The weak cosmic censorship conjecture and the Hoop conjecture in the case of the axially symmetric Einstein-Vlasov system

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In a paper titled "Geons" by Wheeler from 1955 one finds:

> *The simple toroidal geon forms the most elementary object of geon theory much as a simple circular orbit constitutes the first concept of planetary theory. But the simplest physics does not go in the geon case with the simplest mathematics.*

Motivation for studying the axially symmetric case: II

The New York Times

Computer Defies Einstein's Theory

By JOHN NOBLE WILFORD
Published: March 10, 1991

A supercomputer at Cornell University, simulating a tremendous gravitational collapse in the universe, has startled and confounded astrophysicists by producing results that should not be possible according to Einstein's general theory of relativity.

This notice refers to a work by Shapiro and Teukolsky on the weak cosmic censorship conjecture for the axially symmetric Einstein-Vlasov system.

Cosmic censorship: Black holes were disputed

The Schwarzschild solution from 1916 reads

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{1}{1 - \frac{2M}{r}}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

It is a vacuum solution. What happens when matter is included?

Schwarzschild studied a *static* star modeled as an incompressible fluid and concluded that $2M/R < 8/9$. The "singularity" is thus avoided.

Einstein studied a static star described by a compressible fluid and concluded: "The Schwarzschild singularity does not appear for the reason that matter cannot be concentrated arbitrarily" (Ann. Math. 1939).

Oppenheimer-Snyder studied spherically symmetric *gravitational collapse* of dust in 1939 and concluded that a singular spacetime is obtained and that an event horizon forms.

What is realistic in the Schwarzschild solution and in the Oppenheimer-Snyder dust collapse?

Could the singularity and the event horizon be a result of the assumption of (perfect) spherical symmetry?

The Nobel prize in physics 2020 was awarded to Roger Penrose for his celebrated singularity theorem from 1965 which (partially) answers this question.

Nobel prize motivation: "For the discovery that black hole formation is a robust prediction of the general theory of relativity".

The Penrose singularity theorem

Penrose shows in his singularity theorem that any spacetime (spherically symmetric or not) which satisfies some rather general conditions contains a *singularity*.

Penrose wanted to understand the *nature* of the singularity. Is it a black hole or a naked singularity, i.e., is the singularity clothed by an event horizon and is curvature blowing up at the singularity? This led him to propose the cosmic censorship conjecture.

Penrose suggests a "cosmic censor"

Penrose, 1969:

> *We are thus presented with what is perhaps the most fundamental question of general-relativistic collapse theory, namely: does there exist a "cosmic censor" who forbids the appearance of naked singularities, clothing each one in an absolute event horizon? In one sense, a "cosmic censor" can be shown not to exist. For it follows from a theorem of Hawking that the "big bang" singularity is, in principle, observable.* **But it is not known whether singularities observable from outside will ever arise in a generic collapse which starts off from a perfectly reasonable nonsingular initial state.**

Weak cosmic censorship (physical formulation)

The conjecture that Penrose proposes is called the weak cosmic censorship conjecture: The complete gravitational collapse of a body results in a black hole rather than a naked singularity, i.e. all singularities of gravitational collapse are "hidden" within black holes and cannot be "seen" by distant observers.

Need to specify what conditions the matter fields must satisfy. It is natural to assume:

- T_{ab} satisfy an energy condition (such as the dominant energy condition).
- The coupled Einstein-matter system admits a well posed initial value formulation.

Kinetic theory

In kinetic theory an ensemble of particles (atoms, molecules, ions, stars, galaxies) is described by a density function f on phase space, i.e.,

$$f = f(t, x, p),\ t \in \mathbb{R},\ x \in \mathbb{R}^3,\ p \in \mathbb{R}^3.$$

Examples of equations in kinetic theory are:

- The Boltzmann equation (collisional neutral gases)
- The Vlasov-Maxwell system (collisionless plasmas)
- The Vlasov-Poisson system (collisionless Newtonian gravity)
- The Einstein-Vlasov system describes a collisionless ensemble of particles (typically stars, galaxies or clusters of galaxies) which interact through the gravitational field created collectively.

