Infinite-dimensional Geometry: Theory and Applications
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Banach-Lie groupoids part II

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Lie algebroid of a Lie groupoid $G \rightrightarrows M$

1. $R_g : G_{\mathbf{t}(g)} \to G_{\mathbf{s}(g)}$ - right translation, $R_g(h) = hg$ for $\mathbf{s}(h) = \mathbf{t}(g)$, where $G_m := \mathbf{s}^{-1}(m)$;
2. $TR_g : TG_{\mathbf{t}(g)} \to TG_{\mathbf{s}(g)}$;
3. The bundle $T^{\mathbf{s}}G$ tangent to the fibres of $\mathbf{s} : G \to M$ is an involutive subbundle;
4. R-invariant sections of $T^{\mathbf{s}}G$ form Lie subalgebra of $\Gamma^\infty TG$;
5. The section $\xi \in \Gamma^\infty T^{\mathbf{s}}_R G$ is defined by its restriction to $\varepsilon(M) \subset G$, where $\varepsilon : M \to G$ is the identity map.

Definition: A **Lie algebroid** $AG \to M$ **of a groupoid** $G \rightrightarrows M$ is a vector bundle over M

$$AG := \varepsilon^* T^{\mathbf{s}} G \xrightarrow{q} M$$

with an anchor map $a := T\mathbf{t}|_{\varepsilon(M)} : AG \to TM$
and a bracket $[\mathcal{X}|_{\varepsilon(M)}, \mathcal{Y}|_{\varepsilon(M)}] := [\mathcal{X}, \mathcal{Y}]|_{\varepsilon(M)}$.

A Lie **algebroid** on manifold M is a vector bundle $q : A \to M$ together with:

1. $a : A \to TM$ (anchor map)
2. $[\ ,\ ] : \Gamma A \times \Gamma A \to \Gamma A$ (Lie bracket) such that

$[X, fY] = f[X, Y] + a(X)(f)Y$
$a([X, Y]) = [a(X), a(Y)]$
for all $X, Y \in \Gamma A,\ f \in C^\infty(M)$.

One has the functorial correspondence
between a groupoid $G \rightrightarrows M$ and its algebroid $AG \to M$.

One has

$$\begin{array}{ccccc}
\mathcal{J}(\mathfrak{M}) & \xrightarrow{\hookrightarrow} & \mathcal{G}(\mathfrak{M}) & \xrightarrow{(\mathbf{t},\mathbf{s})} & \mathcal{L}(\mathfrak{M}) \times_{\mathcal{R}} \mathcal{L}(\mathfrak{M}) \\
\downdownarrows & & \downdownarrows & & \downdownarrows \\
\mathcal{L}(\mathfrak{M}) & \xrightarrow{\sim} & \mathcal{L}(\mathfrak{M}) & \xrightarrow{\sim} & \mathcal{L}(\mathfrak{M}),
\end{array} \qquad (1)$$

where

- $\mathcal{J}(\mathfrak{M}) := ker\ (\mathbf{t},\mathbf{s}) = \{x \in \mathcal{G}(\mathfrak{M}); \quad \mathbf{t}(x) = \mathbf{s}(x)\}$ is the inner (totaly intransitive) subgroupoid of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$,
- $\mathcal{L}(\mathfrak{M}) \times_{\mathcal{R}} \mathcal{L}(\mathfrak{M}) \ni (q,p)$ iff $q \sim p$.

Using the above functorial correspondence we obtain the short exact sequence of algebroids

$$
\begin{array}{ccccc}
\mathcal{A}\mathcal{J}(\mathfrak{M}) & \xrightarrow{\ \iota\ } & \mathcal{A}\mathcal{G}(\mathfrak{M}) & \xrightarrow{\ a\ } & T\mathcal{L}(\mathfrak{M}) \\
\Big\downarrow T\mathbf{t} & & \Big\downarrow T\mathbf{t} & & \Big\downarrow \\
\mathcal{L}(\mathfrak{M}) & \xrightarrow{\ \sim\ } & \mathcal{L}(\mathfrak{M}) & \xrightarrow{\ \sim\ } & \mathcal{L}(\mathfrak{M})
\end{array}
\qquad (2)
$$

Algebroid bracket

The Lie bracket of $\mathcal{X}_1, \mathcal{X}_2 \in \Gamma^\infty \mathcal{AG}(\mathfrak{M})$ assumes the form

$$[\mathcal{X}_1, \mathcal{X}_2] = a_p \frac{\partial}{\partial y_p} + b_p z_{p\tilde{p}} \frac{\partial}{\partial z_{p\tilde{p}}}, \tag{3}$$

where

$$a_p = \left\langle \frac{\partial a_{2p}}{\partial y_p}, a_{1p} \right\rangle - \left\langle \frac{\partial a_{1p}}{\partial y_p}, a_{2p} \right\rangle \tag{4}$$

and

$$b_p = \left\langle \frac{\partial b_{2p}}{\partial y_p}, a_{1p} \right\rangle - \left\langle \frac{\partial b_{1p}}{\partial y_p}, a_{2p} \right\rangle + [b_{2p}, b_{1p}]. \tag{5}$$

