Symmetry and Renormalisation in Regularity Structures

Ajay Chandra (Imperial)
joint w/ Ilya Chevyrev, Martin Hairer and Hao Shen

Erwin Schrödinger Institute

Nov. 16, 2021

Some motivation

$$\frac{1}{Z}\int \cdots e^{-S(\varphi)}\, d\varphi \mapsto \frac{\partial}{\partial t}\Phi = -\frac{\delta}{\delta \Phi} S(\Phi) + \xi$$

Euclidean QFT

Some motivation

$$\frac{1}{Z}\int \cdots e^{-S(\varphi)}\, d\varphi \mapsto \frac{\partial}{\partial t}\Phi = -\frac{\delta}{\delta \Phi} S(\Phi) + \xi$$

Euclidean QFT

$$\frac{1}{Z}\int \cdots e^{-\int \varphi^4(x) + |\nabla \varphi(x)|^2 dx}\, d\varphi \mapsto \partial_t \Phi = \Delta \Phi - \Phi^3 + \xi$$

ill-posed

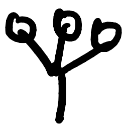
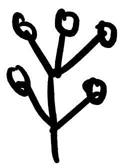
Some motivation

$$\frac{1}{Z}\int \cdots\, e^{-S(\varphi)}\,d\varphi \mapsto \frac{\partial}{\partial t}\Phi = -\frac{\delta}{\delta\Phi}S(\Phi)+\xi$$

Euclidean QFT

$$\frac{1}{Z}\int \cdots\, e^{-\int \varphi^4(x)+|\nabla\varphi(x)|^2\,dx}\,d\varphi \mapsto \partial_t\Phi=\Delta\Phi-\Phi^3+\xi$$

ill-posed

$\Phi =$  $+$  $+\ 3$  $+ \ldots$ $\circ = \xi$, $\overset{\tau}{|} = (\partial_t-\Delta)^{-1}\tau$, $\tau\,\bar{\tau}$ product

Some motivation

$$\frac{1}{Z}\int \cdots e^{-S(\varphi)}\,d\varphi \;\mapsto\; \frac{\partial}{\partial t}\Phi = -\frac{\delta}{\delta\Phi}S(\Phi) + \xi$$

Euclidean QFT

$$\frac{1}{Z}\int \cdots e^{-\int \varphi^4(x)+|\nabla\varphi(x)|^2dx}\,d\varphi \;\mapsto\; \partial_t\Phi = \Delta\Phi - \Phi^3 + \xi$$

ill-posed

$$\Phi(z) = \Phi_{\mathbf{1}}(z)\,\underline{\mathbf{1}} + \sum_{j=1}^{d}\Phi_{X^j}(z)\,\underline{X^j} + \Phi(z) + \Phi(z) + \Phi(z)$$

Some motivation

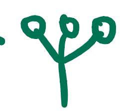
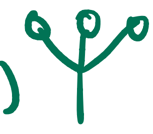
$$\frac{1}{Z}\int \sim e^{-S(\varphi)}\, d\varphi \;\mapsto\; \frac{\partial}{\partial t}\Phi = -\frac{\delta}{\delta \Phi} S(\Phi) + \xi$$

Euclidean QFT

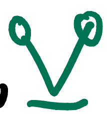
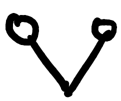
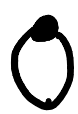
$$\frac{1}{Z}\int \sim e^{-\int \varphi^4(x) + |\nabla \varphi(x)|^2 dx}\, d\varphi \;\mapsto\; \partial_t \Phi = \Delta \Phi - \Phi^3 + \xi$$

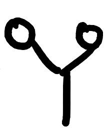
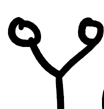
ill-posed

$\Phi(z) = \Phi_1(z)\underline{1} + \sum_{j=1}^{d} \Phi_{X^j}(z)\,\underline{X^j} + \Phi$(z) 

$+\, \Phi$(z)  $+\, \Phi$(z) 

$\Pi_{z,w}$  $=$ (z)

