Graded Manifolds: Some Issues

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$\mathbb{Z}$-graded manifolds: overview

$\mathbb{Z}$-graded (or simply graded) manifolds, $\mathbb{Z}$-graded supermanifolds:

Suitable mathematical theory of "generalized manifolds", where functions locally depend on $\mathbb{Z}$-**graded variables** (which do commute accordingly).

There are two basic approaches to this problem:

1. Consider (smooth) $\mathbb{Z}_2$-supermanifolds with an additional $\mathbb{Z}$-grading. Kontsevich (1997), Ševera (2001), Roytenberg, Voronov (2002).
2. Define $\mathbb{Z}$-graded (henceforth just graded) manifolds without an underlying supermanifold. Mehta (2006), Cattaneo & Schätz (2010), **Fairon (2017)**.

Optimal scenario

Mimic Berezin-Leites approach to supermanifolds (1983) with "purely graded" coordinates of all degrees (not just N-manifolds).

Graded algebra: usual approach

Definition (graded "object")

- An object (vector space, ring, algebra) V is **graded**, if it can be written as as a direct sum $V = \bigoplus_{k \in \mathbb{Z}} V_k$ of its subspaces $V_k \subseteq V$. Elements of $V^h := \cup_{k \in \mathbb{Z}} V_k - \{0\}$ have defined degree $|v| \in \mathbb{Z}$ and are called **homogeneous elements**.
- Algebraic structures (e.g. ring multiplication) satisfy additional grading requirements (e.g. $|v \cdot w| = |v| + |w|$).
- Object morphisms $\varphi : V \to W$ have to preserve the grading, that is $|\varphi(v)| = |v|$ (or shift it by a given number, when required).

Example

$V = \bigoplus_{k \in \mathbb{Z}} V_k$ a graded vector space. **Symmetric algebra** $S(V)$ is

$$S(V) = T(V)/J, \quad J = \langle \{ v \otimes w - (-1)^{|w||v|} w \otimes v \mid v, w \in V^h \} \rangle.$$

It is a graded commutative associative algebra.

Graded manifolds: usual approach

Definition (graded manifold)

- Let M be a second countable Hausdorff space.
- We have a **structure sheaf** of graded commutative associative algebras $\mathcal{C}^{\infty}_{\mathcal{M}}$ on M.
- $\mathcal{M} = (M, \mathcal{C}^{\infty}_{\mathcal{M}})$ is a **locally ringed space**, that is stalks $\mathcal{C}^{\infty}_{\mathcal{M},m}$ are local rings for every $m \in M$.
- There is a finite-dimensional graded vector space V, such that $V_0 = \{0\}$ and $\mathcal{C}^{\infty}_{\mathcal{M}}$ is **locally isomorphic** to the sheaf

$$U \subseteq \mathbb{R}^n \mapsto \mathcal{C}^{\infty}_{n,V}(U) := C^{\infty}(U) \otimes_{\mathbb{R}} S(V),$$

Remark (they are just polynomials!)

If $(\xi_\mu)_{\mu=1}^{\dim(V)}$ is a homogeneous basis of V, then elements of $\mathcal{C}^{\infty}_{n,V}(U)$ are polynomials in $\{\xi_\mu\}_{\mu=1}^{\dim(V)}$ with coefficients in $C^{\infty}(U)$, and

$$\xi_\mu \xi_\nu = (-1)^{|\xi_\mu||\xi_\nu|} \xi_\nu \xi_\mu.$$

Issue number one

I. $\mathcal{C}^\infty_{n,V}$ is *not* a sheaf.

- Consider $V = V_{-2} \oplus V_2$, where $V_{-2} = V_2 = \mathbb{R}$.
- Let $n = 1$ and consider an open subset $U \subseteq \mathbb{R}$ in the form

$$U := \cup_{m \in \mathbb{N}} U_m, \quad U_m := (m-1, m).$$

  $\{U_m\}_{m \in \mathbb{N}}$ is an open cover of U.
- V has a basis (ξ_1, ξ_2) with $|\xi_1| = -2$ and $|\xi_2| = 2$. Define local sections $f_m \in \mathcal{C}^\infty_{1,V}(U_m)$ as

$$f_m := (\xi_1)^m (\xi_2)^m.$$

- $\{f_m\}_{m \in \mathbb{Z}}$ *cannot be glued* to $f \in \mathcal{C}^\infty_{1,V}(U)$, hence $\mathcal{C}^\infty_{1,V}$ is not a sheaf.

The same problem arises even for non-negatively graded V, if we do not restrict the degree of glued local sections.