$\mathcal{VB}$-groupoids and dualization

♠ $\mathcal{VB}$-**groupoid** $(\Omega, \mathcal{G}, E, M)$ is a structure

$$\begin{array}{ccc} \Omega & \underset{\tilde{\pi}}{\overset{\tilde{0}}{\leftrightarrows}} & \mathcal{G} \\ \tilde{1}\uparrow \; \tilde{t}\downdownarrows \tilde{s} & & t \downdownarrows s \; \uparrow 1 \\ E & \underset{0}{\overset{\pi}{\rightleftarrows}} & M \end{array},$$

in which Ω is a vector bundle over $\mathcal{G}$, E is a vector bundle over M, $\mathcal{G}$ is a Lie groupoid over M, and Ω is a Lie groupoid over E, the structure maps of the groupoid (source, target, identity, multiplication, inversion) are vector bundle morphisms, the map $(\tilde{s}, \tilde{\pi})(\Omega) \mapsto E \times_M \mathcal{G} := \{(\zeta, g);\ \pi(\zeta) = s(g)\}$ is surjective submersion;
it holds $(\xi_1 + \xi_2)(\eta_1 + \eta_2) = \xi_1\eta_1 + \xi_2\eta_2$ (interchange law).

♠ If in the above $\mathcal{VB}$-groupoid the left part we change to $\Omega^* \rightrightarrows K^*_\Omega$, where

$$K_\Omega = \{\xi \in \Omega; \quad \tilde{s}(\xi) = 0_m, \quad \tilde{\pi}(\xi) = 1_m \; for \; m \in M\}$$

is the core of the groupoid Ω, then we obtain the $\mathcal{VB}$-groupoid

$$\begin{array}{ccc} \Omega^* & \longrightarrow & \mathcal{G} \\ \tilde{t}^* \downarrow\downarrow \tilde{s}^* & & t \downarrow\downarrow s \\ K^*_\Omega & \longrightarrow & M \end{array}$$

$$\langle \tilde{t}^*(\Phi), k\rangle = \langle \Phi, k\tilde{0}_g\rangle, \;\; k \in K_{t(g)}$$

$$\langle \tilde{s}^*(\Phi), k\rangle = \langle \Phi, -\tilde{0}_g k^{-1}\rangle, \;\; k \in K_{s(g)}$$

dual to the above one.

$\mathcal{VB}$-groupoids and dualization

♠ The fundamental example:

$$\begin{array}{ccc} T\mathcal{G} & \longrightarrow & \mathcal{G} \\ \downdownarrows & & \downdownarrows \\ TM & \longrightarrow & M \end{array} \qquad \begin{array}{ccc} T^*\mathcal{G} & \longrightarrow & \mathcal{G} \\ \downdownarrows & & \downdownarrows \\ A^*\mathcal{G} & \longrightarrow & M \end{array}$$

$$K_{T\mathcal{G}} = A\mathcal{G}$$

$\mathcal{VB}$-groupoids and dualization

♠ The example: for $\mathcal{G} = G$ a Lie group

$$\begin{array}{ccc} TG & \longrightarrow & G \\ \downdownarrows & K_{TG} = T_eG & \downdownarrows \\ T\{e\} & \longrightarrow & \{e\} \end{array} \qquad\qquad \begin{array}{ccc} T^*G & \longrightarrow & G \\ \downdownarrows & & \downdownarrows \\ T^*_eG & \longrightarrow & \{e\} \end{array}$$

♠ The example: for $\mathcal{G} = P \times P$ a pair groupoid

$$\begin{array}{ccc} T(P\times P) & \longrightarrow & P\times P \\ \downarrow\downarrow & & \downarrow\downarrow \\ TP & \longrightarrow & P \end{array} \qquad \begin{array}{ccc} T^*(P\times P) & \longrightarrow & P\times P \\ \downarrow\downarrow & & \downarrow\downarrow \\ T^*P & \longrightarrow & P \end{array}$$

$$K_{T(P\times P)} = \{(v_p, 0_p) \in TP \times TP\} \cong TP$$

REMARK: The dual groupoid is not a pair groupoid

Atiyah sequence of principal bundle $P_0 \to \frac{P_0}{G_0}$

The action of TG_0 on TP_0 restricted to G_0 preserves the structure of the short exact sequence

$$T^V P_0 \longrightarrow TP_0 \longrightarrow TP_0/T_eG_0 \tag{6}$$

$(T^V P_0 := kerT\pi)$ of the Banach vector bundles over P_0. So, quotiening it by G_0 we obtain the short exact sequence of Banach vector bundles

$$\begin{array}{ccccc} p_0\mathfrak{M}p_0 \times_{Ad_{G_0}} P_0 & \xrightarrow{\ a\ } & TP_0/G_0 & \xrightarrow{\ \iota\ } & T(P_0/G_0) \\ \downarrow & & \downarrow & & \downarrow \\ P_0/G_0 & \xrightarrow{\ \sim\ } & P_0/G_0 & \xrightarrow{\ \sim\ } & P_0/G_0 \end{array} \tag{7}$$

which is the Atiyah sequence of the principal bundle $\pi_0 : P_0 \to \frac{P_0}{G_0}$.