$\Pi_{z,w}$  $=$ $(z) -$ 

$\Pi_{z,w}$  $=$ $(z) -$ (w)

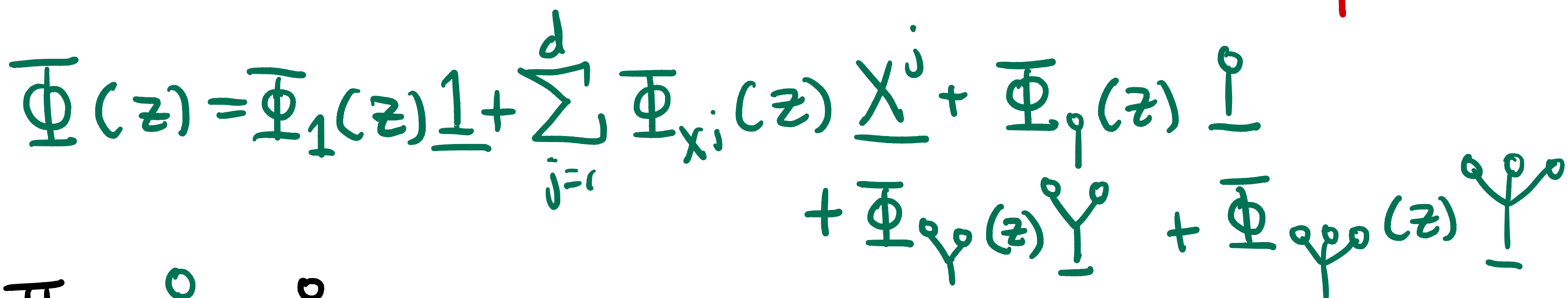
Some motivation

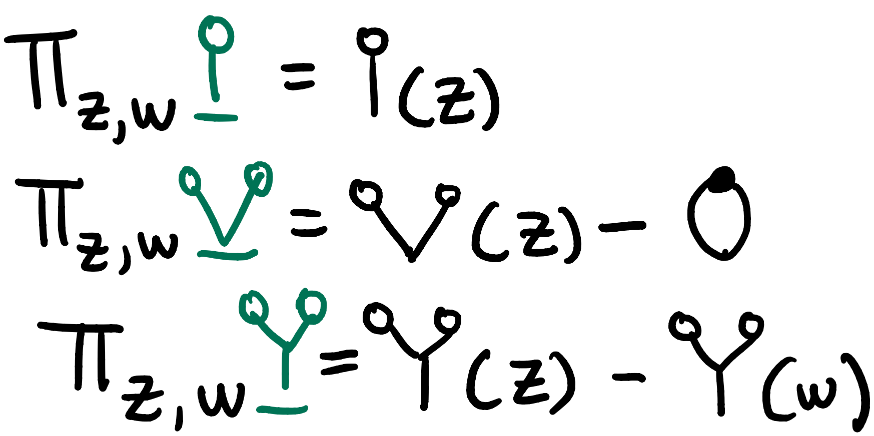
$$\frac{1}{Z}\int \cdots e^{-S(\varphi)}\, d\varphi \;\mapsto\; \frac{\partial}{\partial t}\Phi = -\frac{\delta}{\delta \Phi} S(\Phi) + \xi$$

Euclidean QFT

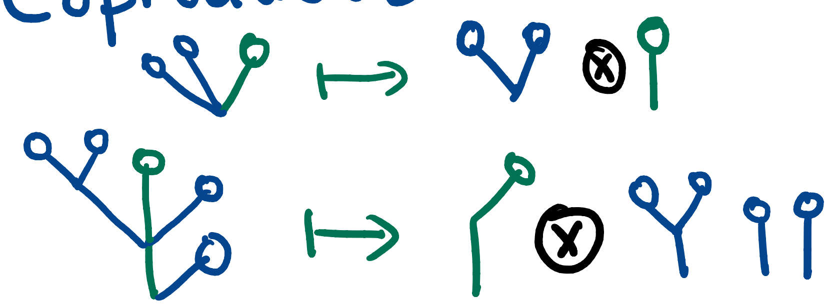
$$\frac{1}{Z}\int \cdots e^{-\int \varphi^4(x) + |\nabla \varphi(x)|^2 dx}\, d\varphi \;\mapsto\; \partial_t \Phi = \Delta \Phi - \Phi^3 + \xi$$

