Shuffle algebra perspective on operator-valued probability theory

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Operator-valued non-commutative probability theory

Definition : Operator-valued n-c. probability space, Voiculescu '85

An (algebraic) operator-valued probability space is a triple $(\mathcal{A}, B, \mathbb{E})$ with

- A complex unital algebra B endowed with an involution $\star$,
- A $\star$-algebra $(\mathcal{A}, \star)$, which is a B-B bimodule,

$$b_1 \cdot (a \cdot b_2) = (b_1 \cdot a) \cdot b_2, \ (a_1 \cdot b)a_2 = a_1(b \cdot a_2).$$

- A positive B-B module map $\mathbb{E} : \mathcal{A} \to B$:

$$\mathbb{E}(b_1 a b_2) = b_1 \mathbb{E}(a) b_2, \ \mathbb{E}(aa^\star) \in BB^\star.$$

Examples

- Classical commutative case, $\mathcal{A} = L^{\infty}(\Omega, \mathcal{F}, \mathbb{P})$, $B = L^{\infty}(\Omega, \mathcal{G}, \mathbb{P})$, $\mathcal{G} \subset \mathcal{F}$,

$$\mathbb{E}(\cdot) = \mathbb{E}(\ |\mathcal{G})$$

- $\mathcal{A} = \mathcal{M}_n(L^{\infty}(\Omega, \mathcal{A}, \mathbb{C}))$, $B = \mathbb{C}$,

$$\mathbb{E}(A) = \frac{1}{n}\mathrm{Tr}(A)$$

- $\mathcal{A} = \mathcal{M}_n(L^{\infty}(\Omega, , \mathcal{F}, \mathcal{M}_d(\mathbb{C}))$, $B = \bigoplus_{i=1}^{d} \mathbb{C} \cdot p^i$,

$$p^i = \mathrm{diag}(0, \ldots, \mathrm{I}_d, 0, \ldots)$$

$$\mathbb{E}(A) = \sum_{i=1}^{n} \frac{1}{d} Tr(p^i A p^i) p^i.$$

Operator-valued probability theory

Definition (Distribution of random variables)

$$B\langle X_i, X_i^\star,\ i \in [\![1,n]\!]\rangle = \langle b_0 X^{\varepsilon_1} b_2 \cdots X^{\varepsilon_n} b_n, b_1, \ldots, b_n \in B, \varepsilon_i \in \{1, \star\}\rangle$$

Let $a_1, \ldots, a_n \in \mathcal{A}$. The *distribution of* $a_1, \ldots, a_n$ is the map :

$$\Phi_{a_1,\ldots,a_n} : \begin{array}{ccc} B\langle X_i, X_i^\star,\ i \in [\![1,n]\!]\rangle & \to & B \\ P & \mapsto & \mathbb{E}\left[P(a_i, a_i^\star)\right] \end{array}$$

Independences

Definition (Freeness)

We say that $(a_1, \dots, a_n)$ is a free family of random variables if

$$\mathbb{E}(P_1(a_{i_1}) \cdots P_k(a_{i_k})) = 0$$

whenever $i_1 \neq i_2 \neq \cdots \neq i_k$, $\mathbb{E}(P_{i_j}(a_{i_j})) = 0$, $P_k \in B[\![X, X^\star]\!]$.

Proposition

Independent matrices with unitary invariant distributions are asymptotically free.

Definition (Boolean independence)

$$\mathbb{E}(P_1(a_{i_1}) \cdots P_k(a_{i_k})) = \mathbb{E}(P_1(a_{i_1}))\mathbb{E}(P_2(a_{i_2})) \cdots \mathbb{E}(P_k(a_{i_k}))$$

whenever $i_1 \neq i_2 \neq i_3 \neq \cdots \neq i_k$.

