Derived Manifolds in Quantum Field Theory

Transversality: X, Y, M C^∞-mfds

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & M \end{tikzcd}$$

f is transverse to g $(f \pitchfork g)$

if $\forall$ (x,y) with $f(x)=f(y)=p$

$$f_*(T_x X) + g_*(T_y Y) = T_p M$$

Example

1)

$$\begin{tikzcd} f^{-1}(0) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \{0\} \arrow[d] \\ X \arrow[r, "f"'] & \mathbb{R} \end{tikzcd}$$

C^∞-mfd

$f \pitchfork 0 \iff f$ is a submersion around zero

In general:

$f \pitchfork g \Rightarrow X \times_M Y \subseteq X \times Y$ closed submanifold

Without transversality:

This can fail drastically, e.g. $\forall$ $C \subseteq X$ closed, $\exists$ $f \in C^\infty(X)$ s.t. $C = f^{-1}(0)$.

2) Let $s : M \longrightarrow \mathbb{R}$ be smooth, then

$$\mathrm{Graph}(ds) \subset T^*M \supset \mathrm{Graph}(Z)$$ zero section

$$ds \pitchfork Z \iff \det(\mathrm{Hess}(s)) \neq 0$$

$\iff$ s has no degenerate critical points

Stationary Phase Approximation:

Let M be a compact oriented n-mfd w/ volume form μ. Let $S : M \to \mathbb{R}$ have no degenerate critical points. Then:

$$\int_{x\in M} e^{-\frac{i}{\hbar} S(x)}\,\mu = (2\pi\hbar)^{n/2} \sum_{x_0 \in \mathrm{Crit}(S)} e^{-\frac{i}{\hbar} S(x_0)} \frac{e^{\frac{i\pi}{4}\cdot \mathrm{sgn}(\mathrm{Hess}_{x_0} S)}}{|\det(\mathrm{Hess}_{x_0} S)|^{1/2}}\,\mu_{x_0} + \underbrace{\text{higher order terms in } \hbar}$$

higher order terms can be written as a sums over Feynman diagrams.

Classical Field Theory:

Ingredients:

$$\begin{array}{ll} \mathcal{E} & \text{- configuration bundle} \\ \downarrow & \\ M & \text{- spacetime} \end{array}$$

space of fields: $\mathcal{F} = \Gamma_M(\mathcal{E})$ – ∞-dim'l

action functional: $\varphi \in \mathcal{F}$

$$S(\varphi) = \int_M \overbrace{\mathcal{L}(\varphi)}^{\text{Lagrangian density}}$$

classical dynamics:

Euler-Lagrange equations

$\boxed{\delta S = 0}$ same as $\boxed{\delta \mathcal{L} = 0}$

variational derivative

Quantum Field Theory

Obs^q = algebra of quantum observables $\subseteq$ functions on $\mathcal{F}$

$\cup$

X

$$\langle X \rangle = \frac{1}{Z_{\hbar,S}} \int_{\varphi \in \mathcal{F}} X(\varphi)\, e^{-\frac{i}{\hbar} S(\varphi)} \underbrace{\mathcal{D}\varphi}_{\text{"measure" on } \mathcal{F}}$$

partition function

Feynman integral

If Hess(S) is non-degenerate, define this by using stationary phase approximation + higher-order corrections using Feynman diagrams...

What to do if $\mathrm{Hess}(S)$ is degenerate?

Warm up: $\mathcal{E}$ - cpt. mfd + oriented - $\dim = n$; $\mathcal{F} = \mathcal{E}$.

$$\mathcal{E} \to * = \mathbb{R}^0$$

$S : \mathcal{F} \longrightarrow \mathbb{R}$ smooth real valued function

(EL) equations: $\boxed{dS = 0}$

Pick μ a nowhere vanishing top form
`$\mathcal{D}\varphi$

Observe: $\forall\ 0 \le i \le n$

contraction w/ μ

$$\iota_\mu : \mathfrak{X}^i(\mathcal{F}) \xrightarrow{\cong} \Omega^{n-i}_{dR}(\mathcal{F})$$

$$\text{``}\Gamma(\wedge^i TM)\text{''} \ni X \longmapsto \iota_\mu X$$

multi-vector fields

de Rham differentia on $\Omega^\bullet(\mathcal{F})$
$\updownarrow$
Divergence operator on $\mathfrak{X}^\bullet(\mathcal{F})$

$$(\mathfrak{X}^\bullet(\mathcal{F}), \mathrm{Div}_\mu) \cong (\Omega^{n-\bullet}(\mathcal{F}), d_{dR})$$

