Tutorial on Infinite-dimensional hyperkaehler quotients, Nahm's equations and coadjoint orbits

A. Barbara Tumpach

Institut CNRS Pauli, Vienna, Austria,

and

Laboratoire Painlevé, Lille, France

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Definition

A **weak Kähler** manifold is a Banach manifold $\mathcal{M}$ endowed with a weak symplectic form ω and a weak Riemannian metric g, satisfying the following compatibility condition :

(C) the endomorphism I of the tangent bundle of $\mathcal{M}$ defined by $\mathrm{g}(IX, Y) = \omega(X, Y)$ satisfies $I^2 = -1$ and the Nijenhuis tensor N of I vanishes.

Recall that the Nijenhuis tensor has the following expression at x in $\mathcal{M}$:

$$N_x(X, Y) := [\tilde{X}, \tilde{Y}] + I[\tilde{X}, I\tilde{Y}] + I[I\tilde{X}, \tilde{Y}] - [I\tilde{X}, I\tilde{Y}],$$

where X and Y belong to $T_x\mathcal{M}$ and $\tilde{X}$ and $\tilde{Y}$ are arbitrary extensions.

Kähler and hyperkähler quotients

Theorem (Kähler quotient — finite-dimensional version)

Let $(\mathscr{M}, g, \omega)$ be a kähler manifold endowed with an hamiltonian action of a Lie group G (with momentum map μ) preserving the kähler structure. If the following conditions are satisfied:

- *ξ is a regular $Ad^*(G)$-invariant value of the momentum map ;*
- *G acts properly and freely on $\mu^{-1}(\xi)$,*

then the kähler structure on $\mathscr{M}$ induces a kähler structure on $\mu^{-1}(\xi)/G$.

Kähler and hyperkähler quotients

Theorem (T07- Kähler quotient — infinite-dimensional version)

Let $(\mathscr{M}, g, \omega)$ be a weak kähler Banach manifold endowed with an hamiltonian action of a Banach Lie group G (with momentum map μ) preserving the kähler structure. If the following conditions are satisfied:

- *ξ is a regular $Ad^*(G)$-invariant value of the momentum map ;*
- *G acts properly and freely on $\mu^{-1}(\xi)$;*
- *for all x in $\mu^{-1}(\xi)$, one has*

$$T_x G.x \oplus T_x G.x^{\perp_g} = T_x \mu^{-1}(\xi),$$

then the kähler structure of $\mathscr{M}$ induces a weak formally integrable kähler structure on $\mu^{-1}(\xi)/G$.

Definition

A **weak hyperkähler** manifold is a Banach manifold $\mathscr{M}$ endowed with a weak Riemannian metric g and 3 weak symplectic forms $\omega_1, \omega_2, \omega_3$, satisfying the following compatibility condition :

(CH) the endomorphisms I_a of the tangent bundle of $\mathscr{M}$ defined by $\mathrm{g}(I_a X, Y) = \omega_a(X, Y)$ satisfy ${I_a}^2 = -1$ and the Nijenhuis tensor N of I_a vanishes.

Recall that the Nijenhuis tensor has the following expression at x in $\mathscr{M}$:

$$N_x(X, Y) := [\tilde{X}, \tilde{Y}] + I_a[\tilde{X}, I_a\tilde{Y}] + I_a[I_a\tilde{X}, \tilde{Y}] - [I_a\tilde{X}, I_a\tilde{Y}],$$

where X and Y belong to $T_x\mathscr{M}$ and $\tilde{X}$ and $\tilde{Y}$ are arbitrary extensions.

Kähler and hyperkähler quotients

Theorem (HKLR87- Hyperkähler quotient — finite-dimensional version)

Let $(\mathscr{M}, g, \omega_1, \omega_2, \omega_3)$ be a hyperkähler manifold endowed with a tri-hamiltonian action of a Lie group G (with momentum map $\mu = (\mu_1, \mu_2, \mu_3)$) preserving the kähler structure. If the following conditions are satisfied:

- *$\xi = (\xi_1, \xi_2, \xi_3)$ is a regular $Ad^*(G)$-invariant value of the momentum map ;*
- *G acts properly and freely on $\mu^{-1}(\xi)$;*

then the hyperkähler structure of $\mathscr{M}$ induces a hyperkähler structure on $\mu^{-1}(\xi)/G$.

