DVR 0065528

Programme on

"Optimal Transport"

April 15 – June 14, 2019

organized by

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Introductory School on Optimal Transport

May 6 – 10, 2019

- **Monday, May 6, 2019**

  09:00 – 09:30 **Opening & Registration**

  09:30 – 11:00 **Nicola Gigli**
  Schrödinger problem and optimal transport (1)

  11:00 – 11:30 *Coffee / Tea break*

  11:30 – 13:00 **Robert McCann**
  Entropic convexity and the Einstein equation for gravity (1)

  13:00 – 14:30 *Lunch break*

  14:30 – 14:45 **Emanuela Radici**
  Deterministic particle approximation for scalar aggregation-diffusion equations with nonlinear mobility

  14:45 – 15:00 **Farhan Abedin**
  Exponential Convergence of Parabolic Optimal Transport on Bounded Domains

  15:00 – 15:15 **Krzysztof Ciosmak**
  Continuity of extensions of Lipschitz maps

  15:15 – 15:30 **Lorenzo Portinale**
  Homogenization of discrete optimal transport and beyond

  15:30 – 16:00 *Coffee / Tea break*

  16:00 – 16:15 **Johannes Wiesel**
  Continuity of martingale optimal transport on the real line

  16:15 – 16:30 **Gudmund Pammer**
  Existence, Duality, and Cyclical monotonicity for weak transport costs

  16:30 – 16:45 **Caroline Moosmueller**
  Analysis of gene expression profiles via Wasserstein optimal transport on networks

  16:45 – 17:00 **Maryam Pouryahya**
  A Novel Integrative Network-based Clustering of Multiomics Data using the Wasserstein Distance

  17:00 – 19:00 *Reception*

- **Tuesday, May 7, 2019**

  09:00 – 10:30 **Yann Brenier**
  Optimal transport theory, hydrodynamics and mean field games (1)

  10:30 – 11:00 *Coffee / Tea break*

  11:00 – 12:30 **Nicola Gigli**
  Schrödinger problem and optimal transport (2)

  12:30 – 13:45 *Lunch break*

  13:45 – 15:15 **Robert McCann**
  Entropic convexity and the Einstein equation for gravity (2)

  15:15 – 15:30 *Coffee / Tea break*

  15:30 – 15:45 **Katharina Hopf**
  1D Fokker-Planck equations with superlinear drift: a framework to cope with singularities

  15:45 – 16:00 **Mathias Hudoba de Badyn**
  Connections Between Mean-Field Games and Optimal Transport

  16:00 – 16:15 **Jaime Santos Rodríguez**
  Metric measure spaces with lower Ricci curvature bounds and its isometries

  16:15 – 16:30 **Dominik Forkert**
  Optimal Transport on Metric Graphs

  16:30 – 16:45 **Clément Steiner**
  On Intertwinings and Stability of Log-Sobolev Inequalities

- **Wednesday, May 8, 2019**

  09:30 – 11:00 **Nicola Gigli**
  Schrödinger problem and optimal transport (3)

  11:00 – 11:30 *Coffee / Tea break*

  11:30 – 13:00 **Yann Brenier**
  Optimal transport theory, hydrodynamics and mean field games (2)

- **Thursday, May 9, 2019**

  09:30 – 11:00 **Marco Cuturi**
  Computational optimal transport (1)

  11:00 – 11:30 *Coffee / Tea break*

  11:30 – 13:00 **Robert McCann**
  Entropic convexity and the Einstein equation for gravity (3)

  13:00 – 15:00 *Lunch break*

  15:00 – 16:30 **Marco Cuturi**
  Computational optimal transport (2)

- **Friday, May 10, 2019**

  09:30 – 11:00 **Yann Brenier**
  Optimal transport theory, hydrodynamics and mean field games (3)

  11:00 – 11:30 *Coffee / Tea break*

  11:30 – 13:00 **Marco Cuturi**
  Computational optimal transport (3)

All talks take place at ESI, Boltzmann Lecture Hall.