Poset of non-crossing partitions

Definition (Non-crossing partitions)

Let $n \geq 1$ be an integer. A non-crossing partition π is a partition of $[\![1, n]\!]$ such that for $a < b < c < d$,

$$a \sim_\pi c,\ b \sim_\pi d \implies b \sim_\pi c.$$

Proposition

The set NC of all non-crossing partitions is a lattice for the refinement order,

$$\pi \prec \tilde{\pi} \Leftrightarrow \forall b \in \pi,\ \exists \tilde{b} \in \tilde{\pi},\ b \subset \tilde{b}.$$

Factorization of non-crossing moments

$B = \mathbb{C}$ and take a, b two free random variables.

$$\mathbb{E}(x_1 \cdots x_n) = \prod_{b \in \pi} \mathbb{E}(\prod_{i \in V} x_i),\ x_i \in \{a, b\},\ (\pi = \mathrm{Ker}(i \mapsto x_i))$$

if π is a *non-crossing partition*.

Free cumulants – Möbius inversion

Free cumulants (Speicher '93)

$a_1, \ldots, a_n \in \mathcal{A}$. The free cumulants are multilinear maps on the algebra $\mathcal{A}$ with values in B,

$$n \geq 1,\ \kappa_n : \mathcal{A}^{\otimes n} \to B,$$

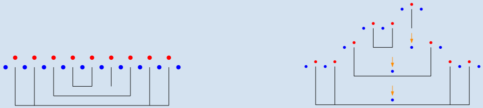
The free cumulants linearize free independence : $(a_1, \ldots, a_n)$ is a free family if and only if

$$\kappa_n(a_{j_1}, \ldots, a_{j_n}) = 0,\ \text{if } \exists k, q \in [\![1, n]\!] \text{ with } j_k \neq j_q.$$

Boolean cumulants can also be defined and linearize boolean independence.

Möbius inversion

We fix random variables $a_1, \ldots, a_n$.



$$E^{a_1,\ldots,a_9}(\pi) = \mathbb{E}(a_1 a_2 \mathbb{E}(a_3 \mathbb{E}(a_4 a_5 \mathbb{E}(a_6)) a_7) a_8 a_9)$$

Moments-cumulants relations (Speicher '93, Ebrahimi-Fard, Patras '14)

$$E^{a_1\cdots a_n}(\pi) = \sum_{\alpha \leq \pi \in \mathrm{NC}(n)} \kappa^{a_1,\ldots,a_n}(\alpha)$$

$$\kappa^{a_1,\ldots,a_n}(\pi) = \sum_{\alpha \leq \pi \in \mathrm{NC}(n)} E^{a_1\cdots a_n}(\alpha)\nu(\alpha,\pi)$$

Then $\kappa_n(a_1 \otimes \cdots \otimes a_n) = \kappa_n^{a_1,\ldots,a_n}(\{\{1,\ldots,n\}\})$ and $\kappa^{a_1,\ldots,a_n}$ factorises over blocks as E^a does.

Gap-insertion operad (Ebrahimi-Fard, Foissy, Kock, Patras 2020)

Definition : Gap insertion operad

A partition $\pi \in \mathrm{NC}(n)$ is an operator with $n+1$ inputs. Each input is a gap between two consecutive elements.

$$\gamma_{\mathcal{NC}}(\pi \otimes \alpha_1 \otimes \cdots \otimes \alpha_{|\pi|}) = \bigcup_{i=1}^{|\pi|} \{i-1+b,\ b \in \pi_i\} \cup \tilde{\pi}$$

where $\tilde{\pi}$ is the partition of $\{|\alpha_1|, |\alpha_1|+|\alpha_2|, \ldots, |\alpha_1|+\cdots+|\alpha_n|\}$ induced by π. The partition of the empty set acts as the unit.