The Vlasov equation for $f = f(t, x^a, p^a)$ reads

$$\partial_t f + \frac{p^a}{p^0}\partial_{x^a} f - \frac{1}{p^0}\Gamma^a_{\beta\gamma}p^\beta p^\gamma \partial_{p^a} f = 0.$$

Define the energy momentum tensor by

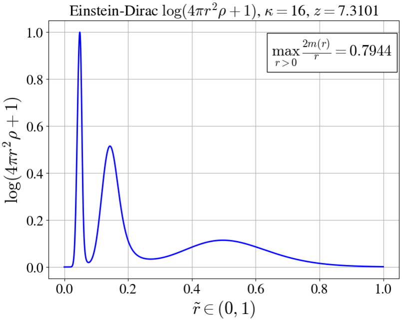
$$T_{\alpha\beta} := \sqrt{|g|}\int p_\alpha p_\beta f \frac{dp^1 dp^2 dp^3}{-p_0}.$$

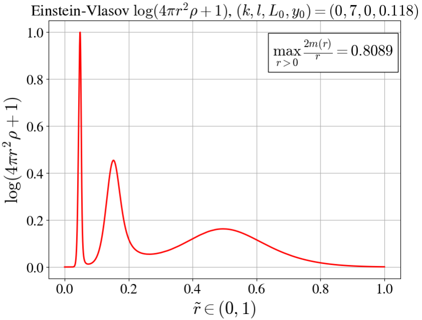
The Einstein-Vlasov system reads

$$R_{\alpha\beta} - \frac{1}{2}R g_{\alpha\beta} = 8\pi T_{\alpha\beta}.$$

It has nice mathematical properties!

Comparing Einstein-Dirac and Einstein-Vlasov





The energy density of a static solution of the spherically symmetric ED system for 16 particles compared with a solution of the EV system. (In preparation together with J. Blomqvist.)

We have carried out two studies on gravitational collapse.

- In paper one the focus was to characterize the end states of the evolution and to investigate critical collapse. The weak cosmic censorship was challenged in the case $|J| > M^2$.
- In paper two the focus was to reconsider the simulations by Shapiro and Teukolsky from 1991 (and by Yoo et al. (2017) and by East (2019)). In fact these studies concern dust rather than Vlasov matter. Moreover, we argue that the original motivation for this study is not relevant. We investigate gravitational collapse for the *regular* EV system in the case of highly prolate initial data where we challenge weak cosmic censorship in view of the Hoop conjecture.

Horizon finder

We detect black hole formation by looking for an apparent horizon. This is defined as the outermost 2-surface in a spatial slice whose outgoing null expansion vanishes.

In our dimensional reduction, such a surface corresponds to a curve in the (r, z) plane. We parametrize this curve as

$$r = R(\theta)\sin\theta, \quad z = R(\theta)\cos\theta,$$

where R is the spherical polar radius, and θ is the polar angle.

The function $R(\theta)$ obeys a second-order ODE with the boundary conditions $R'(0) = R'(\pi) = 0$. We solve this two-point boundary value problem using the shooting method.

If an apparent horizon is found (so that $R'(\pi) = 0$), we compute its irreducible mass and angular momentum.

The Kerr black hole

The Kerr black hole is a generalization of the Schwarzschild black hole. It is described by two parameters, M and J, where J is the angular momentum of the black hole.

In Boyer-Lindquist coordinates the metric is given by

$$
\begin{aligned}
ds^2 &= -\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)dt^2 - \frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}\Big)dtd\Phi \\
&\quad + \left(\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta\, d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2.
\end{aligned}
$$

Here

$$\Sigma = r^2 + a^2\cos^2\theta,\;\; \Delta = r^2 + a^2 - 2Mr \text{ and } a = J/M.$$

If $|J| > M^2$ then $\Delta > 0$ and it follows that spacetime possesses a naked singularity.

We try to collapse a body with an "excess" of angular momentum, i.e. $|J| > M^2$. Note that angular momentum and mass are conserved quantities.

If weak cosmic censorship holds then the body cannot collapse to form a Kerr spacetime where $|J| > M^2$ since such a spacetime possess a naked singularity.

Hence we ask: What happens when you try to force a such a body to collapse?

Axisymmetric collapse of a highly prolate configurations

The New York Times

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A supercomputer at Cornell University, simulating a tremendous gravitational collapse in the universe, has startled and confounded astrophysicists by producing results that should not be possible according to Einstein's general theory of relativity.

Alan Rendall criticized their work (1992) since he found indications that they study dust rather than the regular Einstein-Vlasov system.

Dust is a pressureless fluid which can be approximated by Vlasov matter.

