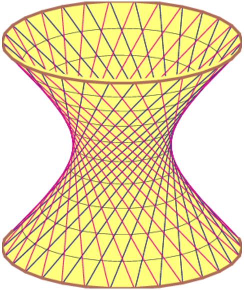
Kinematic Varieties IV
Lines in 3-Space

Bernd Sturmfels
MPI Leipzig

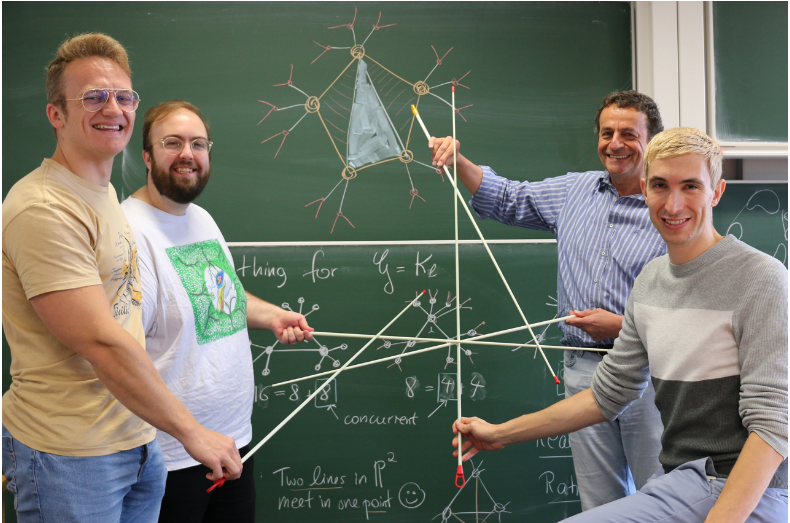


Mini-Course at ESI Vienna, within
Amplitudes and Algebraic Geometry
February 19, 2026

Source

This lecture is based on the article

Ben Hollering, Elia Mazzucchelli, Matteo Parisi, BSt:
Varieties of Lines in 3-Space arXiv:2511.21333



Plücker Coordinates

Lines in $\mathbb{P}^3$ are points $A = (a_{12} : a_{13} : a_{14} : a_{23} : a_{24} : a_{34})$ in the *Grassmannian* $\mathrm{Gr}(2,4)$. This is the hypersurface in $\mathbb{P}^5$ defined by

$$a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} \;=\; 0.$$

The *Plücker coordinates* a_{ij} are the 2×2 minors of any 2×4 matrix $\mathbf{A}$ whose rows span the line. If $\mathbf{B}$ is another matrix, we set

$$AB = \det\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} \;=\; a_{12}b_{34} - a_{13}b_{24} + a_{14}b_{23} + a_{23}b_{14} - a_{24}b_{13} + a_{34}b_{12}.$$

Fact: $AB = 0$ *if and only if the lines A and B intersect in $\mathbb{P}^3$.*

Quiz: What if $B = A$?

Configurations of ℓ lines in $\mathbb{P}^3$ are points in the product $\mathrm{Gr}(2,4)^\ell$.

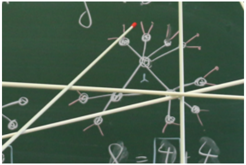
We study subvarieties of $\mathrm{Gr}(2,4)^\ell$ defined by equations $AB = 0$.

Three Lines

The 12-dim'l variety $\mathrm{Gr}(2,4)^3 \subset (\mathbb{P}^5)^3$ encodes triples of lines.
Work in a polynomial ring $\mathbb{C}[A, B, C]$ with 18 variables.

The ideal for pairwise intersecting triples:

$$I_G \;=\; \langle AB, AC, BC \rangle \;=\; I_{[3]} \,\cap\, I^*_{[3]}$$



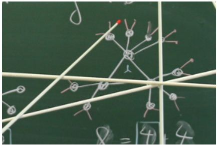
Prime ideals = irreducible varieties:

- The prime ideal $I_{[3]}$ represents triples of concurrent lines
- The prime ideal $I^*_{[3]}$ represents triples of coplanar lines

Each prime ideal has 10 additional cubic generators:

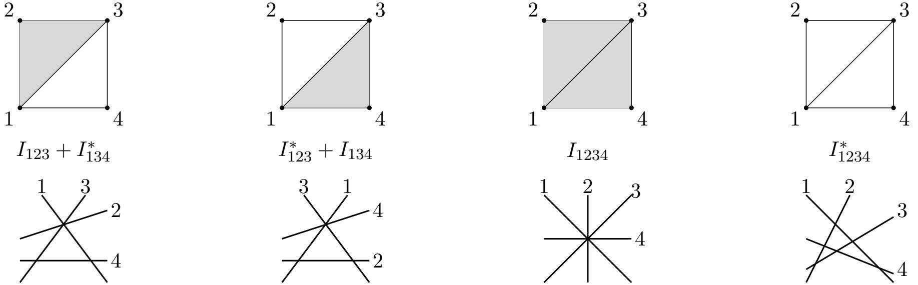
J. Ponce, BSt, M. Trager:
Congruences and concurrent lines in multi-view geometry
Advances in Applied Math, 2017

Four Lines



Let G be the graph with $\ell = 4$ and $|G| = 5$.