Easy solutions? Issue number two

- Can we simply relax the gluing axiom to finite open covers? Not a good idea! A lot of constructions require infinite open covers (e.g. those using partition of unity).
- Can one somehow employ the "sheafification mantra"? However, sheafification does preserve stalks, hence it does not cure the following:

II. Stalks of $\mathcal{C}^{\infty}_{n,V}$ are *not* local rings.

- For each $m \in M$, $(\mathcal{C}^{\infty}_{n,V})_m$ is a local ring, iff its non-invertible elements form an ideal.
- With $n = 1$ and V as above, consider sections $s_{\pm} := 1 \pm \xi_1 \xi_2$ in $\mathcal{C}^{\infty}_{1,V}(\mathbb{R})$. Their germs $[s_{\pm}]_m$ are not invertible.
- Their sum $[s_+]_m + [s_-]_m = [2]_m$ is invertible. Whence $(\mathcal{C}^{\infty}_{1,V})_m$ is not a local ring.

Morphisms of "non-locally" ringed spaces are not so easy to work with.

Issue number three

III. One cannot calculate pullbacks.

- Let us again consider the example above. We have coordinates (x, ξ_1, ξ_2), where $|x| = 0$, $|\xi_1| = -2$ and $|\xi_2| = 2$.
- A (graded) smooth map should be (locally) determined by pullback of coordinate functions. Define e.g.

$$\varphi^*(x) = x + \xi_1 \xi_2, \;\; \varphi^*(\xi_{1,2}) = \xi_{1,2}.$$

- What is a pullback of $f = f(x)$? The formula (e.g. in supergeometry) would be (for above φ):

$$\varphi^*(f) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\mathrm{d}^m f}{\mathrm{d}x^m} (\xi_1)^m (\xi_2)^m.$$

But this is does not converge (in any sense).

Almost a solution, issue number four

Definition (Graded manifold by Fairon (2017))

Replace $\mathcal{C}^{\infty}_{n,V}$ with a sheaf

$$U \subseteq \mathbb{R}^n \mapsto \bar{\mathcal{C}}^{\infty}_{n,V}(U) := C^{\infty}(U) \otimes_{\mathbb{R}} \bar{S}(V),$$

where $\bar{S}(V) = \prod_{p=0}^{\infty} S^p(V)$ (formal power series in variables $\{\xi_\mu\}_{\mu=1}^{\dim(V)}$).

There is a minor oversight - $C^{\infty}(U) \otimes_{\mathbb{R}} \bar{S}(V)$ is not a space of formal power series in $\{\xi_\mu\}_{\mu=1}^{\dim(V)}$ with coefficients in $C^{\infty}(U)$, it should be

$$\bar{\mathcal{C}}^{\infty}_{n,V}(U) = \prod_{p=0}^{\infty} C^{\infty}(U) \otimes_{\mathbb{R}} S^p(V).$$

IV: $\bar{\mathcal{C}}^{\infty}_{n,V}(U)$ is not a graded vector space.

It cannot be written as a direct sum of its "elements of degree k" subspaces, that is $\bar{\mathcal{C}}^{\infty}_{n,V}(U) \neq \bigoplus_{k \in \mathbb{Z}} (\bar{\mathcal{C}}^{\infty}_{n,V}(U))_k$.

Graded algebra revisited

There are other inconveniences with the graded algebra:

- For $V = \bigoplus_{k \in \mathbb{Z}} V_k$ and $W = \bigoplus_{k \in \mathbb{Z}} W_k$, $\mathrm{Lin}(V, W)$ is not a graded vector space, instead $\mathrm{Lin}(V, W) = \prod_{k \in \mathbb{Z}} \mathrm{Lin}_k(V, W)$.
- In particular, $V^* = \mathrm{Lin}(V, \mathbb{R})$ is not a graded vector space.

Definition (graded "object" again)

- A **graded object** V is a sequence $V = \{V_k\}_{k \in \mathbb{Z}}$. We write $v \in V$, if $v \in V_k$ for some $k \in \mathbb{Z}$, and write $|v| := k$. There is no space for inhomogeneous elements!
- A graded morphism $\varphi : V \to W$ is a collection $\varphi = \{\varphi_k\}_{k \in \mathbb{Z}}$ where $\varphi_k : V_k \to W_k$ for all $k \in \mathbb{Z}$. We write $\varphi(v)$ for $\varphi_{|v|}(v)$.
- Algebraic structures are introduced naturally.