Abstracts

Minicourses

Yann Brenier (CNRS, DMA-ENS Paris)
Optimal transport theory, hydrodynamics and mean field games

Optimal transport theory is a very successful field of mathematics connecting calculus of variations, probability theory, differential geometry, partial differential equations, functional analysis, statistics and computer sciences. Going back to Monge around 1780, this theory has deep connections with the earlier work of Euler on Hydrodynamics around 1750. This connection has recently known a strong revival on many different sides, leading to various non trivial generalizations of the concept of optimal transport:

1. continuous multimarginal optimal transport problems (Euler equations of incompressible fluids, models of congestion, sprays...) and their close relationship with the mean-field game theory launched by Lasry and Lions about 10 years ago;
2. optimal transportation of fields of nonnegative symmetric matrices and related problems, such as the solution by convex minimization of the initial value problem for systems of conservation laws with convex entropy (including the Euler equations of compressible fluids) and some of their parabolic limits (such as the porous medium equations).

Marco Cuturi (Google Brain, ENSAE)
Computational optimal transport

TBA

Nicola Gigli (SISSA)
Schrödinger problem and optimal transport

Aim of the course is to give an introduction to the Schrödinger problem. I will highlight its analogies with the classical optimal transport problem and present some applications of it in terms of regularisation of W_2-geodesics.

Robert McCann (University of Toronto)
Entropic convexity and the Einstein equation for gravity

On a Riemannian manifold, lower Ricci curvature bounds are known to be characterized by geodesic convexity properties of various entropies with respect to the Kantorovich-Rubinstein-Wasserstein square distance from optimal transportation. These notions also make sense in a (nonsmooth) metric measure setting, where they have found powerful applications.

In this series of talks I describe the development of an analogous theory for lower Ricci curvature bounds in timelike directions on a Lorentzian manifold. In particular, by lifting fractional powers of the Lorentz distance (a.k.a. time separation function) to probability measures on spacetime, I show the strong energy condition of Penrose is equivalent to geodesic concavity of the Boltzmann-Shannon entropy there. I shall also describe parallel work of Mondino and Suhr which gives the complementary lower bound, hence a reformulation of the Einstein field equations of general relativity in terms of the convexity properties of the entropy.

Participant talks

Farhan Abedin (Michigan State University)
Exponential Convergence of Parabolic Optimal Transport on Bounded Domains

I will speak about joint work with Jun Kitagawa on the asymptotic behavior of solutions to a parabolic version of the optimal mass transport problem. Our main result is an exponential rate of convergence for solutions of the evolution equation to the stationary solution of the optimal transport problem. The key ingredient we use is a global differential Harnack inequality for a special class of functions that solve the linearized problem. I will discuss the proof of this differential Harnack inequality in the case of domains with boundary, and show how it implies the desired exponential convergence result.

Krzysztof Ciosmak (University of Oxford)
Continuity of extensions of Lipschitz maps

Continuity of extensions of 1-Lipschitz real-valued functions plays a role in the L^1 theory of optimal transport, where it may be used to prove the mass balance condition. We study the analogous problem for 1-Lipschitz maps with values in multidimensional spaces. We establish the sharp rate of continuity of extensions of such maps from a subset A of $\mathbb{R}^n$ to a 1-Lipschitz maps on $\mathbb{R}^n$. We consider several cases when there exists an extension with preserved Lipschitz constant and preserved uniform distance to a given 1-Lipschitz map. We prove that if $m > 1$ then a map is 1-Lipschitz and affine if and only if such extension exists for any 1-Lipschitz map defined a subset of $\mathbb{R}^n$. This shows a striking difference from the case $m = 1$, when any 1-Lipschitz map has such property. Another example when it is possible to find an extension with the same Lipschitz constant and the same uniform distance to another Lipschitz map u is when the difference between the two maps belongs to a fixed one-dimensional subspace of $\mathbb{R}^m$ and the set A is geodesically convex with respect to a Riemannian pseudo-metric associated with u.

Dominik Forkert (IST Austria)
Optimal Transport on Metric Graphs

A metric graph is a graph where each edge has been associated with a finite closed interval such that the end points of the interval correspond to the end vertices of the edge. Every metric graph has a natural metric, defined as the shortest distance between two points, measured along the edges of the graph. This means that every metric graph is also a metric space which allows for studying optimal transport on it. In this talk, we will study Wasserstein spaces over metric graphs, highlighting similarities and challenges when compared to the well-established optimal transport theory in Euclidean space. In particular, we will formulate the dynamic characterisation of the 2-Wasserstein distance via the Benamou-Brenier formula. To this aim, we will investigate the continuity equation on metric graphs, which will also serve in showing convergence of the JKO scheme for the logarithmic entropy — a functional which usually lacks displacement convexity in this setting — and identify the limit curves as heat flow on the metric graph.