ill-posed





Coproducts



Equations of the form

$$(\partial_t - \mathcal{L}_b) A_b = F_b(A, \eta), \quad b \in L_+$$

Smooth function of A, η and derivatives

- $(\partial_t - \mathcal{L}_b)$ parabolic operator
- $A = (A_b : b \in L_+)$ components of soln
- $\eta = (\eta_b : b \in L_-)$ noise
- A_b / η_b takes values in W_b (ptwise)

finite dim

Equations of the form

$$(\partial_t - \mathcal{L}_b) A_b = F_b(A, \eta), \quad b \in L_+$$

Smooth function of A, η and derivatives

- $(\partial_t - \mathcal{L}_b)$ parabolic operator
- $A = (A_b : b \in L_+)$ components of soln
- $\eta = (\eta_b : b \in L_-)$ noise
- A_b / η_b takes values in W_b (ptwise)

finite dim

- Have a robust algebraic theory
  - Regularity Structure $\mathcal{T} = \langle \mathcal{T} \rangle$
  - space of models $\ell \in \mathcal{T}_-^*$
  - action of of $\mathcal{T}_-$ on equation

$$(\partial_t - \mathcal{L}) A_b = F(A, \eta) + \sum_{\tau \in \mathcal{T}_-} \ell(\tau) \Upsilon_b[\tau](A, \eta)$$

Depends on $W_b = \mathbb{R} \ \forall \ b \in L_-$

Example: Stochastic Yang-Mills Equation

$L_+ = \mathcal{E} + \mathfrak{Z}$
$L_- = \mathcal{E} - \mathfrak{Z}$

$$(\partial_t - \Delta) A_i = \sum_{j=1}^{d} [A_j, 2\partial_j A_i - \partial_i A_j + [A_j, A_i]] + \eta_i$$

$1 \le i \le d$

A and η take values in $W_+ \simeq W_- \simeq \mathfrak{g}^d$

Lie Algebra

F takes values in W_+

Example: Stochastic Yang-Mills Equation $\quad L_+ = \{+\}$, $L_- = \{-\}$

$$(\partial_t - \Delta) A_i = \sum_{j=1}^{d} [A_j, 2\partial_j A_i - \partial_i A_j + [A_j, A_i]] + \eta_i$$

$1 \le i \le d$

A and η take values in $W_+ \simeq W_- \simeq \mathfrak{g}^d$

Lie Algebra

F takes values in W_+

To apply existing theory we could expand W_- in a basis so we are in a situation where $|L_-| = d \cdot \dim(\mathfrak{g})$ scalar noises

Example: Stochastic Yang-Mills Equation

$L_+ = \{\xi + \}$
$L_- = \{\xi - \}$

$$(\partial_t - \Delta) A_i = \sum_{j=1}^{d} [A_j, 2\partial_j A_i - \partial_i A_j + [A_j, A_i]] + \xi_i$$

$1 \le i \le d$

A and ξ take values in $W_+ \simeq W_- \simeq \mathfrak{g}^d$

Lie Algebra

F takes values in W_+

To apply existing theory we could expand W_- in a basis so we are in a situation where $|L_-| = d \cdot \dim(\mathfrak{g})$ scalar noises

Problems:
- choice of basis not canonical
- Hard to see structure of renormalised eqn

Will introduce framework that:

- allows us to build regularity structures generated by vector valued noise without ref to a basis

Will introduce framework that :

- allows us to build regularity structures generated by vector valued noise without ref to a basis

- Behaves well with respect to decompositions, so we can use results from the scalar noise setting

Will introduce framework that:

- allows us to build regularity structures generated by vector valued noise without ref to a basis

- Behaves well with respect to decompositions, so we can use results from the scalar noise setting

- Allows us to easily verify that if F and z play well with a local symmetry then renormalised equation does too!