Kähler and hyperkähler quotients

Theorem (T07- Hyperkähler quotient — infinite-dimensional version)

Let $(\mathscr{M}, g, \omega_1, \omega_2, \omega_3)$ be a weak hyperkähler Banach manifold endowed with an hamiltonian action of a Banach Lie group G (with momentum map $\mu = (\mu_1, \mu_2, \mu_3)$) preserving the hyperkähler structure. If the following conditions are satisfied:

- *ξ is a regular $Ad^*(G)$-invariant value of the momentum map ;*
- *G acts properly and freely on $\mu^{-1}(\xi)$;*
- *for all x in $\mu^{-1}(\xi)$, there exists a closed linear subspace H_x of $T_x\mu^{-1}(\xi)$ such that*

$$T_x G.x \oplus I_1 T_x G.x \oplus I_2 T_x G.x \oplus I_3 T_x G.x \oplus H_x = T_x \mathscr{M},$$

then the hyperkähler structure of $\mathscr{M}$ induces a weak formally integrable hyperkähler structure on $\mu^{-1}(\xi)/G$.

An infinite dimensional example

Let $H = H_+ \oplus H_-$ be a polarized Hilbert space and define

$$\mathscr{M}_k = \left\{ \begin{array}{ll} x \in B(H_+, H), & pr_+ \circ x - k\mathrm{Id}_{H_+} \in L^1(H_+), \\ & pr_- \circ x \in L^2(H_+, H_-) \end{array} \right\}$$

and $G = U(H_+) \cap \{\mathrm{Id}_{H_+} + L^1(H_+)\}$.

Proposition (T07)

The natural action of G on $T\mathscr{M}$ is hamiltonian with momentum map $\mu = (\mu_1, \mu_2, \mu_3)$ where

$$\begin{array}{llll} \mu_1 : & T\mathscr{M}_k & \rightarrow & \mathfrak{g}' \\ & (x, X) & \mapsto & \left(\mathfrak{a} \mapsto \frac{i}{2}\ Tr\,(x^*x - X^*X)\mathfrak{a}\right) \\ \mu_2 : & T\mathscr{M}_k & \rightarrow & \mathfrak{g}' \\ & (x, X) & \mapsto & \left(\mathfrak{a} \mapsto \frac{1}{2}\ Tr\,(X^*x - x^*X)\mathfrak{a}\right) \\ \mu_3 : & T\mathscr{M}_k & \rightarrow & \mathfrak{g}' \\ & (x, X) & \mapsto & \left(\mathfrak{a} \mapsto -\frac{i}{2}\ Tr\,(x^*X + X^*x)\mathfrak{a}\right). \end{array}$$

Theorem (T07)

For all $k \in \mathbb{R}^*$, $\mathcal{Q}_k := \mu^{-1}\left(\left(\frac{i}{2}k^2\,\mathcal{T}r, 0, 0\right)\right)/G$ *is a smooth hyperkähler quotient.*

Theorem (T07)

For all $k \in \mathbb{R}^*$, $\mathcal{Q}_k$ *is diffeomorphic to the cotangent space* $T'Gr^0_{res}$ *via the map:*

$$\begin{array}{rccc} \Phi_1 : & \mathcal{Q}_k & \to & T'Gr^0_{res} \\ & [(x, X)] & \mapsto & (\mathcal{I}mx, \frac{1}{k^2}x \circ X^*) \end{array}$$

Theorem (T07)

For all $k \in \mathbb{R}^*$, $\mathcal{Q}_k$ *is diffeomorphic to the complexified orbit* $\mathscr{O}^{\mathbb{C}} = \{(P, Q) \in Gr^0_{res} \times Gr^{0*}_{res}, P \oplus_B Q = H\}$ *of* Gr^0_{res} *via the map:*

$$\begin{array}{rccc} \Phi_2 : & \mathcal{Q}_k & \to & \mathscr{O}^{\mathbb{C}} = GL_2(H)/(GL_2(H_+) \times GL_2(H_-)) \\ & [(x, X)] & \mapsto & z = i(x + X)(x^* - X^*) \end{array}$$

$$
\begin{aligned}
\mathscr{N}_k &:= \mu^{-1}\left(\left(\tfrac{i}{2}k^2\mathrm{Tr}, 0, 0\right)\right) = \text{level set contained in } T\mathscr{M}_k;\\
\mathscr{W}_k^{s_1} &:= G^{\mathbb{C}} \cdot_1 \mathscr{N}_k \text{ for the } I_1 - \text{holomorphic action of } G^{\mathbb{C}} \text{ on } T\mathscr{M}_k;\\
\mathscr{W}_k^{s_3} &:= G^{\mathbb{C}} \cdot_3 \mathscr{N}_k \text{ for the } I_3 - \text{holomorphic from } G^{\mathbb{C}} \text{ to } T\mathscr{M}_k.\\
\mathscr{O}^{\mathbb{C}} &:= \left\{(P, Q) \in Gr_{res}^0 \times Gr_{res}^{*0} \;\middle|\; P \cap Q = \{0\}\right\}
\end{aligned}
$$