Example (category gVect)

$V = \{V_k\}_{k \in \mathbb{Z}}$ is **graded (real) vector space**, iff V_k is a vector space for each $k \in \mathbb{Z}$. φ is **graded linear**, iff $\varphi_k : V_k \to W_k$ are linear.

Graded commutative associative algebras

Many constructions are "more natural" from a categorical viewpoint.

- **gVect** is a symmetric monoidal category where

$$(V \otimes_{\mathbb{R}} W)_k := \bigoplus_{j \in \mathbb{Z}} V_j \otimes_{\mathbb{R}} W_{k-j},$$

  where the unit object is $\mathbb{R}$ (with the trivial grading), and $\tau_{VW}(v \otimes w) = (-1)^{|v||w|} w \otimes v$ is the braiding.
- $\mathrm{Lin}(V, W) = \{\mathrm{Lin}_k(V, W)\}_{k \in \mathbb{Z}}$ where

$$\mathrm{Lin}_k(V, W) = \{\varphi = \{\varphi_j\}_{j \in \mathbb{Z}} \mid \varphi_j : V_j \to W_{j+k}\},$$

  is the internal hom in **gVect**. $V^* = \mathrm{Lin}(V, \mathbb{R})$ is a dual object.

Definition (category gcAs)

Graded (commutative) associative algebras are (commutative) monoids in **gVect**, that is (A, μ, η) with $\mu : A \otimes_{\mathbb{R}} A \to A$ and $\eta : \mathbb{R} \to A$.

- In plain English, we write $v \cdot w := \mu(v \otimes w)$ and $1_A := \eta(1)$, finding

$$v \cdot (w \cdot x) = (v \cdot w) \cdot x, \;\; v \cdot 1_A = 1_A \cdot v = v, \;\; \text{distributivity.}$$

  Graded commutativity reads $v \cdot w = (-1)^{|v||w|} w \cdot v$.
- An ideal is a graded abelian subgroup $I \subseteq A$ with $\mu(A \otimes_{\mathbb{R}} I) \subseteq I$.

Definition (local graded commutative ring)

For any $A \in$ **gcAs**, we get a graded ring multiplication $\bar{\mu} : A \otimes_{\mathbb{Z}} A \to A$.
$(A, \bar{\mu}, \eta)$ is a graded commutative ring.
It is a **local graded commutative ring**, if it has a unique maximal ideal.
Local graded ring morphisms preserve those ideals.

- This has many equivalent definitions. Let $\mathfrak{U}(A) = \{\mathfrak{U}_k(A)\}_{k \in \mathbb{Z}}$

$$\mathfrak{U}_k(A) = \{v \in A_k \mid (\exists w \in A_{-k})(v \cdot w = w \cdot v = 1_A)\}.$$

  Then A is local, iff $A - \mathfrak{U}(A)$ is an ideal.
- In this case $\mathfrak{J}(A) := A - \mathfrak{U}(A)$ is the unique maximal ideal, called the **Jacobson radical of A**.
- The ring $A = \oplus_{k \in \mathbb{Z}} A_k$ is not necessarily local!

Sheaves of graded commutative algebras

Let X be a topological space. $\mathbf{Op}(X)$ a category of open subsets.

Definition

- A **presheaf of graded commutative algebras on** X is a functor
$$\mathcal{F} : \mathbf{Op}(X)^{\text{op}} \to \mathbf{gcAs}\,.$$
For each $k \in \mathbb{Z}$, we have a presheaf $\mathcal{F}_k(U) := (\mathcal{F}(U))_k$. Together with natural transformations, we have a category $\mathbf{PSh}(X, \mathbf{gcAs})$.
- $\mathcal{F} \in \mathbf{PSh}(X, \mathbf{gcAs})$ is a **sheaf**, iff $\mathcal{F}_k$ is a sheaf of vector spaces for every $k \in \mathbb{Z}$. We have a full subcategory $\mathbf{Sh}(X, \mathbf{gcAs})$.

- The definition of a sheaf coincides with the "category friendly" definition using products and equalizers.

Definition

For each $x \in X$, a **stalk** $\mathcal{F}_x$ **of a presheaf** $\mathcal{F}$ **at** x is defined as a filtered colimit of $\mathcal{F}$ over the opposite to $\mathbf{Op}_x(X) = \{U \in \mathbf{Op}(X) \mid U \ni x\}$.

Graded locally ringed spaces

Definition

A **graded locally ringed space** $(X, \mathcal{O}_X)$ is a pair, where

- X is a topological space.
- $\mathcal{O}_X \in \mathbf{Sh}(X, \mathbf{gcAs})$ is a sheaf of graded commutative algebras.
- Stalks of $\mathcal{O}_X$ are local graded rings.