The ideal I_G is radical and has four associated primes:



The variety V_G is a complete intersection: codim 5 in $\mathrm{Gr}(2,4)^4$.

Graphs to Varieties

Setting: Polynomial ring $\mathbb{C}[A_1, A_2, \ldots, A_\ell]$ in 6ℓ Plücker variables modulo the prime ideal $\langle A_i A_i : i \in [\ell] \rangle$ defining $\mathrm{Gr}(2,4)^\ell$ in $(\mathbb{P}^5)^\ell$.

For any graph $G \subseteq \binom{[\ell]}{2}$, the ideal

$$I_G \;=\; \big\langle\, A_i A_j \;:\; ij \in G \,\big\rangle$$

defines the *incidence variety* $V_G \subseteq \mathrm{Gr}(2,4)^\ell$.

Graphs to Varieties

Setting: Polynomial ring $\mathbb{C}[A_1, A_2, \dots, A_\ell]$ in 6ℓ Plücker variables modulo the prime ideal $\langle A_i A_i : i \in [\ell] \rangle$ defining $\mathrm{Gr}(2,4)^\ell$ in $(\mathbb{P}^5)^\ell$.

For any graph $G \subseteq \binom{[\ell]}{2}$, the ideal

$$I_G \;=\; \langle\, A_i A_j \;:\; ij \in G \,\rangle$$

defines the *incidence variety* $V_G \subseteq \mathrm{Gr}(2,4)^\ell$.

Fact: I_G *need not be radical*. Complete graph with $\ell = 4$ satisfies

$$I_{K_4} \;=\; I_{[4]} \,\cap\, I^*_{[4]} \,\cap\, \big(\, I_{K_4} + (I_{[4]})^2 + (I^*_{[4]})^2 \,\big)$$

Graphs to Varieties

Setting: Polynomial ring $\mathbb{C}[A_1, A_2, \ldots, A_\ell]$ in 6ℓ Plücker variables modulo the prime ideal $\langle A_i A_i : i \in [\ell] \rangle$ defining $\mathrm{Gr}(2,4)^\ell$ in $(\mathbb{P}^5)^\ell$.

For any graph $G \subseteq \binom{[\ell]}{2}$, the ideal

$$I_G \;=\; \big\langle\, A_i A_j \,:\, ij \in G \,\big\rangle$$

defines the *incidence variety* $V_G \subseteq \mathrm{Gr}(2,4)^\ell$.

Fact: I_G *need not be radical*. Complete graph with $\ell = 4$ satisfies

$$I_{K_4} \;=\; I_{[4]} \,\cap\, I^*_{[4]} \,\cap\, \big(\, I_{K_4} + (I_{[4]})^2 + (I^*_{[4]})^2 \,\big)$$

The strict realizations form a subvariety W_G:
Zariski closure of points in V_G with $A_i A_j \neq 0$ for non-edges ij of G.

Fact: $\dim(V_G) \geq 2\ell + 3$, *but* W_G *may be empty.*

Five Lines

$G = \{14, 15, 24, 25, 34, 35, 45\}$

I_G is not radical
I_G is a complete intersection of codim 7 in $\mathrm{Gr}(2,4)^5$
I_G has the primary decomposition

$$Q \;\cap\; I_{12345} \;\cap\; I^*_{12345} \;\cap\; (I_{1245} + I^*_{345}) \cap (I^*_{1245} + I_{345}) \;\cap$$
$$(I_{1345} + I^*_{245}) \cap (I^*_{1345} + I_{245}) \;\cap\; (I_{2345} + I^*_{145}) \cap (I^*_{2345} + I_{145})\,.$$

Degrees of the 9 components add up to $2 \cdot 8 + 2 \cdot 8 + 6 \cdot 16 = 128 = 2^7$.

The ideal Q is primary but not prime. We have $\sqrt{Q} = W_G$:
Take the same line for 4 and 5 and arbitrary lines $1, 2, 3$ that intersect line 45.

Conclusion: *Main component has non-reduced double structure*.