Some motivation

If $\{e_n\}$ is a basis for W_-, then can write $y = \sum_n y_n e_n$

If we have $\Xi_n \in \mathbb{T}$ with $\Pi \Xi_n = y_n$, then

$$\Xi = \sum_n \Xi_n e_n \in \mathbb{T} \otimes W_- \text{ satisfies } \Pi \Xi = y$$

Some motivation

If $\{e_n\}$ is a basis for W_-, then can write $y = \sum_n y_n e_n$

If we have $\Xi_n \in \mathcal{T}$ with $\Pi \Xi_n = y_n$, then

$$\Xi = \sum_n \Xi_n e_n \in \mathcal{T} \otimes W_- \text{ satisfies } \Pi \Xi = y$$

Idea: $\operatorname{span}(\Xi_n) = \underbrace{\mathcal{T}[\bullet] \subset \mathcal{T}}_{\text{subspace}}, \quad \mathcal{T} = \bigoplus_{\tau \in \mathcal{T}} \mathcal{T}[\tau]$

set of trees

Some motivation

If $\{e_n\}$ is a basis for W_-, then can write $y = \sum_n y_n e_n$

If we have $\Xi_n \in T$ with $\Pi\, \Xi_n = y_n$, then

$$\Xi = \sum_n \Xi_n e_n \in T \otimes W_- \text{ satisfies } \Pi\, \Xi = y$$

Idea: $\operatorname{span}(\Xi_n) = \underbrace{\mathcal{T}[\bullet]}_{\text{subspace}} \subset T \quad , \quad T = \bigoplus_{\tau \in \mathcal{T}} \mathcal{T}[\tau]$

set of trees

Set $\mathcal{T}[\bullet] = W_-^*$. For $w \in W_-^*$, $\Pi w = w(y)$

Then $\Xi = \sum_n \underbrace{e_n^* \otimes e_n}_{= \, \mathrm{id}_W} \in W_-^* \otimes W_-$

Multiplication and symmetrisation

For $w_1 \in \mathcal{T}[\bullet]$, $w_2 \in \mathcal{T}[\circ]$ we can set

$$
\begin{aligned}
w_1 w_2 &= w_1 \otimes w_2 \in \mathcal{T}[\bullet \vee \circ] = \mathcal{T}[\circ \vee \bullet] \\
\| \quad & \\
w_2 w_1 & \qquad\qquad\qquad\qquad \simeq \mathcal{T}[\circ] \otimes \mathcal{T}[\bullet]
\end{aligned}
$$

Multiplication and symmetrisation

For $w_1 \in T[\bullet]$, $w_2 \in T[\circ]$ we can set

$$\begin{array}{l} w_1 w_2 = w_1 \otimes w_2 \in T[\bullet \vee \circ] = T[\circ \vee \bullet] \\ \quad \| \\ w_2 w_1 \qquad\qquad\qquad\qquad\qquad \simeq T[\circ] \otimes T[\bullet] \end{array}$$

However, if we have $w_1, w_2 \in T[\bullet]$ this doesn't work!

Need to set $T[\bullet \vee \bullet] \simeq T[\bullet] \otimes_s T[\bullet]$

symmetric tensor product

Multiplication and symmetrisation

For $w_1 \in T[\bullet]$, $w_2 \in T[\circ]$ we can set

$$w_1 w_2 = w_1 \otimes w_2 \in T[\bullet \vee \circ] = T[\circ \vee \bullet] \simeq T[\circ] \otimes T[\bullet]$$

$$w_1 w_2 = w_2 w_1$$

However, if we have $w_1, w_2 \in T[\bullet]$ this doesn't work!