A **morphism of graded locally ringed spaces** $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is a pair $\varphi = (\underline{\varphi}, \varphi^*)$, where $\underline{\varphi} : X \to Y$ is continuous and

1. $\varphi^* : \mathcal{O}_Y \to \underline{\varphi}_*(\mathcal{O}_X)$ is a sheaf morphism.
2. For each $x \in X$, the map $[s]_{\underline{\varphi}(x)} \mapsto [\varphi^*_U(s)]_x$ is a morphism of local graded rings, $s \in \mathcal{O}_Y(U)$ any $U \in \mathbf{Op}_{\underline{\varphi}(x)}(Y)$.

We obtain a category **gLRS**.

Remark

If φ is an isomorphism, the condition (2) is satisfied automatically.

Graded domain

- Let $(n_j)_{j\in\mathbb{Z}}$ be a sequence $n_j \in \mathbb{N}_0$, such that $n := \sum_{j\in\mathbb{Z}} n_j < \infty$.
- Construct a graded vector space denoted as $\mathbb{R}_*^{(n_j)}$ by setting

$$(\mathbb{R}_*^{(n_j)})_0 := \{0\}, \ \ (\mathbb{R}_*^{(n_j)})_k := \mathbb{R}^{n_k} \text{ for } k \neq 0.$$

Example (graded domain $\mathbb{R}^{(n_j)}$)

By a **graded domain**, we mean $\mathbb{R}^{(n_j)} := (\mathbb{R}^{n_0}, \mathcal{C}^\infty_{(n_j)}) \in$ **gLRS**, where

$$U \subseteq \mathbb{R}^{n_0} \mapsto \mathcal{C}^\infty_{(n_j)}(U) := \prod_{p=0}^{\infty} C^\infty(U) \otimes_{\mathbb{R}} S^p(\mathbb{R}_*^{(n_j)}).$$

- Let $n_* := n - n_0$. Fix the standard "total basis" $(\xi_\mu)_{\mu=1}^{n_*}$ of $\mathbb{R}_*^{(n_j)}$.
- We define a subset $\mathbb{N}_k^{n_*} \subseteq (\mathbb{N}_0)^{n_*}$. Let $\mathbf{p} := (p_1, \dots, p_{n_*})$. Then

$$\mathbb{N}_k^{n_*} := \{\mathbf{p} \in (\mathbb{N}_0)^{n_*} \mid \sum_{\mu=1}^{n_*} p_\mu |\xi_\mu| = k, \ p_\mu \in \{0,1\} \text{ if } |\xi_\mu| \text{ odd}\}.$$

Graded domain: properties

- Every $f \in (\mathcal{C}^{\infty}_{(n_j)}(U))_k$ can be written as a formal power series

$$f = \sum_{\mathbf{p} \in \mathbb{N}^{n_*}_k} f_{\mathbf{p}} \xi^{\mathbf{p}},$$

for the unique sequence $\{f_{\mathbf{p}}\}_{\mathbf{p}} \subseteq C^{\infty}(U)$, $\xi^{\mathbf{p}} = (\xi_1)^{p_1} \dots (\xi_{n_*})^{p_{n_*}}$.

- Multiplication of f with $g = \sum_{\mathbf{p} \in \mathbb{N}^{n_*}_{\ell}} g_{\mathbf{p}} \xi^{\mathbf{p}}$ is the expected one:

$$f \cdot g = \sum_{\mathbf{p} \in \mathbb{N}^{n_*}_{k+\ell}} (f \cdot g)_{\mathbf{p}} \xi^{\mathbf{p}}, \;\; (f \cdot g)_{\mathbf{p}} := \sum_{\mathbf{q} \leq \mathbf{p}} \epsilon_{\mathbf{q},\mathbf{p}-\mathbf{q}} f_{\mathbf{q}} g_{\mathbf{p}-\mathbf{q}},$$

where $\xi^{\mathbf{p}} =: \epsilon_{\mathbf{q},\mathbf{p}-\mathbf{q}} \xi^{\mathbf{q}} \cdot \xi^{\mathbf{p}-\mathbf{q}}$ obtained by $\xi_{\mu}\xi_{\nu} = (-1)^{|\xi_{\mu}||\xi_{\nu}|} \xi_{\nu}\xi_{\mu}$.