Proposition

$$\mathcal{NC} = \langle \mathbf{1}_n, n \geq 1 \mid \mathbf{1}_n \circ_{n+1} \mathbf{1}_m = \mathbf{1}_m \circ_1 \mathbf{1}_n \rangle$$

$$\mathbf{1}_n = \{\{1, \ldots, n\}\} \in NC(n).$$

Example of a composition

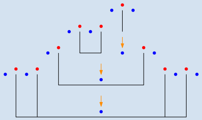
Distribution as an operadic morphism

$$\mathrm{Mult}(B) = \bigoplus_{n \geq 1} \mathrm{Hom}_{\mathrm{Vect}_{\mathbb{C}}}(B^{\otimes n}, B).$$

Proposition (G. '20)

Let $a \in \mathcal{A}$ a random variable. There exists an unique operadic morphism $\mathbb{E}^a : NC \to \mathrm{Mult}(B)$ such that

$$\mathbb{E}^a(\mathbf{1}_n)(b_0, \ldots, b_n) = \mathbb{E}(b_0 a b_1 \ldots b_{n-1} a b_n).$$



$$\mathbb{E}^a(\pi)(b_1, \ldots, b_{10}) = \mathbb{E}(b_1 a b_2 a \mathbb{E}(b_3 a \mathbb{E}(b_4 a b_5 a \mathbb{E}(b_6 a b_7)) a b_8) a b_9 a b_{10})$$

Double bar construction

Back to the scalar case, $B = \mathbb{C}$

$$H = \bar{T}(T(\mathcal{A})) \quad \ni \emptyset,\ a_1 \cdots a_n,\ a^1_1 \cdots a^1_{n_1} \Big| a^2_1 \cdots a^2_{m_1}.$$

$$\Delta^{\sqcup\!\sqcup}(\cdot) = \emptyset \otimes \cdot + \cdot \otimes \emptyset + \bar{\Delta}(\cdot) = \emptyset \otimes \cdot + \cdot \otimes \emptyset + \Delta^{\prec}(\cdot) + \Delta^{\succ}(\cdot).$$

Then $G = (\mathrm{Hom}_{Alg}(H, \mathbb{C}), \sqcup\!\sqcup)$ is a group. Owing to compatibilities between $\Delta^{\prec}$ and $\Delta^{\succ}$, one has in addition to $\exp_{\sqcup\!\sqcup}$ two maps

$$\begin{array}{rcll} \exp_{\prec}: & \mathrm{Lie}(G): & \to & G \\ & k & \mapsto & 1_\star + \sum_{n\geq 1} k^{\prec n} \end{array}$$

$$\begin{array}{rcll} \exp_{\succ}: & \mathrm{Lie}(G) & \to & G \\ & k & \mapsto & 1_\star + \sum_{n\geq 1} k^{\succ n} \end{array}$$

Shuffle and non-commutative probability theory

$$\mathsf{E} \in G, \qquad \mathsf{E}(a_1 \otimes \cdots \otimes a_n) = \mathbb{E}(a_1 \cdot_{\mathcal{A}} \cdots \cdot_{\mathcal{A}} a_n)$$
$$\mathrm{k} \in \mathrm{Lie}(G), \qquad \mathsf{k}(a_1 \otimes \cdots \otimes a_n) = \kappa_n(a_1, \ldots, a_n)$$

Proposition (Ebrahimi-Fard, Patras 2014)

$$\mathsf{E} = \varepsilon + \mathsf{k} \prec \mathsf{E}, \;\; \mathsf{E} = \exp_{\prec}(k^a)$$

- The above equation defines the free cumulants as a half-shuffle logarithm, $k = \log_{\prec}(\mathsf{E}) = \sum_{n \geq 0} (-1)^n (\mathsf{E} - \varepsilon) \prec (\mathsf{E} - \varepsilon)^{⧢ n}$
- The right half-shuffle can be interpreted in the the framework of boolean n-c. probability.