The same reason for action of TG_0 on $T(P_0 \times P_0)$ restricted to G_0 leads to the short exact sequence of Banach vector bundles

$$
\begin{array}{ccccc}
p_0\mathfrak{M}p_0 \times_{Ad_{G_0}} (P_0 \times P_0) & \xrightarrow{\iota_2} & \frac{T(P_0 \times P_0)}{G_0} & \xrightarrow{a_2} & T\left(\frac{P_0 \times P_0}{G_0}\right) \\
\downarrow & & \downarrow & & \downarrow \\
\frac{P_0 \times P_0}{G_0} & \xrightarrow{\sim} & \frac{P_0 \times P_0}{G_0} & \xrightarrow{\sim} & \frac{P_0 \times P_0}{G_0}
\end{array}
\tag{8}
$$

which is the Atiyah sequence of the G_0-principal bundle $\pi_{02} : P_0 \times P_0 \to \frac{P_0 \times P_0}{G_0}$, where ι_2 and a_2 are defined by the quotienting of $I_2 : p_0\mathfrak{M}p_0 \times P_0 \times P_0 \to TP_0 \times TP_0$ and $A_2 : TP_0 \times TP_0 \to \frac{TP_0 \times TP_0}{T_e G_0}$, respectively.

Consider $\mathcal{VB}$-groupoid tangent to the pair groupoid $P_0 \times P_0 \rightrightarrows P_0$

$$\begin{array}{ccc} T(P_0 \times P_0) & \longrightarrow & P_0 \times P_0 \\ Tpr_1 \downarrow\downarrow Tpr_2 & & pr_1 \downarrow\downarrow pr_2 \\ TP_0 & \longrightarrow & P_0 \end{array} \quad , \tag{9}$$

The tangent groupoid $T(P_0 \times P_0) \rightrightarrows TP_0$ can be considered as the pair groupoid $TP_0 \times TP_0 \rightrightarrows TP_0$. The quotient groupoid of it is the gauge groupoid (if $P_0 \to P_0/G_0$ is a principal bundle).

Short exact sequence of tangent (prolongation) groupoids

The vertical subbundle

$$T^V(P_0 \times P_0) := \{(T\kappa_p(e)X,\ T\kappa_q(e)X);\ X \in T_eG_0,\ p, q \in P_0\}$$

of the tangent bundle $T(P_0 \times P_0)$. Note $T^V(P_0 \times P_0) \rightrightarrows T^V P_0$ is a subgroupoid of $T(P_0 \times P_0) \rightrightarrows TP_0$.
We have the short exact sequence of $\mathcal{VB}$-groupoids over the pair groupoid $P_0 \times P_0 \rightrightarrows P_0$

$$\begin{array}{ccccc} T^V(P_0 \times P_0) & \longrightarrow & TP_0 \times TP_0 & \longrightarrow & \frac{TP_0 \times TP_0}{T_eG_0} \\ \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\ T^V P_0 & \longrightarrow & TP_0 & \longrightarrow & TP_0/T_eG_0 \end{array} \quad (10)$$

where $\frac{TP_0 \times TP_0}{T_eG_0} \rightrightarrows \frac{TP_0}{T_eG_0}$ is obtained by taking a quotient of $TP_0 \times TP_0 \rightrightarrows TP_0$ by T_eG_0.

Using the trivialization of the vertical bundle $T^V(P_0 \times P_0) \cong P_0 \times T_eG_0 \times P_0$ we obtain the short exact sequence of $\mathcal{VB}$-groupoids over $P_0 \times P_0 \rightrightarrows P_0$

$$\begin{array}{ccccc} P_0 \times T_eG_0 \times P_0 & \longrightarrow & TP_0 \times TP_0 & \longrightarrow & \frac{TP_0 \times TP_0}{T_eG_0} \\ \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\ P_0 \times T_eG_0 & \longrightarrow & TP_0 & \longrightarrow & TP_0/T_eG_0 \end{array} \qquad (11)$$

in which all maps are commutative with the action of G_0. So, we can take the quotient:

Short exact sequence of $\mathcal{VB}$-groupoids

$$\frac{TP_0 \times TP_0}{G_0}$$

$$TP_0/G_0$$

Short exact sequence of $\mathcal{VB}$-groupoids

$$\frac{TP_0 \times TP_0}{G_0} \qquad \frac{P_0 \times P_0}{G_0}$$

$$TP_0/G_0 \qquad \frac{P_0}{G_0}$$

Short exact sequence of $\mathcal{VB}$-groupoids

$$
\begin{array}{ccc}
\frac{TP_0 \times TP_0}{G_0} & \longrightarrow & \frac{P_0 \times P_0}{G_0} \\
\Downarrow & & \Downarrow \\
TP_0/G_0 & \longrightarrow & \frac{P_0}{G_0}
\end{array}
$$