Need to set $T[\bullet \vee \bullet] \simeq T[\bullet] \otimes_S T[\bullet]$

symmetric tensor product

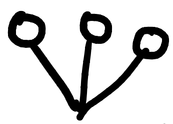
<u>Integration</u> $T[\tau] \simeq T[\overset{\tau}{|}]$, <u>Poly</u> $T[X^k] = \mathbb{R}$

General Prescription

$$\Upsilon[\tau] = \bigotimes_{\substack{\text{noise} \\ \text{in } \tau}} W^*_{\text{noise}} \Big/ \text{symmetries of } \tau$$

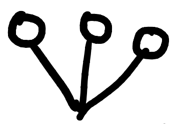
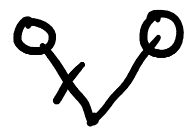
General Prescription

$$\mathcal{T}[\tau] = \bigotimes_{\substack{\text{noise} \\ \text{in } \tau}} W^*_{\text{noise}} \Big/ \begin{array}{c}\text{symmetries} \\ \text{of } \tau\end{array}$$

$\mathcal{T}[$  $] \simeq (W_0^*)^{\otimes_s 3}$

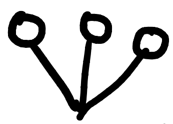
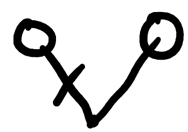
General Prescription

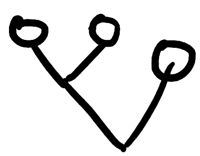
$$\mathcal{T}[\tau] = \bigotimes_{\substack{\text{noise} \\ \text{in } \tau}} W^{*}_{\text{noise}} \Big/ {\substack{\text{symmetries} \\ \text{of } \tau}}$$

$\mathcal{T}[$  $] \simeq (W_0^{*})^{\otimes_s 3}$, $\mathcal{T}[$  $] \simeq W_0^{*} \otimes W_0^{*}$

General Prescription

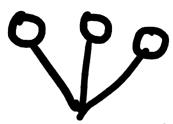
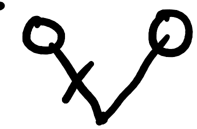
$$\Upsilon[\tau] = \bigotimes_{\substack{\text{noise} \\ \text{in } \tau}} W^*_{\text{noise}} \Big/ \text{symmetries of } \tau$$

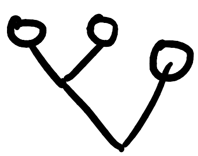
$\Upsilon\Big[$  $\Big] \simeq (W_0^*)^{\otimes_s 3}$, $\Upsilon\Big[$  $\Big] \simeq W_0^* \otimes W_0^*$

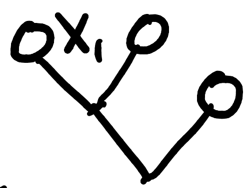
$\Upsilon\Big[$  $\Big] = (W_0^* \otimes_s W_0^*) \otimes W_0^*$

General Prescription

$$T[\tau] = \bigotimes_{\substack{\text{noise} \\ \text{in } \tau}} W^*_{\text{noise}} \Big/ {\scriptstyle \text{symmetries of } \tau}$$

$T[$$] \simeq (W_0^*)^{\otimes_s 3}$, $T[$$] \simeq W_0^* \otimes W_0^*$

$T[$$] = (W_0^* \otimes_s W_0^*) \otimes W_0^*$

$T[$$] = W_0^{\otimes 3}$

The Construction

$L \leftarrow$ finite set of types

$(T, \ell) \leftarrow$ typed set, T a finite set, $\ell : T \to L$ map

The Construction

$L \leftarrow$ finite set of types

$(T, \ell) \leftarrow$ typed set, T a finite set, $\ell : T \to L$ map

Symmetric Set

the instances of our typed set

- an index set A
- for each $a \in A$, a typed set (T^a, ℓ^a)
- For each pair $a, b \in A$, a non-empty set of type preserving bijections $\Gamma^{a,b}$ from T^b to T^a

$$\gamma \in \Gamma^{a,b} \Rightarrow \gamma^{-1} \in \Gamma^{b,a}$$

$$\gamma \in \Gamma^{a,b}, \bar{\gamma} \in \Gamma^{b,c} \Rightarrow \gamma \circ \bar{\gamma} \in \Gamma^{a,c}$$