Relation between Möbius inversion and Shuffle algebra

Shuffle Approach $\rightsquigarrow$ Gap insertion operad of non-crossing partitions

Operad $\mathcal{NC} \rightsquigarrow$ incidence bialgebra (N, Δ) on words on non-crossing partitions :

$$N = \langle \pi_1, \dots, \pi_n, n \geq 1, \pi_0 \in NC \rangle / I, \text{I ideal generated by } \{\emptyset\} - 1$$

N is an algebra for the concatenation product and $\Delta : N \to N \otimes N$,

$$\Delta(\pi) = \sum_{\pi = q \circ (p_1, \dots, p_n)} q \otimes (p_1 \dots p_n) = \Delta^+_{\prec}(\pi) + \Delta^+_{\succ}(\pi).$$

$$(\mathbb{E}(a^n))_{n\geq 1}, (\kappa_n(a))_{n\geq 1} \longrightarrow F : \mathrm{NC} \to \mathbb{C}, \ \text{multiplicative}$$

$$\searrow \quad \mathsf{F} : N \to \mathbb{C}, \text{morphism}, \ \mathsf{F} = \varepsilon_{\mathsf{N}} + f \prec \mathsf{F}.$$

In the scalar case, factorization of F over block is embodied into a morphism for the concenation product, the map F.

In the operator-valued case, we need a second product ∇ on N, obtained from the gap-insertion operad on N, for which the extension of $\mathbb{E}^a$ as an algebra morphism is also an algebra morphism with respect to ∇.

... But ... The relation $\{\emptyset\} - 1$ kills grading.

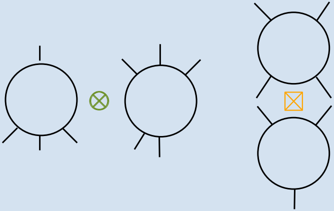
We omit this relation. This allows us to consider a word $\pi_1, \ldots, \pi_n$ as a many-to-many operator.

Duoidal category of bicollections

$$n, m \geq 0, \;\; C_{n,m} \in \mathrm{Vect}_{\mathbb{C}}, \;\; \boldsymbol{C} = (C_{n,m})_{n,m\geq 0}$$

Horizontal product $\otimes$ and Vertical product $\boxtimes$

$$(C \otimes D)_{n,m} = \bigoplus_{\substack{n_c+n_d=n \\ m_c+m_d=m}} C_{n_c,m_c} \otimes D_{n_d,m_d}, \;\; (C \boxtimes D)_{n,m} = \bigoplus_k C_{n,k} \otimes D_{k,m}$$



$$(\mathbb{C}_{\boxtimes})_{n,m} = \delta_{n=m}\mathbb{C}, \quad (\mathbb{C}_{\otimes})_{n,m} = \delta_{n=m=0}\mathbb{C}$$

$$\iota : \mathbb{C}_{\otimes} \to \mathbb{C}_{\boxtimes}$$

Lax property

$$R : (A \boxtimes B) \otimes (C \boxtimes D) \to (A \otimes C) \boxtimes (B \otimes D)$$

Consequences :

The category $\mathrm{Alg}_{\otimes}$ of horizontal algebras endowed with $\boxtimes$ is monoidal with unit $\mathbb{C}_{\boxtimes}$. The product $m_{A_1 \boxtimes A_1}$ on $A_1 \boxtimes A_2$, given two algebras (A_1, m_{A_1}) and (A_2, m_{A_2}) is

$$m_{A_1 \boxtimes A_2} = m_{A_1} \boxtimes m_{A_2} \circ R_{A_1, A_2, A_1, A_2}$$

The category $\mathrm{CoAlg}_{\boxtimes}$ of vertical co-algebras endowed with $\otimes$ is monoidal with unit $\mathbb{C}_{\otimes}$. The coproduct $\Delta_{A_1 \boxtimes A_1}$ on $A_1 \boxtimes A_2$, given two coalgebras (A_1, Δ_{A_1}) and (A_2, Δ_{A_2}) is

$$\Delta_{A_1 \boxtimes A_2} = R_{A_1, A_2, A_1, A_2} \circ \Delta_{A_1} \boxtimes \Delta_{A_2}$$

Definition (PROS)

A PROS is an algebra in the monoidal category $(\mathrm{Alg}_{\otimes}, \boxtimes, \mathbb{C}_{\boxtimes})$