- For $V \subseteq U$, the restriction is obviously $f|_V := \sum_{\mathbf{p} \in \mathbb{N}^{n_*}_k} f_{\mathbf{p}}|_V \xi^{\mathbf{p}}$. It is now easy to see that $\mathcal{C}^{\infty}_{(n_j)} \in \mathbf{Sh}(\mathbb{R}^{n_0}, \mathbf{gcAs})$.
- $[f]_x = [g]_x$, iff $\exists W \in \mathbf{Op}_x(X)$, such that $f_{\mathbf{p}}|_W = g_{\mathbf{p}}|_W$, $\forall \mathbf{p} \in \mathbb{N}^{n_*}_k$.

Proposition ($\mathbb{R}^{(n_j)} \in$ **gLRS**)

For each $x \in \mathbb{R}^{n_0}$, the stalk $\mathcal{C}^{\infty}_{(n_j),x}$ is a local graded ring.

- For $k \neq 0$, one has $\mathfrak{U}_k(\mathcal{C}^{\infty}_{(n_j),x}) = \{0\}$.
- $\mathfrak{U}_0(\mathcal{C}^{\infty}_{(n_j),x}) = \{[f]_x \mid f_{\mathbf{0}}(x) \neq 0\}$, $\mathbf{0} = (0, \dots, 0) \in \mathbb{N}_0^{n_*}$.
  1. One may assume that $f \in (\mathcal{C}^{\infty}_{(n_j)}(U))_0$ and $f_{\mathbf{0}}(y) \neq 0$, $\forall y \in U$.
  2. Then write $f = f_{\mathbf{0}}(1 + f')$ and define $g \in (\mathcal{C}^{\infty}_{(n_j)}(U))_0$ by
  $$g := \frac{1}{f_{\mathbf{0}}} \sum_{q=0}^{\infty} (-1)^q f'^q.$$
  3. g is well-defined and $[f]_x \cdot [g]_x = 1$.
- Clearly $\mathfrak{J}(\mathcal{C}^{\infty}_{(n_j),x}) := \mathcal{C}^{\infty}_{(n_j),x} - \mathfrak{U}(\mathcal{C}^{\infty}_{(n_j),x})$ is an ideal, Q.E.D.
- In fact, one has a direct sum decomposition of graded vector spaces
$$\mathcal{C}^{\infty}_{(n_j),x} = \mathbb{R} \oplus \mathfrak{J}(\mathcal{C}^{\infty}_{(n_j),x}).$$

For any $U \subseteq \mathbb{R}^{n_0}$, we have $U^{(n_j)} := (U, \mathcal{C}^{\infty}_{(n_j)}|_U) \in$ **gLRS** called a **graded domain over** U.

Morphisms of graded domains

Theorem (the most important one)

The following data are equivalent:

- *A morphism* $\varphi \equiv (\underline{\varphi}, \varphi^*) : U^{(n_j)} \to V^{(m_j)}$ *of graded domains.*
- *A* **smooth** *map* $\underline{\varphi} : U \to V$ *together with*
  1. *A collection* $\{\theta^*_\nu\}_{\mu=1}^{m_*}$ *where* $\theta^*_\nu \in \mathcal{C}^\infty_{(n_j)}(U)_{|\theta_\nu|}$ *and* $(\theta_\nu)_{\nu=1}^{m_*}$ *denotes the standard total basis for* $\mathbb{R}^{(m_j)}_*$.
  2. *A collection* $\{f^j_*\}_{j=1}^{m_0}$ *where* $f^j_* \in \mathcal{C}^\infty_{(n_j)}(U)_0$ *and* $(f^j_*)_{\mathbf{0}} = 0$.

- They are pullbacks of $y^j \in \mathcal{C}^\infty_{(m_j)}(V)_0$ and $\theta_\nu \in \mathcal{C}^\infty_{(m_j)}(V)_{|\theta_\nu|}$:

$$\varphi^*_V(\theta_\nu) =: \theta^*_\nu, \;\; \varphi^*_V(y^j) =: y^j \circ \underline{\varphi} + f^j_*.$$

- A pullback of general $f = \sum_{\mathbf{p}} f_{\mathbf{p}} \theta^{\mathbf{p}} \in \mathcal{C}^\infty_{(m_j)}(V)$ given by a formula

$$\varphi^*_V(f) = \sum_{\mathbf{p}} \varphi^*_V(f_{\mathbf{p}}) \theta^{\mathbf{p}}_*, \;\; \varphi^*_V(f_{\mathbf{p}}) := \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{\mathbf{j}, |\mathbf{j}|=r} \Big(\frac{\partial f_{\mathbf{p}}}{\partial y^{\mathbf{j}}} \circ \underline{\varphi}\Big) f^{\mathbf{j}}_*.$$