Also works w/ $\mu_\hbar := e^{-\frac{i}{\hbar}S} \cdot \mu \quad (\hbar \in \mathbb{C})$.

$$X \in C^\infty(\mathcal{F}) \Longrightarrow \langle X \rangle = \frac{\int_M X \overbrace{e^{-\frac{i}{\hbar}S}\mu}^{\mu_\hbar} \in \mathbb{C}}{\int_M e^{-\frac{i}{\hbar}}\mu}$$

$$\begin{tikzcd}
C^\infty(\mathfrak{F}, \mathbb{C}) \arrow[r, "\cong"', "(\cdot)\cdot\mu_\hbar"] \arrow[d, "\ni X"] & \Omega^n(\mathfrak{F}, \mathbb{C}) \arrow[rr, "\int_{\mathfrak{F}}"] \arrow[d] & & \mathbb{C} \\
H^0(\mathfrak{X}^\bullet(\mathfrak{F}), \mathrm{Div}_{\mu_\hbar}) \arrow[r, "\cong"'] & H^n(\mathfrak{F}, \mathbb{C}) \arrow[rru, "\cong"] & &
\end{tikzcd}$$

$$\langle X \rangle = \frac{[X]_\hbar}{[1]_\hbar} \in \mathbb{C}.$$

Exercise: $i\hbar \, \mathrm{Div}_{\mu_\hbar} = \iota_{dS} + i\hbar \, \mathrm{Div}_\mu$

Classical limit

$\hbar \to 0$

$i\hbar \, \mathrm{Div}_{\mu_\hbar} \longmapsto \iota_{dS}$

$\rightsquigarrow$ chain complex: $(\mathfrak{X}^\bullet, \iota_{dS})$ sheaf of cdga's

$\updownarrow$

cochain complex: $(\Gamma(\underbrace{\Lambda^{-i}TM}_{C^\infty(T^*M[1])}), \iota_{dS})$

- model for "derived intersection" of dS with Z as a dg-manifold.

Roughly: A dg-manifold is a ringed space $(M, \mathcal{O}_M)$ locally isomorphic to a f.d. $\mathbb{Z}$-graded v.s. $/\mathbb{R}$ + a deg $= +1$ vector field D, s.t. $[D, D] = 0$.

$\Rightarrow (\mathcal{O}_M, D)$ is a sheaf of cdga's $/\mathbb{R}$

$T^*M[1] = (M, \underline{\Gamma(\Lambda^{-\bullet} TM)}, \iota_{dS}) \overset{q.i.}{\simeq} dS \cap Z$ when $dS \pitchfork Z$

and $\longrightarrow$ is acyclic away from critical points.

When $\hbar \neq 0$ we have:

$\mathfrak{X}^{-\bullet}$ = multi-vector fields have Schouten bracket $\rightsquigarrow$ 1-shifted Poisson algebra

$-i\hbar \, \mathrm{Div}_{\mu_\hbar}$ is a derivation wrt $[\bullet, \bullet]$

$-i\hbar \, \mathrm{Div}_{\mu_\hbar}$ is NOT a derivation wrt $\wedge$:

$$i\hbar \mathrm{Div}_{\mu_\hbar}(V \wedge W) = V \wedge i\hbar \mathrm{Div}_{\mu_\hbar}(W) + (-1)^{|W|} i\hbar \mathrm{Div}_{\mu_\hbar} V \wedge W + i\hbar [V, W]$$

$\rightarrow$ a BD-algebra which is a deformation of functions on the derived intersection of dS & Z.

Goal: formalize this and adapt to infinite-dimensional case ($M \neq *$).