The Vlasov equation is linear in f and distributional solutions make sense. One class of distributional solutions is given by

$$f(x^\gamma, p^a) = -u_0|g|^{-1/2}\rho(x^\gamma)\delta(p^a - u^a),$$

where $\rho \geq 0$ and $u^a(x^\gamma)$ is a mapping from spacetime into the mass shell and u_0 is given by u^a from the mass shell relation.

Solutions of the EV system where the phase space density f has this form are in one-to-one correspondence with dust solutions of the Einstein equations with density ρ and four-velocity u^α.

Dust may thus be considered as a *singular* case of matter described by the Vlasov equation.

The Euler-Poisson system and the Vlasov-Possion system

Let us first compare dust and Vlasov matter in Newtonian gravity.

The pressureless Euler-Poisson system reads

$$
\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho u) = 0,\\
&\partial_t u + (u \cdot \partial_x) u = -\partial_x U(t,x),\\
&\Delta U = 4\pi\rho, \quad \lim_{|x|\to\infty} U(t,x) = 0.
\end{aligned}
$$

The Vlasov-Possion system reads

$$
\begin{aligned}
&\partial_t f + v \cdot \partial_x f - \partial_x U \cdot \partial_v f = 0,\\
&\Delta U = 4\pi\rho, \quad \lim_{|x|\to\infty} U(t,x) = 0,\\
&\rho(t,x) = \int f(t,x,v)\, dv.
\end{aligned}
$$

A collapsing ball of dust

Let

$$\rho(t,x) := \frac{3}{4\pi}\frac{1}{r^3(t)}\mathbf{1}_{B_{r(t)}(0)},$$

where $r(t)$ solves

$$\ddot{r} = -\frac{1}{r^2},\ r(0) = 1,\ \dot{r} = 0.$$

Also, let

$$u(t,x) = \frac{\dot{r}(t)}{r(t)}x,$$

then (ρ, u, U) is a solution of the Euler-Poisson system above (where U is determined via the Poisson equation).

This solution describes a ball of dust, initially at rest, which collapses under its own gravitational field to a point in finite time, since it can be shown that $\lim_{t\to T} r(t) = 0$ for some $T > 0$.

If we swop matter model, from dust to Vlasov, and consider the Vlasov-Poisson system instead, then the celebrated global existence results for the Vlasov-Poisson system guarantee that no singularity will form (Lions-Perthame 1991, Pfaffelmoser 1992).

The global existence results say nothing about the behaviour for such solutions, only that they will not break down. In a work by Rein and Taegert (2016) this question is investigated carefully.

Theorem

For any constants $C1, C2 > 0$ there exists a smooth, spherically symmetric solution f of the Vlasov-Poisson system such that initially

$$\|\rho(0)\|_\infty < C_1,$$

but at some time $t > 0$

$$\|\rho(t)\|_\infty > C_2.$$

Dust collapse in GR: Oppenheimer-Snyder solution

It is standard to use co-moving coordinates to describe Oppenheimer-Snyder collapse. The metric then takes the form:

$$ds^2 = -\,dt^2 + e^{2\lambda(t,r)}dr^2 + R^2(t,r)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right).$$

The evolution of homogeneous initial data prescribed on an interval

$$\mathring{\rho} = c\,1_{[0,1]},$$

gives rise to the Oppenheimer-Snyder solution which reads ($r \leq 1$):

$$R(t,r) = \left(1 - \sqrt{6\pi c}\, t\right)^{2/3} r =: \gamma(t) r,$$

$$e^{\lambda(t,r)} = \gamma(t), \quad \rho(t,r) = \frac{c}{\gamma^3(t)},$$

Moreover,

$$t_0(r) = \frac{1}{\sqrt{6\pi c}}, \quad t_H(r) = \frac{1}{\sqrt{6\pi c}} - \frac{16\pi c}{9} r^3.$$

Remark on stability and instability

A general feature of dust solutions is that they blow up in finite time for any amplitude c; if the amplitude is taken very small, the blow up occurs later but still after a finite time.

Hence, the Einstein-Dust system might be said to be unstable. In particular, **critical collapse** does not occur for dust.

How does this relate to the stability results for the EV system? For small initial data global existence has been shown in the general case without symmetries:

- Lindblad and Taylor '17
- Fajman, Joudioux and Smulevici '17

These results were not known in 1991 but they were known 2019.

Approximate dust with Vlasov matter

To approximate dust with $\mathring{\rho}(x) = c1_{[0,1]}$ we choose

$$\mathring{f}(x, v) = h_\epsilon(v)\mathring{\rho}(x),$$

where h_ϵ is approximating a Dirac delta function.