Katharina Hopf (University of Warwick)
1D Fokker-Planck equations with superlinear drift: a framework to cope with singularities

We consider a class of nonlinear Fokker-Planck equations of a formal Wasserstein-like gradient flow structure with convex mobility. In the drift-dominant regime, the equations have a finite critical mass above which the measure minimising the associated entropy functional displays a singular component. In the 1D case, based on the concept of viscosity solutions at the level of the pseudo-inverse distribution, we present a novel framework for measure solutions, which enables a detailed understanding of the nature of singularities and long-time asymptotics. The talk is based on joint work with J. A. Carrillo, J. L. Rodrigo, and M.-T. Wolfram.

Mathias Hudoba de Badyn (University of Washington)
Connections Between Mean-Field Games and Optimal Transport

In this talk, we consider connections between several related optimal control problems, in particular mean-field games and Brenier-Benamou mass transport. We will review the introductory material behind mean-field games, namely the coupled Hamilton-Jacobi-Bellman and Fokker-Planck equations that describe individual optimal control and aggregate population dynamics. We will conclude by examining the relations between mass transport and an optimal density control formulation of mean-field games.

Caroline Moosmueller (Johns Hopkins University)

Analysis of gene expression profiles via Wasserstein optimal transport on networks

Biological data sets, such as gene expressions or protein levels, are often high-dimensional, and thus difficult to interpret. Finding important structural features and identifying clusters in an unbiased fashion is a core issue for understanding biological phenomena. In this talk, we describe an unsupervised data analysis methodology based on network analysis via Wasserstein optimal transport, spectral analysis in the form of diffusion maps, and topological data analysis. Applied to gene expression profiles of the sarcomas in the Cancer Genome Atlas, we are able to recover the known subtypes. In addition, we find a new signature, mainly described by inactivation of the tumor suppressor gene TP73. This is joint work with J. C. Mathews, M. Pouryahya, I. G. Kevrekidis, J. O. Deasy, and A. Tannenbaum.

Gudmund Pammer (University of Vienna)
Existence, Duality, and Cyclical monotonicity for weak transport costs

The optimal weak transport problem has recently been introduced by Gozlan et. al. We provide general existence and duality results for these problems on arbitrary Polish spaces, as well as a necessary and sufficient optimality criterion in the spirit of cyclical monotonicity. As an application we extend the Brenier-Strassen Theorem of Gozlan-Juillet to general probability measures on R^d under minimal assumptions.

Lorenzo Portinale (IST Austria)
Homogenization of discrete optimal transport and beyond

In the seminal work of Jordan, Kinderlehrer and Otto ('98) the authors showed that the heat flow on $\mathbb{R}^n$ can be seen as gradient flow of the relative entropy functional in the space of probability measures with respect to the Wasserstein distance W_2. The correspondent discrete counterpart is represented by the work of Maas (2011) and Mielke (2011), where a new notion of discrete dynamical optimal transport has been introduced and a similar result has been obtained.

In this talk we will discuss one natural problem arising in the discrete theory, namely scaling limits of discrete optimal transport towards continuous costs. In short we study the behavior of partitions of a convex domain in $\mathbb{R}^n$ and the correspondent discrete transportation costs as the size of the mesh is getting finer. In particular, it is possible to prove that the limit cost need not coincide with the optimal transport cost W_2, and we present a complete and explicit description of such limits in the one dimensional periodic setting, in terms of a suitable homogenization formula. Finally, we discuss a conjecture related to a possible generalization of our result in general dimension and for general costs.
This is a joint work with Peter Gladbach, Eva Kopfer and Jan Maas.