Graded manifolds: definition

Definition (graded chart)

Let M be a second countable Hausdorff. Let $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}}) \in \mathbf{gLRS}$. A pair (U, φ) is called a **graded chart for** M, if

- $U \in \mathbf{Op}(M)$.
- $\varphi : \mathcal{M}|_U \to \hat{U}^{(n_j)}$ is an isomorphism (in **gLRS**) for some sequence $(n_j)_{j \in \mathbb{Z}}$ with $\sum_{j \in \mathbb{Z}} n_j < \infty$ and $\hat{U} \in \mathbf{Op}(\mathbb{R}^{n_0})$.

Definition (graded manifold)

Let M be second countable Hausdorff. Let $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}}) \in \mathbf{gLRS}$. We say that $\mathcal{M}$ is a **graded manifold**, if one can find a collection $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ of graded charts, such that $M = \cup_{\alpha \in I} U_\alpha$ and there is a common sequence $(n_j)_{j \in \mathbb{Z}}$ such that $\varphi_\alpha : \mathcal{M}|_{U_\alpha} \to \hat{U}_\alpha^{(n_j)}$.

$\mathcal{A}$ is called a **graded smooth atlas for** $\mathcal{M}$ and $(n_j)_{j \in \mathbb{Z}}$ is called a **graded dimension** of $\mathcal{M}$. We usually write $\mathcal{C}^\infty_{\mathcal{M}}$ instead of $\mathcal{O}_{\mathcal{M}}$.

Graded manifolds: basic properties

- Graded manifolds form a full subcategory $\mathbf{gMan}^\infty$ of **gLRS**. A **gLRS** morphism $\varphi : \mathcal{M} \to \mathcal{N}$ is called a **graded smooth map**.
- It follows that $\mathcal{A}_0 := \{(U_\alpha, \underline{\varphi_\alpha})\}_{\alpha \in I}$ is a smooth atlas on M, making it into an ordinary smooth n_0-dimensional manifold. Write $M \equiv \underline{\mathcal{M}}$.
- Viewing M as a graded manifold $M = (M, \mathcal{C}^\infty_M)$, there is a canonical graded smooth map $i_M : M \to \mathcal{M}$. For each each $f \in \mathcal{C}^\infty_{\mathcal{M}}(U)$, we define the **body of the function** f as

$$\underline{f} := (i_M)^*_U(f) \in \mathcal{C}^\infty_M(U).$$

- In general, there is no canonical projection $\pi_M : \mathcal{M} \to M$. This is not the case for non-negatively graded manifolds (N-manifolds).
- In fact, for N-manifolds, $(S^p(\mathbb{R}^{(n_j)}))_k = \{0\}$ for $p > k$ and so

$$\mathcal{C}^\infty_{(n_j)}(U) = \bigoplus_{p=0}^{\infty} C^\infty(U) \otimes_{\mathbb{R}} S^p(V) = C^\infty(U) \otimes_{\mathbb{R}} S(V),$$

and formal power series reduce to polynomials.

Graded manifolds: the example

Definition (graded vector bundle)

Let $\mathcal{M} = (M, \mathcal{C}^\infty_\mathcal{M})$ be a graded manifold. A sheaf $\Gamma(\mathcal{E}) \in \mathbf{Sh}(M, \mathbf{gVect})$ is called a **graded vector bundle over** $\mathcal{M}$, if

- It is a sheaf of $\mathcal{C}^\infty_\mathcal{M}$-modules, that is $\Gamma_U(\mathcal{E})$ is a $\mathcal{C}^\infty_\mathcal{M}(U)$-module and restrictions are compatible with the module structures.
- As a sheaf of $\mathcal{C}^\infty_\mathcal{M}$-modules, it is locally isomorphic to the sheaf $U \in \mathbf{Op}(M) \mapsto \mathcal{C}^\infty_\mathcal{M}(U) \otimes_\mathbb{R} \mathbb{R}^{(m_j)}$ for some sequence $(m_j)_{j \in \mathbb{Z}}$, $\sum_{j \in \mathbb{Z}} m_j < \infty$. $(m_j)_{j \in \mathbb{Z}}$ is called the **graded rank** of $\Gamma(\mathcal{E})$.