Symbolic Computation

Use 4ℓ affine coordinates

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & \alpha_1 & \alpha_2 \\ 0 & 1 & \alpha_3 & \alpha_4 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 1 & 0 & \beta_1 & \beta_2 \\ 0 & 1 & \beta_3 & \beta_4 \end{pmatrix}, \ \mathbf{C} = \begin{pmatrix} 1 & 0 & \gamma_1 & \gamma_2 \\ 0 & 1 & \gamma_3 & \gamma_4 \end{pmatrix}, \ldots .$$

Incidence equation

$$\widetilde{AB} \ = \ \det \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix} \ = \ \det \begin{pmatrix} \alpha_1 - \beta_1 & \alpha_2 - \beta_2 \\ \alpha_3 - \beta_3 & \alpha_4 - \beta_4 \end{pmatrix}$$

Example: Three lines

$$\widetilde{I}_{K_3} = \langle \widetilde{AB}, \widetilde{AC}, \widetilde{BC} \rangle = \left\langle 2\times 2\text{-minors of } \begin{pmatrix} \alpha_1-\beta_1 & \alpha_2-\beta_2 & \alpha_1-\gamma_1 & \alpha_2-\gamma_2 \\ \alpha_3-\beta_3 & \alpha_4-\beta_4 & \alpha_3-\gamma_3 & \alpha_4-\gamma_4 \end{pmatrix} \right\rangle$$

$$\cap \left\langle 2\times 2\text{-minors of } \begin{pmatrix} \alpha_1-\beta_1 & \alpha_3-\beta_3 & \alpha_1-\gamma_1 & \alpha_3-\gamma_3 \\ \alpha_2-\beta_2 & \alpha_4-\beta_4 & \alpha_2-\gamma_2 & \alpha_4-\gamma_4 \end{pmatrix} \right\rangle .$$

Proposition

Primary decomposition of I_G is found by homogenizing that of $\widetilde{I}_G$.

Spanning Tree Coordinates

Given any graph G on $[\ell]$, fix a spanning tree T and a 2×2 matrix X_e for edge $e \in T$. The polynomial ring $\mathbb{C}[X]$ has $4\ell - 4$ variables.

The 2×2 determinants $\det(X_e)$ use distinct variables: toric complete intersect.

Spanning Tree Coordinates

Given any graph G on $[\ell]$, fix a spanning tree T and a 2×2 matrix X_e for edge $e \in T$. The polynomial ring $\mathbb{C}[X]$ has $4\ell - 4$ variables.

The 2×2 determinants $\det(X_e)$ use distinct variables: toric complete intersect.

For any edge g of $G \backslash T$, the graph $T \cup \{g\}$ has a unique cycle, consisting of g and edges $e_1, \ldots, e_r$ from the tree T. Set

$$Y_g \;=\; \sum_{i=1}^{r} X_{e_i}\,.$$

We define the ideal

$$I_{G,T} \;=\; \left\langle \det(X_e) \,:\, e \in T \right\rangle + \left\langle \det(Y_g) \,:\, g \in G \backslash T \right\rangle.$$

Spanning Tree Coordinates

Given any graph G on $[\ell]$, fix a spanning tree T and a 2×2 matrix X_e for edge $e \in T$. The polynomial ring $\mathbb{C}[X]$ has $4\ell - 4$ variables.

The 2×2 determinants $\det(X_e)$ use distinct variables: toric complete intersect.

For any edge g of $G \backslash T$, the graph $T \cup \{g\}$ has a unique cycle, consisting of g and edges $e_1, \ldots, e_r$ from the tree T. Set

$$Y_g = \sum_{i=1}^{r} X_{e_i}.$$

We define the ideal

$$I_{G,T} = \langle \det(X_e) : e \in T \rangle + \langle \det(Y_g) : g \in G \backslash T \rangle.$$

Proposition

The ideal $\widetilde{I}_G$ of the incidence variety V_G is isomorphic to $I_{G,T}$.

Spanning Tree Coordinates

Example (Cycle)

The following ideal is prime:

$$I_{G,T} = \left\langle \det(X_1), \det(X_2), \ldots, \det(X_{\ell-1}), \det(X_1 + X_2 + \cdots + X_{\ell-1}) \right\rangle$$

Spanning Tree Coordinates

Example (Cycle)

The following ideal is prime:

$$I_{G,T} = \left\langle \det(X_1), \det(X_2), \ldots, \det(X_{\ell-1}), \det(X_1+X_2+\cdots+X_{\ell-1}) \right\rangle$$

Example (Complete graph)

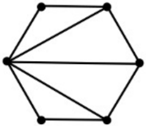
The following ideal is not radical for $\ell \geq 4$:

$$I_{G,T} = \left\langle \det(X_i) : 1 \leq i \leq \ell-1 \right\rangle + \left\langle \det(X_i+X_j) : 1 \leq i < j \leq \ell-1 \right\rangle$$