$\boxtimes\otimes$ - Hopf algebras

Definition : $\boxtimes\otimes$ - Hopf algebras

An algebra $(C, m^{\otimes} : C \otimes C \to C,)$ and maps in $\mathrm{Alg}_{\otimes}$:

$$\Delta^{\boxtimes} : C \to C \boxtimes C, \quad \varepsilon : C \to \mathbb{C}_{\boxtimes}$$

$$\nabla^{\boxtimes} : C \boxtimes C \to C, \; S : C \to C, \; \eta : \mathbb{C}_{\boxtimes} \to C$$

$$\nabla^{\boxtimes} \circ (S \boxtimes \mathrm{id}_C) \circ \Delta^{\boxtimes} = \varepsilon \circ \eta, \quad \nabla^{\boxtimes} \circ (\mathrm{id}_C \boxtimes S) \circ \Delta^{\boxtimes} = \varepsilon \circ \eta$$

Notice that we de not require for $\Delta^{\boxtimes}$ to be a $\nabla^{\boxtimes}$ morphism... because it does not make sense !

Monoid of bicollection morphisms

Pick $(C, \Delta^{\boxtimes}, m_{\otimes})$ a $\boxtimes\otimes$ -bialgebra.

$$\alpha, \beta \in \mathrm{Hom}_{\mathrm{Coll}_2}(C, T(\mathrm{Hom}(B))).$$

$$\alpha \star \beta = \nabla^{\boxtimes}_{\mathrm{Hom(B)}} \circ (\alpha \boxtimes \beta) \circ \Delta^{\boxtimes} \in \mathrm{Hom}_{\mathrm{Coll}_2}(C, T(\mathrm{Hom}(B)))$$

If $\alpha, \beta \in \mathrm{Alg}_{\otimes}$, then $\alpha \star \beta \in \mathrm{Alg}_{\otimes}$.

- If C is Hopf, $\alpha \in \mathrm{Hom}_{\mathrm{Alg}_{\boxtimes}} \subset \mathrm{Hom}_{\mathrm{Alg}_{\otimes}}$ then $\alpha^{-1} = \alpha \circ S$,
- but $\alpha^{-1} \notin \mathrm{Hom}_{\mathrm{Alg}_{\boxtimes}}$,
- but $S^2 \neq \mathrm{id}_C$, $(\alpha^{-1})^{-1} \neq \alpha^{-1} \circ S$.
- If β is another PROS morphism, $\alpha \star \beta$ is in general not a PROS morphism !

$\boxtimes\otimes$-Hopf algebras of non-crossing partitions.

We denote by $T(\mathcal{NC})$ the free algebra on $\mathcal{NC}$ for the monoidal product $\otimes$.

$$\nabla^{\boxtimes} = T(\rho_{\mathcal{NC}}) : T(\mathcal{NC} \circ \mathcal{NC}) \simeq (T(\mathcal{NC}) \boxtimes T(\mathcal{NC}), \cdot) \to (T(\mathcal{NC}), \cdot)$$

$$\nabla^{\boxtimes}((\pi_1 \cdot \pi_2) \boxtimes w_1 \cdots w_{|\pi_1|} w_{|\pi_1|+1} \cdot w_{|\pi_1|+|\pi_2|})$$
$$= \big(\gamma_{\mathcal{NC}}(\pi_1 \otimes w_1 \cdots w_{|\pi_1|})\big) \cdot \big(\gamma_{\mathcal{NC}}(\pi_2 \otimes w_{|\pi_1|} \cdots w_{|\pi_1|+|\pi_2|})\big)$$

The above ismorphism is given by the natural transformation R and hold for PROS in the image of the free functor T.

$$\begin{array}{cccc} \Delta^{\boxtimes} : & T(\mathcal{NC}) & \to & T(\mathcal{NC}) \boxtimes T(\mathcal{NC}) \\ & \pi & \mapsto & \displaystyle\sum_{\substack{\alpha, \beta_1, \dots, \beta_{|\alpha|} \\ \alpha \circ (\beta_1, \dots, \beta_{|\alpha|}) = \pi}} \alpha \boxtimes (\beta_1 \otimes \cdots \otimes \beta_{|\alpha|}) \end{array}$$

$$S(\pi) = (-1)^{\text{numberOfBlocks}(\pi)} \delta_{\pi \in \mathrm{Int}} \pi.$$

Notice that the square of the "Antipode" is a projector onto interval partitions.