Hence we have two parameters, ϵ and c.

- If we fix c and let $\epsilon \to 0$ then $\mathring{f} \to \infty$.
- If we fix ϵ and let $c \to 0$ then $\mathring{f} \to 0$.

In the former case the global existence results for the EV system do not apply whereas in the latter they do.

Motivation for Shapiro and Teukolsky: Lin-Mestel-Shu instability

The original motivation for Shapiro and Teukolsky to study collapse of highly prolate (cigar-like) matter configurations was due to the Lin-Mestel-Shu instability.

The Lin-Mestel-Shu instability concerns collapse of prolate (and oblate) configurations of dust in Newtonian gravity. A prolate spheroid of dust collapses to a spindle so that the density becomes unbounded similarly to the situation when a uniform ball of dust collapses to a point.

From the global existence results for the Vlasov-Poisson system we can again conclude that the scenario is different in the case of Vlasov matter; the density will become large but it will stay bounded.

The Newtonian regime

Shapiro and Teukolsky consider initial data described by the parameters a, b and M, where a is the equatorial radius, b the semi-major axis. They fix the ratio a/b (fixed eccentricity) and vary the ratio b/M.

The ratio b/M determines how compact the configuration is. A body with a large ratio is close to being Newtonian.

They consider the two cases $b/M = 2$ and $b/M = 10$. It is the latter case, in the Newtonian regime, which indicates violation of cosmic censorship.

Since they consider dust-like initial data it is not surprising that their solution resembles the Lin-Mestel-Shu solution! Their original motivation for investigating the weak cosmic censorship conjecture is thus highly questionable.

The more recent works from 2017 and 2019 also consider dust-like initial data (despite the critisicm raised by Alan Rendall 1992).

Collapse of highly prolate initial data is nevertheless very interesting in view of the Hoop conjecture.

The Hoop conjecture was formulated 1972 by Kip Thorne:

> *Horizons form when and only when a mass M gets compacted into a region whose circumference in EVERY direction is $\mathcal{C} \leq 4\pi GM/c^2$. (Like most conjectures, this one is sufficiently vague to leave room for many different mathematical formations!)*

We will only be concerned with the "only if" part of the conjecture due to its relation to the weak cosmic censorship conjecture.

The “only if” part of the conjecture reads:

> *If a horizon forms then the mass M is compacted into a region whose circumference $\mathcal{C}$ in every direction satisfies $\mathcal{C} \leq 4\pi M$.*

Several questions can be asked. Does it refer to the initial data? No, apparent horizons may form in the evolution of initial data which do not satisfy this inequality (A. 2012).

How should one interpret the circumference of a body? Even if the data is chosen in such a way that there is a clear cut boundary initially it may not be so later on in the evolution. Typically there is a core of the matter which is surrounded by a thin atmosphere.

Hence, it is also not clear what the mass M refers to if one cannot naturally define the boundary of the body.

Our formulation of the Hoop conjecture

Assume that there is no apparent horizon initially and that an apparent horizon forms at $t = t_H > 0$. The circumference $\mathcal{C}$ in the conjecture is then the polar and the equatorial circumference of the apparent horizon, which we denote by $\mathcal{C}_{H,e}$ and $\mathcal{C}_{H,p}$ respectively.

For the mass M we choose the horizon mass M_H which we define to be the irreducible mass M_{irr} of the apparent horizon. In the case where the angular momentum vanishes the horizon mass is given by $M_H := \sqrt{\mathcal{A}_H/16\pi}$, where $\mathcal{A}_H$ is the area of the apparent horizon.

Our formulation then reads

> *If an apparent horizon forms then* $\mathcal{C}_{H,e}(t_H) \leq 4\pi M_H$ *and* $\mathcal{C}_{H,p}(t_H) \leq 4\pi M_H$.

A similar interpretation is used by East (2019).

Initial data of highly prolate configurations

The parameter space is enormous. We limit ourselves to three sets of supercritical initial data: ID1, ID2 and ID3.