Short exact sequence of $\mathcal{VB}$-groupoids

$$\frac{P_0 \times T_e G_0 \times P_0}{G_0}$$

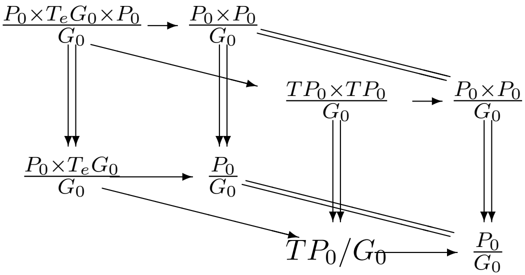
$$\frac{P_0 \times T_e G_0}{G_0}$$

$$\begin{array}{ccc} \frac{TP_0 \times TP_0}{G_0} & \longrightarrow & \frac{P_0 \times P_0}{G_0} \\ \downarrow\downarrow & & \downarrow\downarrow \\ TP_0/G_0 & \longrightarrow & \frac{P_0}{G_0} \end{array}$$

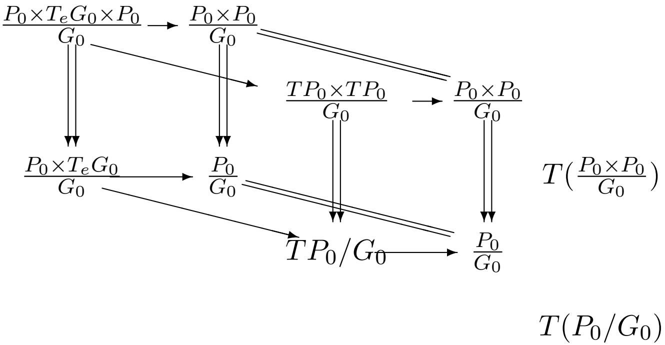
Short exact sequence of $\mathcal{VB}$-groupoids

$$
\begin{array}{ccc}
\frac{P_0 \times T_e G_0 \times P_0}{G_0} & \longrightarrow & \frac{P_0 \times P_0}{G_0} \\
\downarrow\downarrow & & \downarrow\downarrow \\
\frac{P_0 \times T_e G_0}{G_0} & \longrightarrow & \frac{P_0}{G_0}
\end{array}
\qquad
\begin{array}{ccc}
\frac{TP_0 \times TP_0}{G_0} & \longrightarrow & \frac{P_0 \times P_0}{G_0} \\
\downarrow\downarrow & & \downarrow\downarrow \\
TP_0/G_0 & \longrightarrow & \frac{P_0}{G_0}
\end{array}
$$

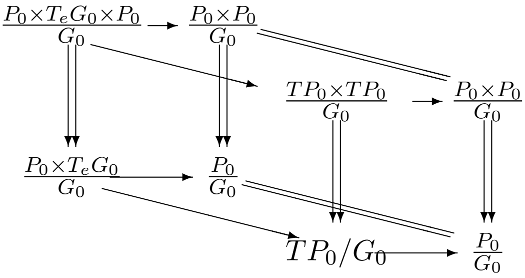
Short exact sequence of $\mathcal{VB}$-groupoids

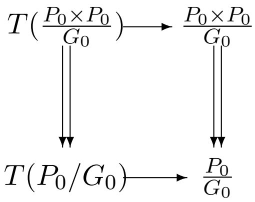


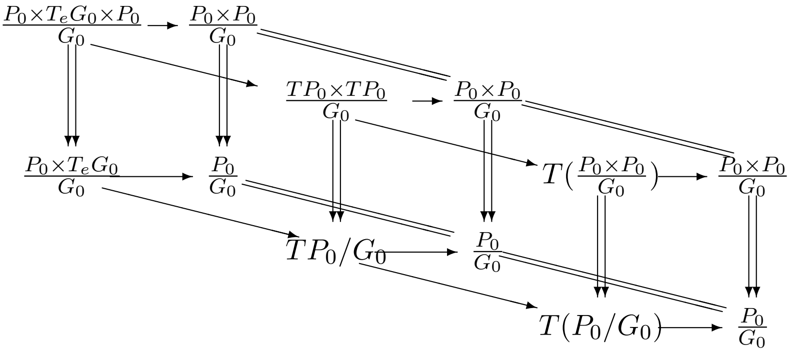
Short exact sequence of $\mathcal{VB}$-groupoids