Unshuffle $\boxtimes$ $\otimes$ Hopf algebras of non-crossing partitions.

Half unshuffe coproducts

$$\Delta_{\prec}(\pi) = \sum_{\substack{\alpha,\beta_1,\ldots,\beta_{|\alpha|} \\ \alpha\circ(\beta_1,\ldots,\beta_{|\alpha|})=\pi \\ 1\in\alpha}} \alpha\boxtimes(\beta_1\otimes\cdots\otimes\beta_{|\alpha|}),\ \pi\neq\{\emptyset\},1$$

$$\Delta_{\succ}(\pi) = \sum_{\substack{\alpha,\beta_1,\ldots,\beta_{|\alpha|} \\ \alpha\circ(\beta_1,\ldots,\beta_{|\alpha|})=\pi \\ 1\notin\alpha}} \alpha\boxtimes(\beta_1\otimes\cdots\otimes\beta_{|\alpha|}),\pi\neq\{\emptyset\},1$$

An unshuffle ($\boxtimes$-co)($\otimes$-al)gebra

Definition (G 2020)

A bigraded collection C with $C_{n,m} = \delta_{n\neq m} C_{n,m}$.

A ($\boxtimes$-co)($\otimes$-al)gebra $(\bar{\boldsymbol{C}} = \boldsymbol{C} \oplus \mathbb{C}_{\boxtimes}, \Delta^{\boxtimes}, m_{\otimes}, \nabla^{\boxtimes})$

$$\Delta(c) = \bar{\Delta}(c) + c \boxtimes 1_m + 1_n \boxtimes c, \quad \bar{\Delta} = \Delta^{\boxtimes}_{\prec} + \Delta^{\boxtimes}_{\succ},$$

$$\mathbb{C}_{\boxtimes} \curvearrowright C, \quad \Delta^{\boxtimes}_{\prec,\succ}(\mathbb{C}_{\boxtimes} \curvearrowright) = \mathbb{C}_{\boxtimes} \curvearrowright \Delta^{\boxtimes}_{\prec,\succ}$$

$$(\Delta^{\boxtimes}_{\prec,\succ} \circ m_{\otimes})(p \otimes q) = m_{\otimes}^{C \boxtimes C} \circ (\Delta^{\boxtimes}_{\prec,\succ} \otimes \Delta)(p \otimes q), \quad p \notin \mathbb{C}_{\boxtimes}, \; q \in C.$$

Proposition

With $H = \langle \pi_1 \cdots \pi_n, \quad \pi_i \in \mathrm{NC} : \exists \pi_i \neq \{\emptyset\} \rangle$, one has that $(H \oplus \mathbb{C}_{\boxtimes}, \Delta_{\prec}, \Delta_{\succ})$ is a $\boxtimes\otimes$ unshuffle algebras and then

$$\mathrm{Hom}_{\mathrm{Coll}_2}(H, T_{\otimes}(\mathrm{Mult}(B))) + \mathbb{C}\eta_{\mathrm{Mult} B} \circ \epsilon_{\boxtimes}$$

endowed with the duals of $\Delta_{\prec,\succ}$ is an (augmented) shuffle algebra.

Let $k : T_{\otimes}(NC) \to T_{\otimes}(\mathrm{Hom}(B))$ an infinitesimal morphism with

$$k(\pi) = 0,\ \pi \neq \mathbf{1}_n,\ k(\mathbf{1}_n) \circ_1 k(\mathbf{1}_m) = k(\mathbf{1}_m) \circ_m k(\mathbf{1}_n)$$

Proposition (Left half-shuffle) (G. 2020)

The solution $\mathbb{K}$ of

$$\mathbb{K} = T_{\otimes}(\varepsilon_{\mathcal{NC}}) + k \prec \mathbb{K}$$

is an algebra morphism (for the concatenation product) AND an operadic morphism.