C^∞-rings:

A C^∞-ring is a commutative $\mathbb{R}$-algebra

A with extra structure: $\forall\ f: \mathbb{R}^n \longrightarrow \mathbb{R}$

$\rightsquigarrow A(f): A^n \longrightarrow A$

n-ary operation + compatibility

Proto-typical example:

M manifold, $C^\infty(M)$ is a C^∞-ring

$f: \mathbb{R}^n \longrightarrow \mathbb{R}$

$\rightsquigarrow$

$$C^\infty(M) \times C^\infty(M) \times \cdots C^\infty(M) \longrightarrow C^\infty(M)$$
$$(g_1, g_2, \ldots, g_n) \longmapsto f(g_1, g_2, \ldots, g_n).$$

Formally Let C^∞ = subcat of Mfd on Euclidean manifolds $\mathbb{R}^n$

A C^∞-ring is a finite product-preserving functor $\mathcal{A}: C^\infty \longrightarrow \mathrm{Set}$.

$\mathbb{R}$ is a ring object in mfds:

$+: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$
$\cdot: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$
are C^∞

$\rightsquigarrow$ $\mathcal{A}(\mathbb{R})$ is a comm. ring.
$\quad\ \ \shortparallel$
$\quad\ \ A$

coproduct

$C^\infty\mathrm{Alg}$ = cat of C^∞-rings: has tensor product $\otimes_\infty$: $C^\infty(M) \otimes_\infty C^\infty(N) = C^\infty(M \times N)$

& $\mathrm{Mfd} \longrightarrow C^\infty\mathrm{Alg}^{op}$
$\quad M \longmapsto C^\infty(M)$

is fully faithful & preserves $\pitchfork$-pullbacks

Homotopical C^∞-rings

(C. Roytenberg)

Def A dg-C^∞-algebra is a cdga A $/\mathbb{R}$ s.t. A_0 is a C^∞-ring.

Proto-typical example: $C^\infty(\mathcal{M})$, $\mathcal{M}$ a dg-mfd.

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{Model cat.}\\ dgC^\infty Alg_{\leq 0}\end{array}} \arrow[r, "C^\infty\text{-DK}", "\sim"'] & \boxed{\begin{array}{c}\text{Model cat.}\\ \text{simplicial}\\ C^\infty\text{-rings}\end{array}} \arrow[d, "\sim"', "\text{Alg Thy}"] \\
& \boxed{\begin{array}{c}C^\infty\text{-ring objects}\\ \text{in the } \infty\text{-cat of}\\ \text{spaces}\end{array}}
\end{tikzcd}$$

(Roytenberg, Nuiten)

(Berger, Lurie)

$\mathcal{C}$ an ∞-cat, a C^∞-ring object is a finite product preserving functor $C^\infty \xrightarrow{A} \mathcal{C}$.

Derived Manifolds

Universal property:

$$\begin{tikzcd}
\mathrm{Mfd} \arrow[r, "\forall F"] \arrow[d, hook, "i"'] & \mathcal{C} \\
\mathrm{DMfd} \arrow[ur, dashed, "\exists! \tilde{F}"'] &
\end{tikzcd}$$

F – preserves $\pitchfork$ pullbacks + terminal object

$\mathcal{C}$ – ∞-cat. w/ fin. limits & idempotent complete

f&f

i – preserves $\pitchfork$-pullbacks + terminal object

$\exists! \tilde{F}$ which pres. f.n. limits

∞-cat. has finite limits & idempotent complete $\rightsquigarrow$ $\exists!$ ∞-cat with this property.

Theorem: (C.-Steffens)

$\mathcal{C}$ any ∞-cat w/ finite limits + idempotent complete

p. finite limits

$$\mathrm{Fun}^{\mathrm{lex}}(\mathrm{DMfd}, \mathcal{C}) \xrightarrow{\sim} C^\infty\mathrm{Alg}(\mathcal{C})$$

∞-cat of C^∞-ring objects in $\mathcal{C}$

Corollary

– homotopically finitely presented.

$$\mathrm{DMfd} \simeq \left(\mathrm{Alg}^{f.p.}_{C^\infty}(\mathcal{S}pc)\right)^{op}$$

= affine derived C^∞-schemes of finite type.

Upshot: Derived differential geometry
"
algebraic geometry of dg-C^∞-algebras.