It has two associated primes, namely

$$\left\langle 2 \times 2\text{-minors of } \left(X_1 \mid X_2 \mid \cdots \mid X_{\ell-1} \right) \right\rangle$$
$$\text{and} \quad \left\langle 2 \times 2\text{-minors of } \left(X_1^T \mid X_2^T \mid \cdots \mid X_{\ell-1}^T \right) \right\rangle$$

Macaulay2



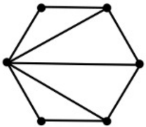
Example (Triangulated hexagon)

For $\ell = 6$ we use 20 spanning tree coordinates:

```
R = QQ[a1,a2,a3,a4,a5,b1,b2,b3,b4,b5,c1,c2,c3,c4,c5,d1,d2,d3,d4,d5];
X1 = matrix {{a1,b1},{c1,d1}}; X2 = matrix {{a2,b2},{c2,d2}};
X3 = matrix {{a3,b3},{c3,d3}}; X4 = matrix {{a4,b4},{c4,d4}};
                               X5 = matrix {{a5,b5},{c5,d5}};
IG = ideal( det(X1), det(X2), det(X3), det(X4), det(X5),
           det(X1+X2), det(X2+X3), det(X3+X4), det(X4+X5) );
betti mingens IG, codim IG, degree IG
DIG = decompose IG; toString DIG
intersect(DIG) == IG
```

Output:

Macaulay2



Example (Triangulated hexagon)

For $\ell = 6$ we use 20 spanning tree coordinates:

```
R = QQ[a1,a2,a3,a4,a5,b1,b2,b3,b4,b5,c1,c2,c3,c4,c5,d1,d2,d3,d4,d5];
X1 = matrix {{a1,b1},{c1,d1}}; X2 = matrix {{a2,b2},{c2,d2}};
X3 = matrix {{a3,b3},{c3,d3}}; X4 = matrix {{a4,b4},{c4,d4}};
                               X5 = matrix {{a5,b5},{c5,d5}};
IG = ideal( det(X1), det(X2), det(X3), det(X4), det(X5),
           det(X1+X2), det(X2+X3), det(X3+X4), det(X4+X5) );
betti mingens IG, codim IG, degree IG
DIG = decompose IG; toString DIG
intersect(DIG) == IG
```

Output: The ideal I_G is radical.

Get 16 prime ideals of codim 9.

Quiz: What are the 16 components of V_G?

Elekes-Sharir Framework

Orit Raz:

Configurations of lines in space and combinatorial rigidity

Discrete and Computional Geometry, 2017

Identify $\mathbb{R}^4$ with the space $\mathbb{R}^{2,2}$ of pairs of points in the Euclidean plane. For two pairs (s, t) and (u, v), we write

$$\alpha_1 = s_2 - t_2, \quad \alpha_2 = t_1 - s_1\,, \quad \beta_1 = u_2 - v_2\,, \quad \beta_2 = v_1 - u_1\,,$$
$$\alpha_3 = s_1 + t_1\,, \quad \alpha_4 = s_2 + t_2\,, \quad \beta_3 = u_1 + v_1\,, \quad \beta_4 = u_2 + v_2$$

Here $\alpha - \beta$ are spanning tree coordinates.

Elekes-Sharir Framework

Orit Raz:
Configurations of lines in space and combinatorial rigidity
Discrete and Computional Geometry, 2017

Identify $\mathbb{R}^4$ with the space $\mathbb{R}^{2,2}$ of pairs of points in the Euclidean plane. For two pairs (s, t) and (u, v), we write

$$\begin{aligned} \alpha_1 = s_2 - t_2, \quad \alpha_2 = t_1 - s_1\,, \quad \beta_1 = u_2 - v_2\,, \quad \beta_2 = v_1 - u_1\,, \\ \alpha_3 = s_1 + t_1\,, \quad \alpha_4 = s_2 + t_2\,, \quad \beta_3 = u_1 + v_1\,, \quad \beta_4 = u_2 + v_2 \end{aligned}$$

Here $\alpha - \beta$ are spanning tree coordinates. The equation $\widetilde{AB} = 0$ becomes

$$||s-u||^2 = (s_1-u_1)^2 + (s_2-u_2)^2 = (t_1-v_1)^2 + (t_2-v_2)^2 = ||t-v||^2$$

Lemma

The Grassmannian $\mathrm{Gr}(2,4)$ is a natural compactification of $\mathbb{R}^{2,2}$. For two point pairs, the two Euclidean distances agree if and only if the two corresponding lines intersect in $\mathbb{P}^3$.

Rigidity Theory

Let $\mathbb{R}^{2\ell}$ be the space of configurations $s = (s^{(1)}, \ldots, s^{(\ell)})$ of ℓ points in the Euclidean plane. We record the pairwise distances indexed by the graph G via

$$\delta_G \,:\, \mathbb{R}^{2\ell} \to \mathbb{R}^G \,, \;\; s \mapsto \big(\, ||s^{(i)} - s^{(j)}||^2 \,\big)_{ij \in G} \,.$$

Rigidity theory studies the image and fibers of the map δ_G.