Theorem

For any $k \in \mathbb{R}^*$, the quotient $\mathscr{Q}_k := \mathscr{N}_k / G$ is a hyperkähler smooth manifold and one has the following commutative diagram

$$
\begin{tikzcd}
 & \mathscr{W}_k^{s_1} \arrow[r, two heads, "q_1"] \arrow[dl, "p_1"', "I_1\text{-holo.}"] \arrow[d] & \mu^{-1}\left(\left(\tfrac{i}{2}k^2\mathrm{Tr}, 0, 0\right)\right) \arrow[d] & \mathscr{W}_k^{s_3} \arrow[l, two heads, "q_3"'] \arrow[d] \arrow[dr, "p_3", "I_3\text{-holo.}"'] & \\
T'Gr_{res}^0 \arrow[r, phantom, "\simeq"] & \mathscr{W}_k^{s_1}/G^{\mathbb{C}} \arrow[r, "\sim", "I_1\text{-holo.}"'] & \mathscr{Q}_k & \mathscr{W}_k^{s_3}/G^{\mathbb{C}} \arrow[l, "\sim"', "I_3\text{-holo.}"] \arrow[r, phantom, "\simeq"] & \mathscr{O}^{\mathbb{C}}
\end{tikzcd}
$$

Hyperkähler structure on the cotangent bundle of a complex Lie group

Kähler structure on T^*G

For G a **compact Lie group** with Lie algebra $\mathfrak{g}$, the **cotangent bundle**

$$T^*G \simeq G \times \mathfrak{g} \simeq G^{\mathbb{C}}$$

is a **Kähler manifold** where the symplectic form is the canonical symplectic form of the cotangent space, and the complex structure is the complex structure of the complexification $G^{\mathbb{C}}$ of G with the identification $T^*G \ni (x, \xi) \mapsto x \exp(i\xi) \in G^{\mathbb{C}}$.

Ex: $G = S^1$, $T^*S^1 = \mathbb{C}^*$
$G = SU(2)$, $T^*SU(2) = SL(2, \mathbb{C})$.

Hyperkähler structure on the cotangent bundle of a complex Lie group

hyperkähler structure on $T^*G^{\mathbb{C}}$

For G a compact Lie group with Lie algebra $\mathfrak{g}$, the **cotangent bundle** $T^*G^{\mathbb{C}}$ of the corresponding **complex group** carries a **hyperkähler structure** which is invariant under left- and right-translations by elements of G.

$$T^*G^{\mathbb{C}} \simeq G \times \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \simeq G^{\mathbb{C}} \times \mathfrak{g} \times \mathfrak{g}$$

is a hyperkähler manifold where two real symplectic forms are given by the real and imaginary parts of the canonical complex symplectic form of the complex cotangent space, and one complex structure is induced by the natural complex structure of the complexification $G^{\mathbb{C}}$ of G.

Ex: $G = S^1$, $T^*G^{\mathbb{C}} = S^1 \times \mathbb{R}^3$

Nahm's equations

Definition

Nahm's equations are the following system of equations on four Lie-algebra-valued functions of one variable, $A_i : \mathbb{R} \mapsto \mathfrak{g}$, i $= 0, 1, 2, 3$:

$$\frac{dA_1}{ds} + [A_0, A_1] + [A_2, A_3] = 0 \tag{1}$$
$$\frac{dA_2}{ds} + [A_0, A_3] + [A_3, A_1] = 0 \tag{2}$$
$$\frac{dA_3}{ds} + [A_0, A_4] + [A_1, A_2] = 0. \tag{3}$$

The set of $\mathscr{C}^1$-solutions will be denoted by $\mathscr{N}$.

Nahm's equations

Remark

The gauge group $\mathscr{G}$ *of* $\mathscr{C}^2$ *maps* $g : [0,1] \to G$ *acts on the space of solutions* $\mathscr{N}$ *by*

$$A_0 \mapsto gA_0g^{-1} + g\frac{dg^{-1}}{ds} \tag{4}$$

$$A_i \mapsto gA_ig^{-1}, i = 1, 2, 3. \tag{5}$$

Remark

Nahm's equation correspond to $SU(2)$ *invariant selfdual connexions on a trivial principal* G*-bundle over* $\mathbb{R}^4 \setminus \{0\}$

Nahm's equations

Theorem (K88)

Consider the normal subgroup $\mathscr{G}_0$ of $\mathscr{G}$ consisting of paths g with $g(0) = g(1) = 1$. Then

- *$\mathscr{N}/\mathscr{G}_0$ is a hyperkähler manifold.*
- *For one of the complex structures, the underlying holomorphic symplectic manifold is $T^*G^{\mathbb{C}}$.*

Remark

Replacing the closed interval $[0, 1]$ by $[0, \infty)$ and adding some boundary conditions (and a lot of analysis) allows to construct hyperkähler structures on complex coadjoint orbits.