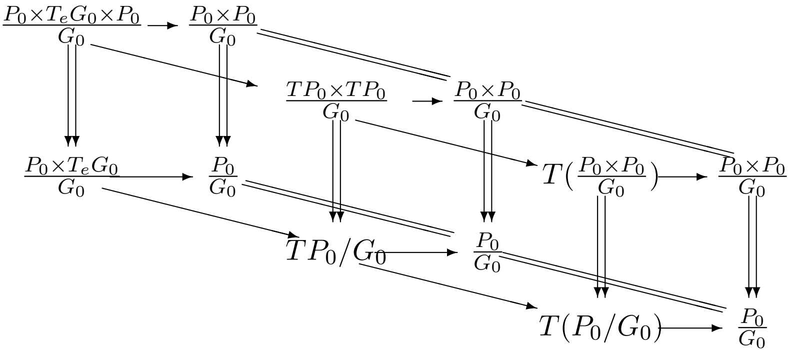
Short exact sequence of $\mathcal{VB}$-groupoids






$\frac{P_0 \times T_e G_0 \times P_0}{G_0}$
$\frac{P_0 \times P_0}{G_0}$
$\frac{TP_0 \times TP_0}{G_0}$
$\frac{P_0 \times P_0}{G_0}$
$\frac{P_0 \times T_e G_0}{G_0}$
$\frac{P_0}{G_0}$
$T(\frac{P_0 \times P_0}{G_0})$
$\frac{P_0 \times P_0}{G_0}$
TP_0/G_0
$\frac{P_0}{G_0}$
$T(P_0/G_0)$
$\frac{P_0}{G_0}$

Short exact sequence of $\mathcal{VB}$-groupoids



(12)

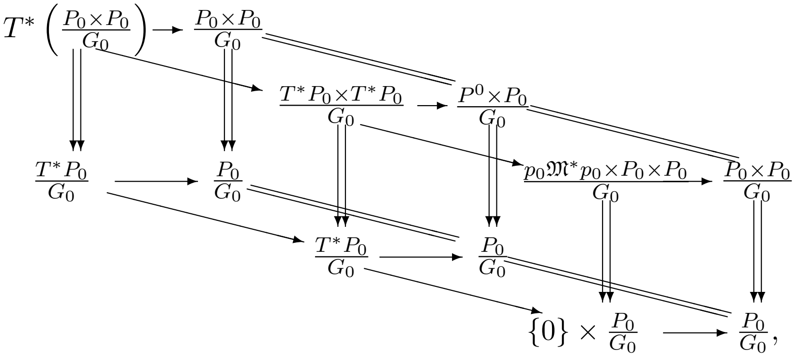
Horizontal arrows (in front) define Atiyah sequences of principal bundles $P_0 \times P_0 \to \frac{P_0 \times P_0}{G_0}$ and $P_0 \to P_0/G_0$.

Dualization

After dualization the above diagram:

Dualization

After dualization the above diagram:



(13)

which have the gauge groupoid $\frac{P_0\times P_0}{G_0} \rightrightarrows P_0/G_0$ as their side groupoid. In the front we obtain a short exact sequence of Poisson bundles, which is dual to the Atiyah sequence of the principal bundle $(P_0 \times P_0, \mu_2 : P_0 \times P_0 \to \frac{P_0\times P_0}{G_0}, G_0)$.

Theorem (in finite dimensional case):

1. The groupoid $T^*(\frac{P_0\times P_0}{G_0}) \rightrightarrows \frac{T^*P_0}{G_0}$ is the symplectic groupoid of the Poisson manifold $\frac{T^*P_0}{G_0}$.
2. The symplectic groupoid $T^*(\frac{P_0\times P_0}{G_0}) \rightrightarrows \frac{T^*P_0}{G_0}$ is the subgroupoid as well as symplectic leaf of the Poisson groupoid $\frac{T^*P_0\times T^*P_0}{G_0} \rightrightarrows \frac{T^*P_0}{G_0}$.
3. Let $J_0 : T^*P_0 \to p_0\mathfrak{M}^*p_0, \quad J_0(\varphi,\eta) = \varphi\eta.$
   Then the symplectic leaves of $\frac{T^*P_0}{G_0}$ defined as $J_0^{-1}(\{0\})/G_0$ one can consider as orbits of the standard action of the groupoid $T^*(\frac{P_0\times P_0}{G_0}) \rightrightarrows \frac{T^*P_0}{G_0}$ on its base $\frac{T^*P_0}{G_0}$.