Morphisms

Given symmetric sets $s, \bar{s}$

$$\mathrm{Hom}(s, \bar{s}) = \left\{ \begin{array}{c} \Phi = \left(\Phi_{\bar{a},a} : \begin{array}{l} a \in A_s \\ \bar{a} \in A_{\bar{s}} \end{array} \right) \text{ with} \\ \Phi_{\bar{a},a} \in \left\langle \begin{array}{c} \text{type preserving biject.} \\ (T^a, \ell^a) \text{ to } (T^{\bar{a}}, \ell^{\bar{a}}) \end{array} \right\rangle \\ \underline{\text{Such that}} \\ \forall\, \gamma \in \Gamma_{\Delta}^{a,b},\ \bar{\gamma} \in \Gamma_{\bar{\Delta}}^{\bar{a},\bar{b}}, \\ \Phi_{\bar{a},a} \circ \gamma_{a,b} = \gamma_{\bar{a},\bar{b}} \circ \Phi_{\bar{b},b} \end{array} \right\}$$

For trees

i.e, same tree, different vertex set

- $A \leftarrow$ all "instances" of tree
- typed sets (N^a, ℓ^a)

  $N^a \leftarrow$ set of noises

  $\ell \leftarrow$ type of noise

- $\Gamma_{a,a} \leftarrow$ symmetries of tree a

  $\Gamma_{b,a} \leftarrow$ symmetries between different instances

Given $V = (V_b : b \in L)$ and symmetric set s

we define $$F_V(s) = \left\{ (v^{(a)})_{a \in A_s} \in \prod_{a \in A_s} V^{\otimes T^a} : \begin{array}{c} \underline{\text{such that}} \\ \forall a,b \in A_s,\ \gamma \in \Gamma^{a,b} \\ \gamma_{a,b}(v^{(b)}) = v^{(a)} \end{array} \right\}$$

Given $V = (V_b : b \in L)$ and symmetric set s we define

$$F_V(s) = \left\{ (v^{(a)})_{a \in A_s} \in \prod_{a \in A_s} V^{\otimes T^a} : \begin{array}{c} \underline{\text{such that}} \\ \forall a, b \in A_s,\ \gamma \in \Gamma^{a,b} \\ \gamma_{a,b}(v^{(b)}) = v^{(a)} \end{array} \right\}$$

Here, for bijection $f : B \to C$, $u = \bigotimes_{b \in B} u_b \in V^{\otimes B}$

$$f(u) = \bigotimes_{c \in C} u_{f^{-1}(c)} \in V^{\otimes C}$$

Given $V = (V_b : b \in L)$ and symmetric set s we define

$$F_V(s) = \left\{ (v^{(a)})_{a \in A_s} \in \prod_{a \in A_s} V^{\otimes T^a} : \begin{array}{c} \underline{\text{such that}} \\ \forall a, b \in A_s,\ \gamma \in \Gamma^{a,b} \\ \gamma_{a,b}(v^{(b)}) = v^{(a)} \end{array} \right\}$$

Here, for bijection $f : B \to C$, $u = \bigotimes_{b \in B} u_b \in V^{\otimes B}$

$$f(u) = \bigotimes_{c \in C} u_{f^{-1}(c)} \in V^{\otimes C}$$

F_V extends to a functor from Symmetric sets to Vector spaces

For Regularity Structures

$V = W^* = (W_b^* : b \in L_-)$

- We set $\mathcal{T}[\tau] = F_{W^*}(\tau)$, $\mathcal{T} = \bigoplus_{\tau \in \mathfrak{T}} \mathcal{T}[\tau]$

For Regularity Structures

- We set $\mathcal{T}[\tau] = F_{W^*}(\tau)$, $\mathcal{T} = \bigoplus_{\tau \in \mathcal{T}} \mathcal{T}[\tau]$

- Coproducts on trees as symmetric sets $\xmapsto{F_{W^*}}$ coproducts on $\mathcal{T}$

Can organize algebraic operations using trees in vector setting!

