Combinatorics of higher order ren. group equations

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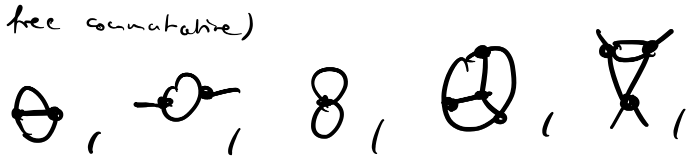
Let H = a $\mathbb{Q}$-algebra generated by graphs
(free commutative)

Ex: $H = \langle$  $, \dots \rangle$

$$X = \sum_{\text{graphs } \Gamma} \frac{\Gamma}{|\mathrm{Aut}\,\Gamma|} \hbar^{|\Gamma|} \in H[[\hbar]]$$

sum over appropriate subset of graphs

automorphism group of the graph

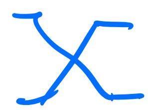
H can be promoted to a Hopf algebra

$$\Delta \Gamma = \sum_{\substack{\gamma \subset \Gamma \\ \text{bridgeless } \gamma}} \gamma \otimes \Gamma/\gamma$$

Connes-Kreimer '98

(or variants of this) $\to$ Hopf algebra

$$\phi : H \to A$$

$$\to \; G(t_1, L) = \phi_L^R\Big($$  $$\Big)$$

$$G(t_1, L) = \sum_{\text{graphs } \Gamma} \frac{\phi_L^R(\Gamma)}{|\mathrm{Aut}\,\Gamma|} t_1^{|\Gamma|}$$

ϕ_L^R are the renormalized Feynman rules

Characters $H \to A$ are very interesting!

They form a group:

$$\phi * \psi = m \circ (\phi \otimes \psi) \circ \Delta$$

The renormalization group eq. demands:

$$\phi^R_{L_1} * \phi^R_{L_2} = \phi^R_{L_1+L_2}$$

which translates to

$$\left(\frac{\partial}{\partial L} + \beta(t_1)\frac{\partial}{\partial t_1} - \gamma(x)\right) G(t_1, L) = 0$$

in the target space of ϕ^R_L!

The surprising effectiveness of these characters in enumerative geometry

Ex:

$$\phi(\Gamma) = (-1)^{|V_\Gamma|}$$

e.g. $\phi(\theta) = (-1)^2$

$\phi(8) = -1$

$$\hookrightarrow \phi(\mathcal{X}) = \sum_{n\geq 1} \chi(\mathrm{Comm}^k)\, t^{k+1}$$

commutative graph complex

Kontsevich '92, '94

$$= \sum_{k\geq 1} \frac{\zeta(-k)}{-k}\, t^{k+1}$$

$\to$ relations to $\mathcal{M}_{g,n}$ + $\mathfrak{grt}_1$

Ex:

$$\phi^{*-1}(\mathcal{X}) = \sum_{n\geq 1} \chi(\mathrm{Out}\, F_{n+1})\, t^{n+1}$$

MB – Vogtmann 19

Evaluations of characters $\qquad \phi(\mathfrak{X}) = \sum_{n \geq 1} a_n \, t_1^{\,n}$

show interesting behaviours.

For instance, resurgence:

$$a_n \sim \alpha^n \, \Gamma(n+\beta) \left(1 + \mathcal{O}\left(\tfrac{1}{n}\right)\right) \quad n \to \infty$$

for large classes of simple characters ϕ

(MB - 18)

Quantum Field Theorists dream

① Def: $B_+ : H \to H$ s.t.

$$\Delta \circ B_+ = B_+ \otimes \mathbb{I} + (\mathrm{id} \otimes B_+) \circ \Delta$$

(appropriately) ↪ analogy to rooted trees

② write an (appropriate) fix-point eq.

$$X = \mathbb{I} + B_+ \circ f(X)$$

③ Apply an appropriate character ϕ

$$\phi(X) = \mathbb{I} + \phi \circ B_+ \circ f(X)$$

④ Evaluate $\phi \circ B_+ \circ f$ to obtain a differential / integral / recursive equation for $\phi(X)$ in the target space of ϕ!

Many examples where this „realized":

- „simple" DSE's → (Gerald Dunne's talk)
  Broadhurst-Kreimer '00,
  Yeats '08, ...

- chord diagram representations Yeats '08, ...

- rooted trees Foissy '11, ...

- Topological recursion Eynard-Orantin '07
  → (talk by Elba Garcia Failde)

  → Hopf algebra interpretation ?

Outlook:

- Tropical Quantum Field theory
  Panzer '19, MB '20
  → (Erik's talk)