Example (prototypical example of $\mathbf{gMan}^\infty$)

For a given graded vector bundle $\Gamma(\mathcal{E})$ over $\mathcal{M}$, there is a canonical (up to an isomorphism) vector bundle $q : E \to M$ and a graded manifold $\mathcal{E} = (E, \mathcal{C}^\infty_\mathcal{E})$ together with a graded smooth map $\pi : \mathcal{E} \to \mathcal{M}$.

The most usual case $M = (M, \mathcal{C}^\infty_M)$ an ordinary manifold together with $\Gamma(\mathcal{E}) := \Gamma(F)[k]$ for an ordinary vector bundle $r : F \to M$. Then $\mathcal{E} \equiv F[k] = (M, \mathcal{C}^\infty_{F[k]})$ is called the **degree shifted vector bundle**.

Relation to supermanifolds

- Let $\mathcal{S} = (M, \mathcal{C}^\infty_\mathcal{S})$ be a supermanifold, that is $\mathcal{C}^\infty_\mathcal{S}$ is a sheaf of superalgebras locally isomorphic to a superdomain sheaf

$$U \subseteq \mathbb{R}^p \mapsto \mathcal{C}^\infty_{p|q}(U) = C^\infty(U) \otimes_\mathbb{R} \Lambda(\mathbb{R}^q).$$

- By Batchelor's theorem, there is a superdiffeomorphism $\mathcal{S} \cong \Pi A$ for some vector bundle $q : A \to M$, that is $\mathcal{C}^\infty_\mathcal{S} \cong \mathcal{C}^\infty_{\Pi A} = \Omega(A^*)$. There is an obvious $\mathbb{Z}$-grading on $\Omega(A^*)$, and we obtain a graded manifold $A[1]$. It is highly **non-canonical** though!
- Conversely, let $\mathcal{M} = (M, \mathcal{C}^\infty_\mathcal{M})$ be a graded manifold. There is the **even part submanifold** $\mathcal{M}_0 = (M, \mathcal{C}^\infty_{\mathcal{M}_0})$ of $\mathcal{M}$. Suppose that it is *non-negatively graded*.
  One can then construct a canonical (up to a superdiffeomorphism) supermanifold $\mathcal{S} = (S, \mathcal{C}^\infty_\mathcal{S})$, together with a smooth surjective submersion $\pi : S \to M$.
- Both S and $\mathcal{C}^\infty_\mathcal{S}$ are glued together by transition maps of $\mathcal{M}$, there is no direct functor.
- There is a kind of Batchelor's theorem for N-manifolds.

Vector fields

Definition (sheaf of vector fields)

Let $\mathcal{M} = (M, \mathcal{C}^\infty_{\mathcal{M}})$ be a graded manifold. $\forall k \in \mathbb{Z}$ and $U \in \mathbf{Op}(M)$ let

$$\mathfrak{X}_{\mathcal{M}}(U)_k := \mathrm{Der}_k(\mathcal{C}^\infty_{\mathcal{M}}(U)).$$

This defines $\mathfrak{X}_{\mathcal{M}} \in \mathbf{Sh}(X, \mathbf{gVect})$. It is a sheaf of $\mathcal{C}^\infty_{\mathcal{M}}$-modules. Its sections are called **vector fields on** $\mathcal{M}$.

- It is locally isomorphic to $\mathcal{C}^\infty_{\mathcal{M}} \otimes_{\mathbb{R}} \mathbb{R}^{(n_{-j})}$, where $(n_j)_{j\in\mathbb{Z}}$ is the graded dimension of $\mathcal{M}$. Local generators are usual $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial \xi_\mu}\}$.
- Hence $\mathfrak{X}_{\mathcal{M}}$ is a graded vector bundle and we can construct a tangent bundle $\tau_{\mathcal{M}} : T\mathcal{M} \to \mathcal{M}$, a graded manifold over $\tau_M : TM \to M$ of graded dimension $(n_j + n_{-j})_{j\in\mathbb{Z}}$.
- For each $m \in M$, there is a **tangent space at** m:

$$(T_m\mathcal{M})_k := \mathrm{Der}_k(\mathcal{C}^\infty_{\mathcal{M},m}, \mathbb{R}).$$

  It has a graded dimension $(n_{-j})_{j\in\mathbb{Z}}$ and there is a canonical (surjective) graded linear map $X \in \mathfrak{X}_{\mathcal{M}}(U) \mapsto X|_m \in T_m\mathcal{M}$.

Differential forms

Let $\mathcal{M} = (M, \mathcal{C}^\infty_\mathcal{M})$ be a graded manifold. Let $s \in \mathbb{Z}$ be an **even** integer, such that $|\xi_\mu| + s > 0$ for all $\mu \in \{1, \dots, n_*\}$.