Proposition (Right half-shuffle) (G. 2020)

The solution $\mathbb{B}$ of

$$\mathbb{B} = T_{\otimes}(\varepsilon_{\mathcal{NC}}) + \mathbb{B} \succ k$$

is an algebra morphism (for the concatenation product) AND and

$$B(\pi) = 0 \text{ if } \pi \text{ is not an interval partition}$$

Splitting map

To obtain the operator-valued moment-cumulants relation, we pullback $\mathbb{K}^a$ by the splitting map, with

$$\mathbb{K}^a = T_{\otimes}(\varepsilon_{\mathcal{NC}}) + \Bbbk^a \prec \mathbb{K}^a, \quad \Bbbk^a(\mathbf{1}_n)(b_0, \ldots, b_n) = \kappa_n(b_0 a, b_1 a, \ldots, a b_n).$$

Definition : the splitting map (Ebrahimi-Fard, Patras, 2015, G. 2020)

The splitting is an algebra morphism $Sp : T_{\otimes}(\mathbb{N}) \to T_{\otimes}(\mathrm{Hom}(B))$ such that

$$Sp(n) = \sum_{\pi \in \mathrm{NC}(n)} \pi$$

The bicollection $T_{\otimes}(\mathbb{N})$ is a $\boxtimes\otimes$ bialgebra, with

$$|n| = n + 1,$$

$$\Delta_{\prec}^{\mathbb{N}}(n) = \sum_{\substack{(m_1,\dots,m_p)\\ m_1+\dots+m_p=n}} p\boxtimes(0, m_1, \dots, m_p)$$

$$\Delta_{\succ}^{\mathbb{N}}(n) = \sum_{\substack{(m_0,m_1,\dots,m_p)\\ m_0+\dots+m_p=n\\ m_0>0}} p\boxtimes(m_0, m_1, \dots, m_p)$$

$$\varepsilon_{\mathbb{N}} : \mathbb{N} \to \mathbb{C}_{\boxtimes}, \quad \varepsilon_{\mathbb{N}}(n) = \delta_{n=0}\mathbf{1}_1$$

Proposition (Ebrahimi-Fard, Patras, 2015, G. 2020)

The Splitting map is a morphism of unshuffle $\boxtimes$ $\otimes$ bi-algebras,

$$Sp\boxtimes Sp \circ \Delta_{\prec,\succ}^{\mathbb{N}} = \Delta_{\prec,\succ} \circ Sp, \quad \varepsilon_{\mathbb{N}} \circ Sp = Sp \circ \varepsilon_{\mathcal{NC}}$$

Moments–cumulants relations

Proposition : operator-valued (Free) moments-cumulants relations

The operator-valued free moments-cumulants relation is equivalent to the left half-shuffle fixed point equation

$$E^a = T_\otimes(\varepsilon_{\mathbb{N}}) + k^a \prec E^a$$

with $k^a(n) = \kappa_n(a^{\otimes n})$.

The proof is simple...

$$\mathbb{K}^a = T(\varepsilon_{\mathcal{NC}}) + k^a \prec \mathbb{K}^a, \quad \mathbb{K}^a \circ Sp = T(\varepsilon_{\mathbb{N}}) + (k^a \circ Sp) \prec (\mathbb{K}^a \circ Sp)$$

$$(\mathbb{K}^a \circ Sp)(n)(b_0, \ldots, b_n) = \mathbb{E}(b_0 a b_1 a \cdots a b_n),$$
$$(k^a \circ Sp)(n)(b_0, \ldots, b_n) = \kappa_n(b_0 a, \ldots, a b_n).$$

On-going work : Wick polynomials

Classical Wick Pol.

If X is a random variable,

$$\frac{d}{dx_i}\mathbb{E}W^{cl}_{X_1,\ldots,X_p}(x_1,\ldots,x_p) = W^{cl}_{X_1,\ldots,\hat{X}_i,\ldots,X_p}(x_1,\ldots,\hat{x}_i,\ldots,x_p)$$

$$\mathbb{E}\left[W^{cl}_{X_1,\ldots,X_p}(x_1,\ldots,x_p)\right] = 0.$$

Scalar Free Wick Pol.