The dimension of the image of δ_G is the *rigidity rank* of G.

J. Sidman and A. Lee–St.John:
Frameworks in Motion – An Introduction to Rigidity
Undergraduate textbook, 2026+

Rigidity Theory

Let $\mathbb{R}^{2\ell}$ be the space of configurations $s = (s^{(1)}, \ldots, s^{(\ell)})$ of ℓ points in the Euclidean plane. We record the pairwise distances indexed by the graph G via

$$\delta_G : \mathbb{R}^{2\ell} \to \mathbb{R}^G, \quad s \mapsto \left(||s^{(i)} - s^{(j)}||^2 \right)_{ij \in G}.$$

Rigidity theory studies the image and fibers of the map δ_G.

The dimension of the image of δ_G is the *rigidity rank* of G.

J. Sidman and A. Lee–St.John:
Frameworks in Motion – An Introduction to Rigidity
Undergraduate textbook, 2026+

Proposition

Our incidence variety equals

$$V_G = \{(s,t) \in \mathbb{R}^{2\ell} \times \mathbb{R}^{2\ell} : \delta_G(s) = \delta_G(t)\}.$$

Rigidity Matroid

The Jacobian matrix J_ℓ of the map δ_{K_ℓ} has format $2\ell \times \binom{\ell}{2}$ and rank $2\ell - 3$. Its matroid on $\binom{[\ell]}{2}$ is the *rigidity matroid* $\mathcal{R}_\ell$.

A graph G is *(a, b)-sparse* if $|\, G[S] \,| \leq a|S| - b$ for all $\emptyset \subsetneq S \subseteq [\ell]$.

Here $a, b \in \mathbb{N}$ and $G[S]$ is the induced subgraph of G on S.

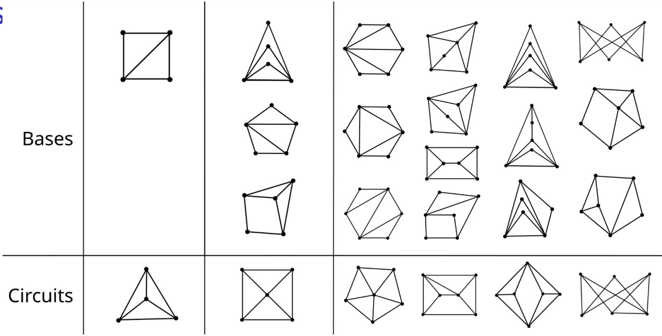
Theorem (Geiringer-Laman)

A graph $G \subseteq \binom{[\ell]}{2}$ is independent in the rigidity matroid $\mathcal{R}_\ell$ if and only if it $(2,3)$-sparse. The bases of $\mathcal{R}_\ell$ are obtained from K_2 by successively performing two operations:

- *add a new vertex and connect it to two old vertices;*
- *subdivide an edge by a vertex and connect to an old vertex.*

G is $(2,4)$-sparse if and only if it contains no induced basis on ≥ 3 vertices.

Bases and Circuits



Theorem

Let G be the edge graph of any triangulation of the ℓ-gon. Then V_G has $2^{\ell-2}$ irreducible components, each of codimension $2\ell - 3$, one for each bicoloring of the triangles.

Theorem

The wheel graph $G = W_\ell$ has precisely $2^{\ell-1} - 2(\ell - 1)$ irreducible components.

Codimension

Define X_G as the Zariski closure in $V_G \subseteq \mathrm{Gr}(2,4)^\ell$ of all configurations of ℓ *distinct* lines satisfying the incidences of G.

Theorem

For any graph G, we have $\mathrm{codim}(X_G) = \mathrm{rank}(G)$. Hence,

$$\mathrm{codim}(V_G) \leq \mathrm{rank}_{\mathcal{R}_\ell}(G) \leq \min\{|G|, 2\ell - 3\}.$$

Corollary

If V_G is a complete intersection, then G is independent in $\mathcal{R}_\ell$.

Codimension

Define X_G as the Zariski closure in $V_G \subseteq \mathrm{Gr}(2,4)^\ell$ of all configurations of ℓ *distinct* lines satisfying the incidences of G.