Definition (differential forms on $\mathcal{M}$)

A **sheaf of differential forms** $\Omega_\mathcal{M}$ is defined as

$$\Omega_\mathcal{M}(U) := \mathcal{C}^\infty_{T[s+1]\mathcal{M}}(U),$$

where $T[s+1]\mathcal{M} = (M, \mathcal{C}^\infty_{T[s+1]\mathcal{M}})$ is a graded manifold obtained from the sheaf $\mathfrak{X}_\mathcal{M}(M)[s+1]$.

- If $\{x^i, \xi_\mu\}$ are local coordinates on $\mathcal{M}$, we have additional local coordinates $\{\mathrm{d}x^i, \mathrm{d}\xi_\mu\}$ on $T[s+1]\mathcal{M}$ with

$$|\mathrm{d}x^i| = s + 1, \;\; |\mathrm{d}\xi_\mu| = |\xi_\mu| + s + 1.$$

- For each $p \in \mathbb{N}_0$, one has a subsheaf $\Omega^p_\mathcal{M}$ of p-**forms**, which are locally sums of monomials of degree p in $\{\mathrm{d}x^i, \mathrm{d}\xi_\mu\}$.

de Rham cohomology

- For $\omega \in \Omega^p_{\mathcal{M}}(U)$, it is convenient to introduce an alternative grading:

$$\deg(\omega) = |\omega| - p(s+1). \;\; \omega \in \Omega^p_{\mathcal{M}}(U)_{(\deg(\omega))}.$$

- Form operations are introduced as vector fields on $\Omega_{\mathcal{M}} = \mathcal{C}^\infty_{T[s+1]\mathcal{M}}$:
  1. **exterior derivative** $\mathrm{d} : \Omega^p_{\mathcal{M}}(U)_{(k)} \to \Omega^{p+1}_{\mathcal{M}}(U)_{(k)}$;
  2. **interior product** $i_X : \Omega^p_{\mathcal{M}}(U)_{(k)} \to \Omega^{p-1}_{\mathcal{M}}(U)_{(k+|X|)}$;
  3. **Lie derivative** $\mathcal{L}_X : \Omega^p_{\mathcal{M}}(U)_{(k)} \to \Omega^p_{\mathcal{M}}(U)_{(k+|X|)}$.

  One obtains a full set of Cartan relations.
- ***p*-th de Rham cohomology of** $\mathcal{M}$ is a sequence $\{H^p_{(k)}(\mathcal{M})\}_{k\in\mathbb{Z}}$

Proposition (...it is not interesting)

- *For $k \neq 0$, one has $H^p_{(k)}(\mathcal{M}) = \{0\}$: For every closed $\omega \in \Omega^p_{\mathcal{M}}(U)_{(k)}$, one has $\omega = \frac{1}{k} i_E(\omega)$, E is the* ***Euler vector field*** *$E(f) := |f| f$.*
- *Using Čech cohomology and double complexes: $H^p_{(0)}(\mathcal{M}) \cong H^p(M)$. Poincaré lemma still works.*

Graded manifolds: summary

Most of the things are working as expected:

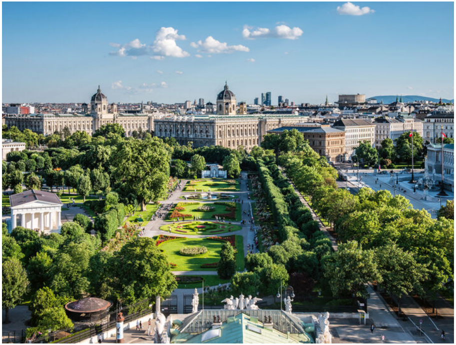
- Inverse function theorem, immersions and submersions;
- Submanifolds (embedded, immersed), transversal submanifolds, level sets, fiber products, intersections;
- Graded Lie groups, graded Lie algebras and their (one way) relation. graded Lie group actions, infinitesimal generators;
- Graded symplectic geometry;
- Multivectors (shifted cotangent bundle), Schouten–Nijenhuis bracket, graded Poisson geometry.

Many things remain to be verified:

- Better justification for coordinates of all degrees (BV?);
- Vector field flows, distributions, Frobenius theorem;
- Integration of graded Lie algebras;
- Darboux theorem for graded symplectic manifolds;

Thank you for your attention slide

Overview paper focused on “differential geometry” of graded manifolds to appear soon (if I am not eaten by founders of the genre).



Thank you for your attention!