta: \Pi^{-1}(U) \rightarrow U \times V_1 \times V_2 \times V_2^*$ is a bijection such that $\Pi = \text{pr}_1 \circ \Theta$.

Two double vector bundle charts c and c' are **compatible** if the “change of chart” $\Theta' \circ \Theta^{-1}$ over $U \cap U'$ sends (x, v_1, v_2, l) to

$$(x, A(x)v_1, B(x)v_2, (B(x)^{-1})^*l + \omega(x)(v_1)(B(x)v_2))$$

with $x \in U \cap U'$, $v_i \in V_i$, $A \in C^\infty(U \cap U', \text{Gl}(V_1))$, $B \in C^\infty(U \cap U', \text{Gl}(V_2))$ and $\omega \in C^\infty(U \cap U', V_1^* \otimes V_2^* \wedge V_2^*)$.

Cocycle conditions: the standard ones for A 's and B 's, and $\omega^{\gamma\beta}(x)(v_1) = B^{\alpha\gamma}(x)^t \cdot \omega^{\alpha\beta}(x)(v_1) \cdot B^{\alpha\gamma}(x) + \omega^{\gamma\alpha}(x)(A^{\alpha\beta}(x)v_1)$.

Geometrisation of [2]-manifolds

Theorem (JL 2018)

The category of positively graded manifolds of degree 2 is equivalent to the category of involutive double vector bundles.

Degree 1:

$$\xi_i^\beta = \sum_{j=1}^{r_1} A_{ji}^{\alpha\beta} \xi_j^\alpha.$$

Degree 2:

$$\eta_i^\beta = \sum_{j=1}^{r_2} B_{ji}^{\alpha\beta} \eta_j^\alpha + \sum_{1 \leq k < l \leq r_1} \omega_{kli}^{\alpha\beta} \xi_k^\alpha \wedge \xi_l^\alpha.$$

Geometrisation of $[n]$ -manifolds

VB- and LA-Courant algebroids

Theorem (JL 2018)

The category of positively graded Poisson manifolds of degree 2 is equivalent to the category of self-dual VB-algebroids.

Theorem (Li-Bland 2012, JL 2019)

The category of Lie 2-algebroids is equivalent to the category of VB-Courant algebroids.

A decomposed VB-Courant algebroid defines the structure objects as on Slide 11:

$$\Theta(\tilde{q}) = \widehat{\nabla}_q \in \mathfrak{X}(B), \quad \llbracket \tilde{q}, \tau^\dagger \rrbracket = (\Delta_q \tau)^\dagger \text{ and} \\ \llbracket \tilde{q}_1, \tilde{q}_2 \rrbracket = \widetilde{\llbracket q_1, q_2 \rrbracket} - R_\omega(q_1, q_2)^\dagger,$$

for all $q, q_1, q_2 \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$, where

$\Delta: \Gamma(Q) \times \Gamma(Q^*) \rightarrow \Gamma(Q^*)$ is the “Lie derivative” that is dual to the dull bracket.

Theorem (Li-Bland 2012, JL 2020)

The category of Poisson Lie 2-algebroids is equivalent to the category of LA-Courant algebroids.

Symplectic Lie 2-algebroids correspond to tangent doubles of Courant algebroids.

$$\begin{array}{ccc} TE & \longrightarrow & TM \\ \downarrow & & \downarrow \\ E & \longrightarrow & M \end{array}$$

Matched pairs of representations

Matched pairs of representations

Matched pairs of representations

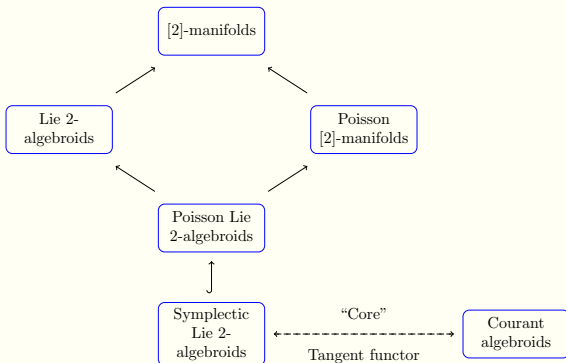
Lie bialgebroids

Thank you for your attention!

References and tables

- ❖ *The geometrisation of \mathbb{N} -manifolds of degree 2*, “Journal of Geometry and Physics” (2018), Volume 133, 113-140.
- ❖ *Lie 2-algebroids and matched pairs of 2-representations – a geometric approach*, “Pacific Journal of Mathematics” (2019), Volume 301, Number 1, 143-188.
- ❖ *On LA-Courant algebroids and Poisson Lie 2-algebroids*, “Mathematical Physics, Analysis and Geometry” (2020), Volume 23, Number 31.
- ❖ *Multiple vector bundles: cores, splittings and decompositions*, with Malte Heuer, “Theory and Applications of Categories” (2020), Vol. 35, No. 19, 665-699.

Table of the supergeometric objects.



Double geometric objects.