Theorem

For any graph G*, we have* $\mathrm{codim}(X_G) = \mathrm{rank}(G)$*. Hence,*

$$\mathrm{codim}(V_G) \leq \mathrm{rank}_{\mathcal{R}_\ell}(G) \leq \min\{|G|, 2\ell - 3\}.$$

Corollary

If V_G *is a complete intersection, then* G *is independent in* $\mathcal{R}_\ell$*.*

Theorem

For any graph G*, we have* $\mathrm{codim}(V_G) = \min_{J \in \mathcal{P}_\ell} \left\{\mathrm{rank}(G_J) + 4s_J\right\}.$

If G *is independent then* $\mathrm{codim}(V_G) = \min_{J \in \mathcal{P}_\ell} \left\{|G_J| + 4s_J\right\}.$

Must define $\mathcal{P}_\ell, G_J, s_J$

Some Definitions

Let $\mathcal{P}_\ell$ be the set of partitions of $[\ell]$. For $J = \{J_1, \ldots, J_r\} \in \mathcal{P}_\ell$, we omit singletons, i.e. $J_1, \ldots, J_r \subseteq [\ell]$ are disjoint and $|J_i| > 1$.

Set $s_J := |J| - r$ where $|J| := |J_1| + \cdots + |J_r|$.

Let G_J be the graph on $[\ell - s_J]$ obtained from G by identifying vertices in each J_i, merging parallel edges, and deleting loops.

If $J = \emptyset$, then $G_J = G$ and $s_J = 0$.

Some Definitions

Let $\mathcal{P}_\ell$ be the set of partitions of $[\ell]$. For $J = \{J_1, \ldots, J_r\} \in \mathcal{P}_\ell$, we omit singletons, i.e. $J_1, \ldots, J_r \subseteq [\ell]$ are disjoint and $|J_i| > 1$.

Set $s_J := |J| - r$ where $|J| := |J_1| + \cdots + |J_r|$.

Let G_J be the graph on $[\ell - s_J]$ obtained from G by identifying vertices in each J_i, merging parallel edges, and deleting loops.

If $J = \emptyset$, then $G_J = G$ and $s_J = 0$.

If $\tilde{X}_{G_J} \subseteq V_G$ is the variety where lines in each J_i coincide, then

$$V_G \quad = \quad \bigcup_{J \in \mathcal{P}_\ell} \tilde{X}_{G_J}.$$

Note: $\operatorname{codim}(\tilde{X}_{G_J}) = \operatorname{codim}(X_{G_J}) + 4s_J$.

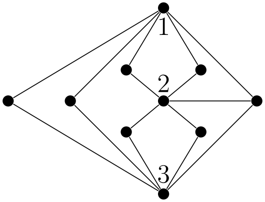
A graph G is *contraction-stable* if

$$|G_J| + 4s_J \;\geq\; |G| \quad \text{for all } J \in \mathcal{P}_\ell.$$

G is *strictly contraction-stable* if $>$ holds for $J \neq \emptyset$.

Three Theorems

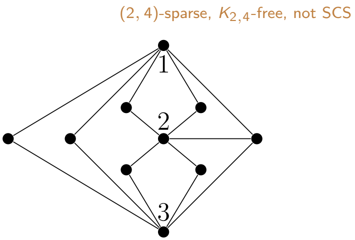
(2, 4)-sparse, $K_{2,4}$-free, not SCS



Theorem (Complete Intersection)

The incidence variety V_G is a complete intersection if and only if the graph G is $(2, 3)$-sparse and contraction-stable.

Three Theorems



Theorem (Complete Intersection)

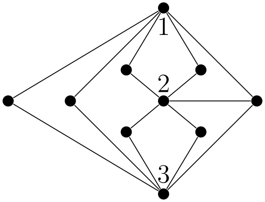
The incidence variety V_G is a complete intersection if and only if the graph G is $(2,3)$-sparse and contraction-stable.

Theorem (Irreducibility)

V_G is irreducible if and only if G is $(2,4)$-sparse and strictly contraction-stable. In this case, $V_G = W_G = X_G$, and this irreducible variety is a unirational complete intersection.

Three Theorems

(2, 4)-sparse, $K_{2,4}$-free, not SCS



Theorem (Complete Intersection)

The incidence variety V_G is a complete intersection if and only if the graph G is $(2,3)$-sparse and contraction-stable.

Theorem (Irreducibility)

V_G is irreducible if and only if G is $(2,4)$-sparse and strictly contraction-stable. In this case, $V_G = W_G = X_G$, and this irreducible variety is a unirational complete intersection.

Theorem

If V_G is irreducible, then the ideal I_G is prime.

Six, Seven and Eight Lines

ℓ	Connected G	Compl Inters	Irreducible	$W_G \neq \emptyset$
4	6	5	3	6
5	21	16	6	21
6	112	69	17	103
7	853	379	52	681

Table: Numerical irreducible decomposition of V_G for connected graphs G with $\ell = 4, 5, 6, 7$. The four columns report the number of all graphs G up to isomorphism, the number whose variety V_G is a complete intersection, whose V_G is irreducible, and whose W_G is non-empty.