Predual

$\mathfrak{M} = (\mathfrak{M}_*)^*$
$\mathcal{A}_*(\mathfrak{M}) \subset \mathcal{A}^*(\mathfrak{M}) \quad \longleftrightarrow \quad (q\mathfrak{M}q)^* \supset (q\mathfrak{M}q)_* \cong q\mathfrak{M}_* q$
$\mathcal{A}_*\mathcal{G}(\mathfrak{M}) \subset \mathcal{A}^*\mathcal{G}(\mathfrak{M}) \quad \longleftrightarrow \quad (\mathfrak{M}q)^* \supset (q\mathfrak{M})_* \cong \mathfrak{M}_* q$
$T_*\mathcal{L}(\mathfrak{M}) \subset T\mathcal{L}^*(\mathfrak{M}) \quad \longleftrightarrow \quad ((1-q)\mathfrak{M}q)^* \supset ((1-q)\mathfrak{M}q)_* \cong q\mathfrak{M}_*(1-q)$
are Banach quasi subbundles, i.e. their fibres are Banach subspaces but without Banach complements.
For every $x \in \mathfrak{M}$

$$\begin{array}{l} \langle R_a^*\varphi,\ x\rangle := \langle \varphi,\ ax\rangle \\ \langle L_a^*\varphi,\ x\rangle := \langle \varphi,\ xa\rangle \end{array}, \tag{14}$$

REMARK: $R_a^*\mathfrak{M}_* \subset \mathfrak{M}_*, \qquad L_a^*\mathfrak{M}_* \subset \mathfrak{M}_*.$

So, one has

$$\begin{array}{ccccc} T_*\mathcal{L}(\mathfrak{M}) & \xrightarrow{a_*} & \mathcal{A}_*\mathcal{G}(\mathfrak{M}) & \xrightarrow{\iota_*} & \mathcal{A}_*\mathcal{J}(\mathfrak{M}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{L}(\mathfrak{M}) & \xrightarrow{\sim} & \mathcal{L}(\mathfrak{M}) & \xrightarrow{\sim} & \mathcal{L}(\mathfrak{M}) \end{array} \qquad (15)$$

Predual

The short exact sequence

$$
\begin{array}{ccccc}
T_*\mathcal{L}_{p_0}(\mathfrak{M}) & \xrightarrow{a_*} & \mathcal{A}_*\mathcal{G}_{p_0}(\mathfrak{M}) & \xrightarrow{\iota_*} & \mathcal{A}_*\mathcal{J}_{p_0}(\mathfrak{M}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{L}_{p_0}(\mathfrak{M}) & \xrightarrow{\sim} & \mathcal{L}_{p_0}(\mathfrak{M}) & \xrightarrow{\sim} & \mathcal{L}_{p_0}(\mathfrak{M})
\end{array}
\tag{16}
$$

is isomorphic to

$$
\begin{array}{ccccc}
T_*(P_0/G_0) & \xrightarrow{a_*} & T_*P_0/G_0 & \xrightarrow{\iota_*} & p_0\mathfrak{M}_*p_0 \times_{Ad_{G_0^*}} P_0 \\
\downarrow & & \downarrow & & \downarrow \\
P_0/G_0 & \xrightarrow{\sim} & P_0/G_0 & \xrightarrow{\sim} & P_0/G_0,
\end{array}
\tag{17}
$$

Fibre-wise linear sub Poisson structure of the predual Atiyah sequence

- Weak symplectic structure of $T_*P_0 \cong p_0\mathfrak{M}_* \times P_0$

$(\varphi,\eta) \in T_*P_0, \qquad \xi_{(\varphi,\eta)} = (\varphi,\eta,\theta,\vartheta)$

$$\omega_{(\varphi,\eta)}(\xi^1_{(\varphi,\eta)}, \xi^2_{(\varphi,\eta)}) = \langle \theta_1, \vartheta_2 \rangle - \langle \theta_2, \vartheta_1 \rangle. \tag{18}$$

By

$$\omega_{(\varphi,\eta)}(\xi_{(\varphi,\eta)}, \cdot) : T_{(\varphi,\eta)}(p_0\mathfrak{M}_* \times P_0) \to T^*_{(\varphi,\eta)}(p_0\mathfrak{M}_* \times P_0) \tag{19}$$

one defines the bundle morphism (quasi-immersion)

$$\flat : T(T_*P_0) \hookrightarrow T^*(T_*P_0)$$

where

$$T^\flat(T_*P_0) := \flat(T(T_*P_0)) \subsetneq T^*(T_*P_0)$$

is a quasi Banach subbundle.

• For $f \in C^\infty(T_*P_0)$ one has
$\frac{\partial f}{\partial \eta}(\varphi, \eta) \in (\mathfrak{M}p_0)^*$ and $\frac{\partial f}{\partial \varphi}(\varphi, \eta) \in (p_0\mathfrak{M}_*)^* = \mathfrak{M}p_0$.
• Thus for $f, g \in C^\infty(T_*P_0)$one defines the bracket

$$\{f, g\} = \langle \frac{\partial g}{\partial \eta}, \frac{\partial f}{\partial \varphi} \rangle - \langle \frac{\partial f}{\partial \eta}, \frac{\partial g}{\partial \varphi} \rangle \tag{20}$$

which is bilinear, anti-symmetric and satisfies the Leibniz property but not satisfies the Jacobi identity for arbitrary smooth functions.
• Therefore we define

$$\mathcal{P}^\infty(T_*P_0) := \left\{ f \in C^\infty(T_*P_0) : \frac{\partial f}{\partial \eta}(\varphi, \eta) \in (\mathfrak{M}p_0)_* \subset (\mathfrak{M}p_0)^* \right\}.$$

Proposition

The space $(\mathcal{P}^\infty(T_*P_0), \{\cdot,\cdot\})$ is a Poisson algebra with respect to the bracket (20). The derivation $\{f,\cdot\}$ defined by $f \in \mathcal{P}^\infty(T_*P_0)$ is a vector field $\xi_f \in \Gamma^\infty T(T_*P_0)$ satisfying

$$\omega(\xi_f,\cdot) = -df, \tag{21}$$

i.e. it is Hamiltonian with respect to the weak symplectic form ω.