Free analogs for Wick polynomials (Anschelevich, 2003) : If $\{X_i,\ i \geq 1\}$ is a random walk with *free* increments, then $\{W^{free}_n(X_i),\ i \geq 1\}$ is a martingale.

$$\frac{d}{dx_i}W^{free}_{a_1,\ldots,a_p}(x_1,\ldots,x_p) = W^{free}_{a_1\ldots a_{i-1}}(x_1,\ldots,x_{i-1})W^{free}_{a_{i+1},\ldots,a_p}(x_1,\ldots,x_p)$$

$$\phi(W^{free}_{a_1,\ldots,a_p}(x_1,\ldots,x_p)) = 0.$$

Shuffle calculus for Wick Pol. (Ebrahimi-Fard, Tapias, Zambotti '20)

$$W^{free} : (T(T(\mathcal{A})), \mid) \to (T(T(\mathcal{A})), \mid),$$
$$W^{free} = (\mathrm{Id} \otimes \phi^{-1}) \circ \Delta = (\mathrm{Id} \circ \exp_{\succ}(-\mathrm{k})) \circ \Delta,$$

$$W^{free}(a_1 a_2 a_3) = \ldots + 2\phi(a_1 a_3)\phi(a_2) + \ldots + a_1\phi(a_2)a_3 + \ldots$$

Operator-valued case ?

$\phi(a_1 a_3)\phi(a_2) \rightsquigarrow \phi(a_1\phi(a_2)a_3)$, ...but... $a_1(\phi(a_2)a_3) \neq (a_1\phi(a_2))a_3$

$$\mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} / \langle (a_1 \cdot b) \otimes a_2 - a_1 \otimes (b \cdot a_2) \rangle = \mathcal{A} \otimes_B \mathcal{A}?$$

... Does not work... because the coproduct Δ does not descend to the quotient. Instead $W^{free}_{a_1 a_2 a_3} \in Mult(B)$.

On going work : T-transform

X,Y two independent random variables,

$$\log(\mathcal{F}(X+Y)) = \log(\mathcal{F}(X)) + \log(\mathcal{F}(Y))$$

$$\rightsquigarrow \text{Free version, the R-transform, } R^{a} = \sum_{n\geq 1} \kappa_n^{a} z^{n-1} \in \mathbb{C}\left[[z]\right].$$

$$\mathcal{FM}(X\cdot Y) = \mathcal{FM}(X)\cdot\mathcal{FM}(Y)$$

$$\rightsquigarrow \text{Free version, the T-transform, } T^{a} = \sum_{n\geq 1} t_n^{a} z^{n-1} \in \mathbb{C}\left[[z]\right].$$

With a,b two free random variables,

$$T^{ab} = T^{a}\cdot T^{b},\ \kappa_n^{a} = \sum_{\pi\in \mathrm{NC}(n-1)} t_\pi^{a}$$

On going work : T-transform

$$\mathbb{C}[[z]] \rightsquigarrow G = \left\{ A = \sum_n A_n,\ A_n \in \mathrm{Hom}_{\mathrm{Vect}_{\mathbb{C}}}(B^{\otimes n}, B) \right\}$$

$$(A \circ B)_n = \sum_{\substack{k \\ n_1 + \cdots + n_k = n}} A_k(B_{n_1}, \ldots, B_{n_k}),\ (A \cdot B)_n = \sum_{s+t=n} A_s B_t$$

$$(A \cdot B) \circ C = (A \circ C) \cdot (B \circ C),\ A, B, C \in G.$$

The R and T transform exist in the operator-valued case,

$$R^a \in \mathrm{Mult}(B),\ \ T^a \in \mathrm{Mult}(B).$$

Proposition (Ken Dykema 2005)

$$T^{ab} = (T^a \circ (T^b \cdot I \cdot [T^b]^{-1})) \cdot T^b$$