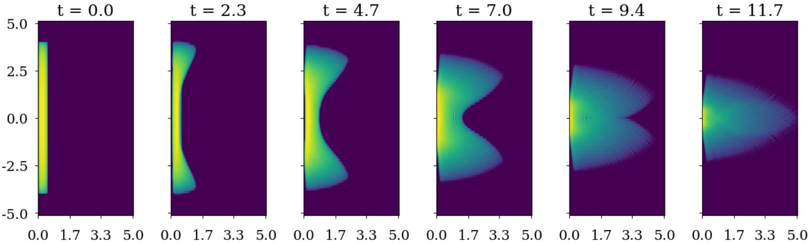
Although our initial configurations are not perfect ellipsoids they are highly elongated and we associate the initial data with an eccentricity e which we define to be

$$e = \sqrt{1 - \frac{r_{max}^2}{z_{max}^2}}.$$

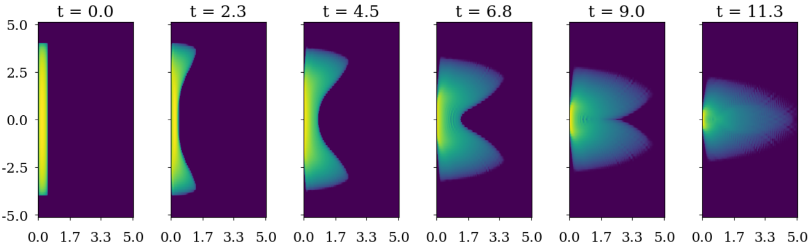
For our choice of parameters the eccentricity is $e = 0.99$ (which is larger than in previous studies).

The range of the angular momentum is smaller for ID2 than for ID1 and ID3. Both ID1 and ID2 are time symmetric whereas ID3 is not since the particles are shot inwards trying to "force" collapse.

Evolution of ID1



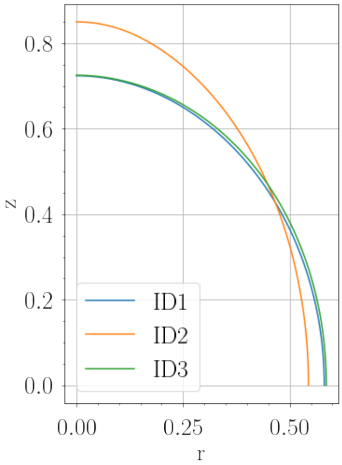
Evolution of ID3



It is remarkable how the shape of the matter configuration changes during the evolution in order not to violate weak cosmic censorship!

Shape of the apparent horizon

The shape of the horizon at the time it forms for the three initial data sets. In each case the horizon is mildly prolate.



Testing the Hoop conjecture

We define the ratios κ_p and κ_e as follows:

$$\kappa_p := \frac{\mathcal{C}_{H,p}}{4\pi M_H} \quad \text{and} \quad \kappa_e := \frac{\mathcal{C}_{H,e}}{4\pi M_H}.$$

The Hoop conjecture holds (in the strict sense) if $\kappa_p \leq 1$ and $\kappa_e \leq 1$. The *spirit* of the Hoop conjecture holds true:

ID Set	t_H	M_H	κ_p	κ_e
ID1	11.71	1.0	1.00	0.87
ID2	9.80	0.99	1.12	0.85
ID3	11.30	0.98	1.00	0.86

East (2019) tests the Hoop conjecture for dust-like initial data and finds that $\kappa_{p,e}(t) \leq 1.25$, $t \geq t_H$.

Generic feature?

Notice that the value of κ_p is (exactly) one for both ID1 and ID3.

Interestingly, this feature also holds in several other cases we have tried.

For initial data ID2, $\kappa_p = 1.12$. In this case the numerical error is however bigger. The particles spend more time close to the axis due to less angular momentum.

Hence it is still a *possibility* that $\kappa_p = 1$ is a generic feature.

The inverse Hoop conjecture

Hod (2020) has suggested "the inverse Hoop conjecture for black holes" which states that $4\pi\mathcal{A} \leq \mathcal{C}^2$ where $\mathcal{A}$ is the area of the black hole and $\mathcal{C}$ is the circumference length of the smallest ring that can engulf the black-hole horizon in every direction.

> *It is straightforward to imagine non-black hole objects that violate the area-circumference relation. For example, a moon-like object whose surface is covered with craters can violate the relation. Likewise, a non-black hole Coronavirus-like object, whose surface is covered with spikes, can violate the area-circumference relation.*

Since we obtain $\max\{\kappa_p, \kappa_e\} \geq 1$ in all our simulations our result supports the inverse Hoop conjecture.

Open problems

- Understand why κ_p is exactly 1 in many simulations. Generic or not?
- Investigate the stability of the highly compact stationary configurations we have obtained which contain ergoregions.
- Stationary case: The transition to a black hole.
- Stationary case: Massless solutions. "Wheeler's dream!"

Thank You!