Remark

1. *The bracket (20) after restriction to $\mathcal{P}^\infty(T_*P_0)$ is equal to the Poisson bracket defined by the weak symplectic form (18).*
2. *If $f \in \mathcal{P}^\infty(T_*P_0)$ then the equality (21) defines a vector field $\xi_f \in \Gamma^\infty T(T^*P_0)$. But if $f \notin \mathcal{P}^\infty(T_*P_0)$ then $\{f,\cdot\}$ is only a section of the bundle $T^{**}(T_*P_0)$ which contains $T(T_*P_0)$ as a Banach subbundle.*
3. *$f \in \mathcal{P}^\infty(T_*P_0)$ if and only if $df \in \Gamma^\infty T^\flat(T_*P_0)$, i.e. the Banach subbundle $T^\flat(T_*P_0)$ is defined by $\mathcal{P}^\infty(T_*P_0)$.*

• Since $T^\flat(T_*P_0) \subsetneq T^*(T_*P_0)$ the bundle map $\# : T^\flat(T_*P_0) \to T(T_*P_0)$ inverse to $\flat : T(T_*P_0) \hookrightarrow T^*(T_*P_0)$ is not defined on the whole of $T^*(T_*P_0)$. This map will be called a **sub Poisson anchor**.

• $\mathcal{P}^\infty(T_*P_0) \supset \mathcal{P}^\infty_{G_0}(T_*P_0)$ - the Poisson subalgebra of G_0-invariant functions

$$\mathcal{P}^\infty_{G_0}(T_*P_0) \cong \mathcal{P}^\infty(T_*P_0/G_0)$$

$(\mathcal{P}^\infty(T_*P_0/G_0), \{\cdot,\cdot\}_{G_0})$ - a Poisson algebra

$$\{F,G\}_{G_0} := \{F \circ \pi_{*G_0}, G \circ \pi_{*G_0}\}\,, \tag{22}$$

where $\pi_{*G_0} : T_*P_0 \to T_*P_0/G_0$.

• Lie-Poisson bracket of $F, G \in C^\infty(p_0\mathfrak{M}_*p_0)$, $\frac{\partial F}{\partial \chi}(\chi) \in p_0\mathfrak{M}p_0$

$$\{F,G\}_{LP}(\chi) := \left\langle \chi, \left[\frac{\partial F}{\partial \chi}(\chi), \frac{\partial G}{\partial \chi}(\chi)\right]\right\rangle \tag{23}$$

• A SUB POISSON STRUCTURE on $p_0\mathfrak{M}_*p_0 \times_{Ad^*_{G_0}} P_0$

For this reason we take the subalgebra of smooth functions

$$\mathcal{P}^\infty_{G_0}(p_0\mathfrak{M}_*p_0 \times P_0) := \tag{24}$$

$$= \left\{ F \in C^\infty(p_0\mathfrak{M}_*p_0 \times P_0) : \; \frac{\partial F}{\partial \eta}(\chi, \eta) \in p_0\mathfrak{M}_* \text{ and } F(Ad^*_g\chi, \eta g) = F(\chi, \eta) \right\}.$$

The Poisson bracket of $F, G \in \mathcal{P}^\infty_{G_0}(p_0\mathfrak{M}_*p_0 \times P_0)$ is defined by

$$\{F, G\}_{SP}(\chi, \eta) := \left\langle \chi, \left[\frac{\partial F}{\partial \chi}(\chi, \eta), \frac{\partial G}{\partial \chi}(\chi, \eta) \right] \right\rangle. \tag{25}$$

Proposition

1. The vector bundle epimorphism $\iota_* : T_*P_0/G_0 \to p_0\mathfrak{M}_*p_0 \times Ad^*_{G_0}P_0$ is a Poisson submersion.
2. One has $ker\ \iota_* = J_0^{-1}(0)/G_0$, where $J_0^{-1}(0)/G_0$ is the weak symplectic leaf in T_*P_0/G_0 obtained by the Marsden-Weinstein symplectic reduction procedure. The predual anchor map $a_* : T_*(P_0/G_0) \xrightarrow{\sim} J_0^{-1}(0)/G_0$ defines an isomorphism of weak symplectic manifolds, where the precotangent bundle $T_*(P_0/G_0)$ is endowed with the canonical weak symplectic structure.