Decomposition

$p : \bar{L} \longrightarrow L$ surjection,

$$V = (V_b : b \in L)$$

$$\bar{V} = (\bar{V}_{\bar{b}} : \bar{b} \in \bar{L})$$

$$V_b = \bigoplus_{\bar{b} \in p^{-1}(b)} V_{\bar{b}}$$

Decomposition

$p : \bar{L} \longrightarrow L$ surjection, $V = (V_b : b \in L)$

$\bar{V} = (\bar{V}_{\bar{b}} : \bar{b} \in \bar{L})$

$$V_b = \bigoplus_{\bar{b} \in p^{-1}(b)} V_{\bar{b}}$$

Have a canonical isomorphism

$$j : F_V(s) \longrightarrow \bigoplus_{\bar{s} \in p(s)} F_V(\bar{s})$$

Allows us to import algebraic results from scalar setting to vector setting

Vectorial Thm on renormalised eqns

$$\Upsilon_b[\tau](A,\mathfrak{z}) \in T[\tau] \otimes W_b$$

$$(\partial_t - \mathcal{L}_b) A_b = F_b(A,\mathfrak{z}) + \sum_{\tau \in \mathcal{T}_-} (\underbrace{\ell[\tau]}_{\in T[\tau]^*} \otimes \mathrm{id}) \Upsilon_b[\tau](A,\mathfrak{z})$$

Advantage: If F is written in terms of Lie brackets, same is true of $\Upsilon_b[\tau]$

Local Symmetry

Fact: if we have a linear map

$L = \bigoplus_{b \in L} L_b$, $L_b \in \mathrm{Lin}(V_b, \overline{V}_b)$, then

L lifts to a map $L \in \mathrm{Lin}(F_V(S), F_{\overline{V}}(S))$

"apply L to every factor"

Local Symmetry

Fact: if we have a linear map

$L = \bigoplus_{b \in L} L_b$, $L_b \in Lin(V_b, \bar{V}_b)$, then

L lifts to a map $L \in Lin(F_V(S), F_{\bar{V}}(S))$

"apply L to every factor"

$\Rightarrow$ Transformations on target space of noise induce transformations on regularity structure

Given transformations $T = (T_b : \underline{b} \in L_+ \sqcup L_-)$

$T_b \in Lin(W_b, W_b)$

we say $F = (F_b)_{b \in L_+}$ is T covariant if

$\forall b \in L_+, \; F_b(TA, Ty) = T_b F_b(A, y)$

Given transformations $T = (T_b : \underline{b \in L_+ \sqcup L_-})$

$T_b \in \mathrm{Lin}(W_b, W_b)$

we say $F = (F_b)_{b \in L_+}$ is $\mathcal{T}$ covariant if

$\forall\, b \in L_+,\ F_b(TA, Ty) = T_b F_b(A, y)$

<u>Lemma</u> If F is $\mathcal{T}$ covariant, then $\forall\, b \in L_+$,

$$(T_{\mathcal{T}} \otimes \mathrm{id}_{W_b})\Psi_b(TA, Ty) = (\mathrm{id}_{\mathcal{T}} \otimes T_b)\Psi_b(A, y)$$

induced transformation on $\mathcal{T}$

Can combine with probabilistic invariance

<u>Lemma</u> If $T\eta \overset{d}{=} \eta$, the $\ell^{BPHZ} \circ T_T = \ell^{BPHZ}$

<u>Thm</u> If F is T covariant and $T\eta \overset{d}{=} \eta$,
then all BPHZ counterms (tree by tree)
are T covariant.

Can combine with probabilistic invariance

<u>Lemma</u> If $T\xi \overset{d}{=} \xi$, the $\ell^{BPHZ} \circ T_{\mathcal{T}} = \ell^{BPHZ}$

<u>Thm</u> If F is T covariant and $T\xi \overset{d}{=} \xi$, then all BPHZ counterms (tree by tree) are T covariant.

<u>BPHZ Renorm of SYM</u>

$$(\partial_t - \Delta)A_i = \sum_{j=1}^{d} [A_j, 2\partial_j A_i - \partial_i A_j + [A_j, A_i]] + C A_i + \xi_i$$

for $1 \le i \le d$

where $C \in L(\mathfrak{g}, \mathfrak{g})$ and commutes with adjoint acton of Lie Group G

Thanks for
listening!