Six, Seven and Eight Lines

ℓ	Connected G	Compl Inters	Irreducible	$W_G \neq \emptyset$
4	6	5	3	6
5	21	16	6	21
6	112	69	17	103
7	853	379	52	681

Table: Numerical irreducible decomposition of V_G for connected graphs G with $\ell = 4, 5, 6, 7$. The four columns report the number of all graphs G up to isomorphism, the number whose variety V_G is a complete intersection, whose V_G is irreducible, and whose W_G is non-empty.

ℓ	Triangle-free G	Compl Inters	Irreducible	$W_G \neq \emptyset$
4	3	3	3	3
5	6	6	6	6
6	19	19	17	19
7	59	57	52	59
8	267	254	219	266

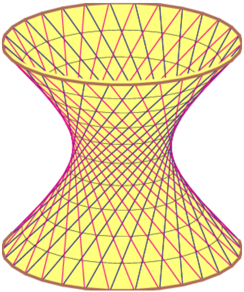
Table: Numerical irreducible decomposition of V_G for triangle-free connected graphs G with $\ell = 5, 6, 7, 8$.

Bipartite Graphs

Example ($\ell = 8$, triangle-free)

Among the 267 graphs, 266 are realizable. The unique graph G with $W_G = \emptyset$ is $K_{4,4}$ with one edge removed. Its variety V_G has 63 irreducible components of codim $13, 14, 15$.

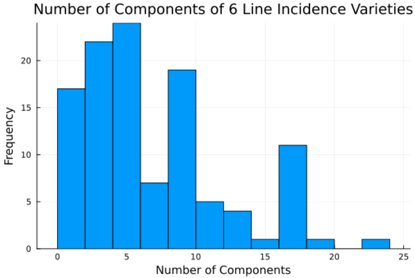
For $\ell \leq 7$, every triangle-free graph is realizable.

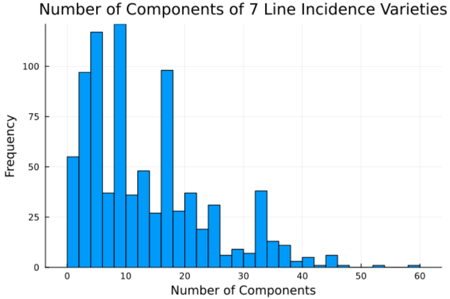


Proposition

Let $G = K_{a,b}$ where $a, b \geq 3$. Then W_G is irreducible, and it parametrizes $a + b$ lines on some quadric surface, with a lines in one ruling and b lines in the other ruling.

How Many Components?





Proposition (Winner for $\ell = 7$)

Let G be the complete graph K_7 with a 7-cycle removed, so $|G| = 14$. Then V_G has 58 irreducible components, which is the maximum among all graphs with $\ell = 7$. The graph G represents an incidence theorem for lines in 3-space, i.e. we have $W_G = \emptyset$.

Multidegree

The incidence variety V_G lives in $\mathrm{Gr}(2,4)^\ell$ and hence in $(\mathbb{P}^5)^\ell$.
The cohomology ring of $(\mathbb{P}^5)^\ell$ is the truncated polynomial ring

$$H^*\big((\mathbb{P}^5)^\ell, \mathbb{Z}\big) \;=\; \mathbb{Z}[t_1, t_2, \ldots, t_\ell]/\langle t_1^6, t_2^6, \ldots, t_\ell^6\rangle\,.$$

The *multidegree* of V_G is the cohomology class

$$[V_G] \;=\; \sum_{u \in \mathbb{N}^\ell} \gamma_u \, t_1^{5-u_1} t_2^{5-u_2} \cdots \; t_\ell^{5-u_\ell}.$$

This polynomial can be computed in `Macaulay2` with the command `multidegree`, or numerically by intersecting in $(\mathbb{P}^5)^\ell$.

Theorem

The coefficient γ_u in the multidegree of V_G is the leading singularity degree of the auxiliary graph G_u.

Fact: Our earlier degree is $2^{-\ell}\sum_u \gamma_u$.

Schubert Problems

Write $d := 4\ell - \mathrm{codim}(V_G)$ for the dimension of V_G.

Fix $u \in \mathbb{N}^\ell$ with $u_1 + \cdots + u_\ell = d$. Create G_u from G
by attaching u_i edges at vertex i, for each $i \in [\ell]$.

The graph G_u has $\ell + d$ vertices and $|G| + d$ edges.

The map $\psi_u : V_{G_u} \to \mathrm{Gr}(2,4)^d$ deletes the ℓ original lines.

Fibers are ψ_u are given by d constraints on a d-dim'l variety,
each a Schubert condition on one of the ℓ lines.