Proposition

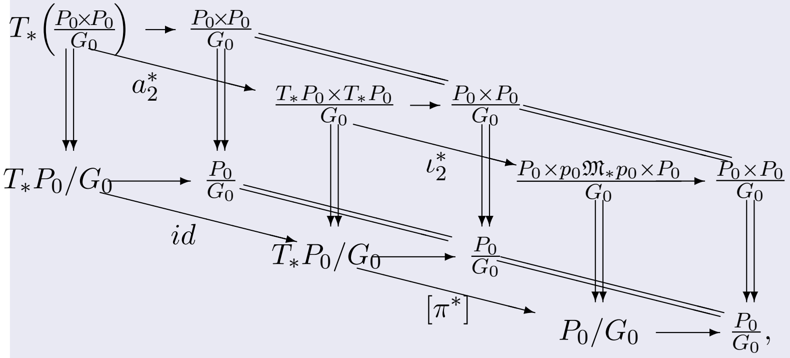
The predual Atiyah sequence

$$\begin{array}{ccccc} T_*(P_0/G_0) & \xrightarrow{a_*} & T_*P_0/G_0 & \xrightarrow{\iota_*} & p_0\mathfrak{M}_*p_0 \times_{Ad_{G_0^*}} P_0 \\ \downarrow & & \downarrow & & \downarrow \\ P_0/G_0 & \xrightarrow{\sim} & P_0/G_0 & \xrightarrow{\sim} & P_0/G_0, \end{array} \quad (26)$$

is a short exact sequence of fibre-wise linear sub Poisson manifolds.

Theorem

All Banach-Lie groupoids in the front of the diagram



are **sub Poisson groupoids** and the corresponding horizontal arrows of its define sub Poisson morphisms between of them.

Sub Poisson groupoid

Definition

The Banach-Lie groupoid $G \rightrightarrows M$ is a **sub Poisson groupoid** with a sub Poisson anchor $\# : T^\flat G \to TG$ if there exists a Banach subgroupoid $T^\flat G \rightrightarrows A^\flat G$ of the Banach groupoid $T^*G \rightrightarrows A^*G$ dual to $TG \rightrightarrows TM$ and Banach bundles morphism $a_* : A^\flat G \to TM$ such that

$$\begin{array}{ccc} T^\flat G & \xrightarrow{\quad \# \quad} & TG \\ \downarrow\downarrow & & \downarrow\downarrow \\ A^\flat G & \xrightarrow{\quad a_* \quad} & TM \end{array} \tag{27}$$

is a morphism of $\mathcal{VB}$-groupoids, where by AG we have denoted the algebroid of $G \rightrightarrows P$.

• The Poisson bracket of $f, g \in \mathcal{P}^\infty\left(\frac{T_*P_0 \times T_*P_0}{G_0}\right)$ defined by the sub Poisson structure of $\frac{T_*P_0 \times T_*P_0}{G_0}$ written in the coordinates assumes the following form

$$\{f, g\} = \left\langle \frac{\partial g}{\partial y_p}, \frac{\partial f}{\partial \alpha_p} \right\rangle - \left\langle \frac{\partial f}{\partial y_p}, \frac{\partial g}{\partial \alpha_p} \right\rangle + \tag{28}$$

$$+\left\langle \frac{\partial g}{\partial \tilde{y}_{\tilde{p}}}, \frac{\partial f}{\partial \tilde{\alpha}_{\tilde{p}}} \right\rangle - \left\langle \frac{\partial f}{\partial \tilde{y}_{\tilde{p}}}, \frac{\partial g}{\partial \tilde{\alpha}_{\tilde{p}}} \right\rangle + \left\langle \beta_p, \left[\frac{\partial g}{\partial \beta_p}, \frac{\partial f}{\partial \beta_p}\right] \right\rangle + \left\langle \tilde{\beta}_{\tilde{p}}, \left[\frac{\partial g}{\partial \tilde{\beta}_{\tilde{p}}}, \frac{\partial f}{\partial \tilde{\beta}_{\tilde{p}}}\right] \right\rangle +$$

$$+\left\langle \frac{\partial g}{\partial z_{p\tilde{p}}}, \frac{\partial f}{\partial \beta_p} z_{p\tilde{p}} - z_{p\tilde{p}} \frac{\partial f}{\partial \tilde{\beta}_{\tilde{p}}} \right\rangle - \left\langle \frac{\partial f}{\partial z_{p\tilde{p}}}, \frac{\partial g}{\partial \beta_p} z_{p\tilde{p}} - z_{p\tilde{p}} \frac{\partial g}{\partial \tilde{\beta}_{\tilde{p}}} \right\rangle,$$

where $(\alpha_p, \beta_{\tilde{p}p}, \tilde{\alpha}_{\tilde{p}})$ are the predual to the variables $(a_p, b_{p\tilde{p}}, \tilde{a}_{\tilde{p}})$.

THANK YOU