Note: $\mathbb{R}$ is a C^∞-ring object in

$\mathrm{Mfd} \subseteq$ spaces ringed in homotopical C^∞-rings

Spec_{C^∞} ⇝ fully faithful

$\rightsquigarrow (\mathrm{Alg}^{f.p}_{C^\infty}(\mathrm{Spc}))^{op} \simeq \mathrm{DMfd}$

recovers Spivak's model.

$\rightsquigarrow \mathrm{Shv}(\mathrm{DMfd})$ - ∞-topos of sheaves on derived manifolds wrt open covers

where DDG happens.

BACK TO QFT's:

$$\begin{array}{c} \mathcal{E} \\ \downarrow \\ M \end{array}, \quad \Gamma_M(\mathcal{E}) =: \mathcal{F} \qquad S(\phi) = \int_M \mathcal{L}(\phi)$$

$\phi \in C^\infty(\mathcal{F})$,

Q: what is the space of derived solutions to $\delta S = 0$?

Hint: It's NOT $\underbrace{dS \cap Z}_{\text{"too big"}}$ in $T^*\mathcal{F}$!

[C. - Gwilliam]

$f : N \to \mathbb{R}$

$$\begin{tikzcd} T^*N \arrow[d] \\ N \arrow[u, bend left, "df"] \end{tikzcd}$$

$\mathcal{L} : \mathcal{F} \to \mathrm{Dens}_M$

$$\begin{tikzcd} T^*_{var}\mathcal{F} \arrow[d] \\ \mathcal{F} \arrow[u, bend left, "\delta\mathcal{L}"] \end{tikzcd}$$

$T^*_{var}\mathcal{F}$ = variational cotangent bundle

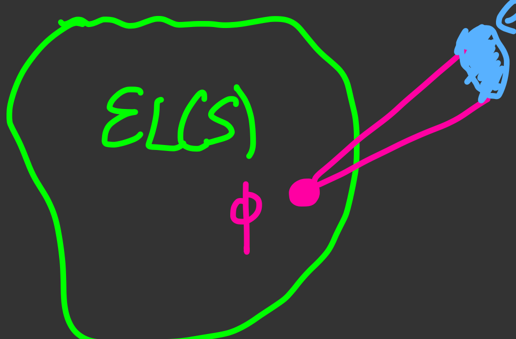
"relative Verdier dual of $T\mathcal{F}$"

$$\begin{tikzcd} EL(S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{F} \arrow[d, "z"] \\ \mathcal{F} \arrow[r, "\delta\mathcal{L}"'] & T^*_{var}\mathcal{F} \end{tikzcd}$$

When $\mathcal{L}$ is a gauge theory,

$T^*_{var}\mathcal{F} \to \mathcal{F}$ is a "2-vector bundle"

(perturbation theory)



formal nbd

$\Updownarrow$

L_∞-algebra

$\Big\{$

sheaf of L_∞-algebra / M

" (derived) classical field theory in sense of Costello - Gwilliam

Example 3D Chern-Simons

$$\mathcal{F}^{pre} = \Gamma_M(T^*M \otimes \mathfrak{g}) \quad \text{connection 1-forms}$$

$$\mathcal{G} = C^\infty(M, G)$$

$$\mathcal{F} = \mathcal{F}^{pre} /\!/ \mathcal{G}$$

$$\delta\mathcal{L}(\phi) = 0 \iff \underline{\mathrm{curv}(\phi) = 0} \Rightarrow d_\phi^2 = 0$$

$$L_\infty\text{-algebra} = (\Omega^\bullet(M, \mathfrak{g}), d_\phi)[2]$$

Example Yang-Mills

L_∞-algebra =

$$\underset{-2}{\Omega^0(M,\mathfrak{g})} \xrightarrow{d_\phi} \underset{-1}{\Omega^1(M,\mathfrak{g})} \xrightarrow{d_\phi * d_\phi} \underset{0}{\Omega^{n-1}(M,\mathfrak{g})} \xrightarrow{d_\phi} \underset{1}{\Omega^n(M,\mathfrak{g})}$$

Agree w/ adhoc computations by Costello-Gwilliam.