Maryam Pouryahya (Memorial Sloan Kettering Cancer Center)
A Novel Integrative Network-based Clustering of Multiomics Data using the Wasserstein Distance

The remarkable growth of multi-platform genomic profiles has led to the multiomics data integration challenge. The effective integration of such data provides a comprehensive view of the molecular complexity of cancer tumors and can significantly improve clinical outcome predictions. In this study, we present a novel network-based integration method of multiomics data as well as a clustering technique founded on the Wasserstein (Earth Mover's) distance from the theory of optimal mass transport (OMT). We utilized the Wasserstein distance to measure the similarity between every pair of probability measures Π^1 and Π^2 of integrative measures assigned to every two samples. The integrative measures are derived from the invariant measures of the stochastic matrix associated with the interaction network of genes and they aggregate the gene expression, copy number alteration and methylation in a network-based fashion. We applied our proposed method of aggregating multiomics and Wasserstein distance clustering (aWCluster) to perform the hierarchical clustering of invasive breast carcinoma from The Cancer Genome Atlas (TCGA) project. The subtypes were characterized by the concordant effect of mRNA expression, DNA copy number alteration, and DNA methylation as well as the interaction network connectivity of the gene products. Applying aWCluster to breast cancer TCGA data successfully recovered the known PAM50 molecular subtypes. In addition, aWCluster preserves the gene-specific data, which enables us to interpret the results and perform further analysis of significant genes for a specific cluster. A gene ontology enrichment analysis of significant genes in the low survival subgroup leads to the well-known phenomenon of tumor hypoxia and the transcription factor ETS1 whose expression is induced by hypoxia. Consequently, we believe aWCluster has the potential to discover novel subtypes and biomarkers by accentuating the genes that have concordant multiomics measurements in their interaction network, which are challenging to find without the network inference or with single omics analysis.

Emanuela Radici (Università degli Studi dell'Aquila)
Deterministic particle approximation for scalar aggregation-diffusion equations with nonlinear mobility

We aim to describe the one dimensional dynamic of a biological population influenced by the presence of a nonlocal attractive potential and a diffusive term, under the constraint that no over crowding can occur. It is well known that this setting can be expressed by a class of aggregation-diffusion PDEs with nonlinear mobility. We investigate the existence of weak type solutions obtained as large particle limit of a suitable nonlocal version of the follow-the-leader scheme, which is interpreted as the discrete Lagrangian approximation of the target continuity equation. We restrict the analysis to nonnegative bounded initial with finite total variation, away from vacuum and supported in a closed interval with zero-velocity boundary conditions. The main novelties of this work concern the presence of a nonlinear mobility term and the non strict monotonicity of the diffusion function, thus, our result applies also to strongly degenerate diffusion equations. We also address the pure attractive regime, where we are able to achieve a stronger notion of solution. Indeed, in this case our scheme converges towards the unique entropy solution to the target PDE as the number of particles tends to infinity. This is a joint work with Marco Di Francesco and Simone Fagioli.

Jaime Santos Rodríguez (Universidad Autónoma de Madrid)
Metric measure spaces with lower Ricci curvature bounds and its isometries

Lott, Villani and Sturm defined the notion of synthetic Ricci curvature bound on a metric measure space. This definition is given in terms of the convexity of an entropy functional along geodesics in the Wasserstein space. Several refinements have been made in order to avoid pathological behaviour and have spaces that are Riemannian, the so called $RCD^*(K,N)$ spaces.

Isometric actions on Riemannian manifolds have been a useful tool to investigate the interaction between the topology and the Riemannian metric a manifold might admit. A major result in this area is the theorem of Myers-Steenrod stating that the isometry group of a Riemannian manifold is a Lie group.

In this talk I will discuss some structural properties of $RCD^*(K,N)$ spaces as well as some examples. I will also sketch the proof that the isometry group of an $RCD^*(K,N)$ space is a Lie group.

Clément Steiner (Institut de Mathématiques de Toulouse)
On Intertwinings and Stability of Log-Sobolev Inequalities

In the scope of the study of Markov processes, the so-called intertwinings between gradients and Markov semigroups recently turned out to be an efficient tool, leading for example to quantitative estimates for the exponential decay in some suitable functional spaces and related functional inequalities of spectral flavour, such as PoincarÃ© and Brascamp-Lieb ones. In this talk, we focus on a somewhat different functional inequality, the log-Sobolev inequality, and provide some stability results for small perturbations of Boltzmann measures.

Johannes Wiesel (St. John's College, Oxford University)
Continuity of martingale optimal transport on the real line

We show continuity of the martingale optimal transport optimisation problem as a functional of its marginals. This is achieved via an estimate on the projection in the nested/causal Wasserstein distance of an arbitrary coupling on to the set of martingale couplings with the same marginals. As a corollary we obtain an independent proof of sufficiency of the monotonicity principle established in [Beiglböck, Juillet 2016] for functions of polynomial growth.