The *LS degree* of G_u is the cardinality of the generic fiber of ψ_u
if this is finite; and 0 otherwise.

Schubert Problems

Write $d := 4\ell - \mathrm{codim}(V_G)$ for the dimension of V_G.

Fix $u \in \mathbb{N}^\ell$ with $u_1 + \cdots + u_\ell = d$. Create G_u from G by attaching u_i edges at vertex i, for each $i \in [\ell]$.

The graph G_u has $\ell + d$ vertices and $|G| + d$ edges.

The map $\psi_u : V_{G_u} \to \mathrm{Gr}(2,4)^d$ deletes the ℓ original lines.

Fibers are ψ_u are given by d constraints on a d-dim'l variety, each a Schubert condition on one of the ℓ lines.

The *LS degree* of G_u is the cardinality of the generic fiber of ψ_u if this is finite; and 0 otherwise.

Example ($\ell = 1$)

G = one vertex and no edge. Then $d = u_1 = 4$, and $G_u = \{12, 13, 14, 15\}$ is a star tree. Points on V_{G_u} are ... ? quiz

The map ψ_u is onto $\mathrm{Gr}(2,4)^4$. Generic fibers = two lines that meet four given lines in $\mathbb{P}^3$. The LS degree of G_u is 2.

Multidegree Formulas

Corollary

If V_G is a complete intersection then

$$[V_G] \;=\; 2^{\ell} \cdot \prod_{i=1}^{\ell} t_i \cdot \prod_{ij \in G} (t_i + t_j)\,.$$

Corollary (Escobar-Knutson)

The multidegree of the concurrent lines variety equals

$$[V_{[\ell]}] \;=\; t_1^3 t_2^3 \cdots t_\ell^3 \;\cdot\; \left(4 \cdot \sum_{(i,j)} t_i^{-2} t_j^{-1} \;+\; 8 \cdot \sum_{\{i,j,k\}} t_i^{-1} t_j^{-1} t_k^{-1} \right).$$

L. Escobar and A. Knutson:
The multidegree of the multi-image variety,
Combinatorial Algebraic Geometry,
Fields Institute Comm, 2017.

Triangle Revisited

The complete graph K_3 has multidegree

$$[V_{K_3}] = [V_{123}] + [V_{123}^*] = 8 \cdot t_1 t_2 t_3 (t_1 + t_2)(t_1 + t_3)(t_2 + t_3)$$

$$= 8 \cdot \left(\sum_{\pi \in S_3} t_{\pi(1)}^3 t_{\pi(2)}^2 t_{\pi(3)} \right) + 16 \cdot t_1^2 t_2^2 t_3^2$$

Here $u = (4, 3, 2)$ and $u = (3, 3, 3)$.

Quiz: Explain the Schubert problem on G_u with these 8 resp. 16 solutions.

Triangle Revisited

The complete graph K_3 has multidegree

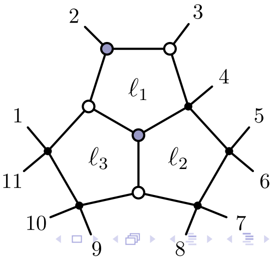
$$[V_{K_3}] = [V_{123}] + [V^*_{123}] = 8 \cdot t_1 t_2 t_3 (t_1 + t_2)(t_1 + t_3)(t_2 + t_3)$$

$$= 8 \cdot \left(\sum_{\pi \in S_3} t^3_{\pi(1)} t^2_{\pi(2)} t_{\pi(3)} \right) + 16 \cdot t_1^2 t_2^2 t_3^2$$

Here $u = (4, 3, 2)$ and $u = (3, 3, 3)$.

Quiz: Explain the Schubert problem on G_u with these 8 resp. 16 solutions.

J. Bourjaily, C. Vergu, M. von Hippel:
Landau singularities and higher-order roots,
Physical Review D, 2023



Sounds like physics?

Towards Physics

Identify $\mathbb{R}^4$ with Minkowski spacetime $\mathbb{R}^{1,3}$ instead of $\mathbb{R}^{2,2}$

Wick rotation: Replace our earlier 2×4 matrices **A** and **B** with

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & p_0 - p_3 & p_1 + ip_2 \\ 0 & 1 & p_1 - ip_2 & p_0 + p_3 \end{pmatrix} \text{ and } \mathbf{Q} = \begin{pmatrix} 1 & 0 & q_0 - q_3 & q_1 + iq_2 \\ 0 & 1 & q_1 - iq_2 & q_0 + q_3 \end{pmatrix}$$

Incidence equation

$$\widetilde{PQ} \;=\; \det \begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix} \;=\; (p_0 - q_0)^2 - (p_1 - q_1)^2 - (p_2 - q_2)^2 - (p_3 - q_3)^2$$

Towards Physics

Identify $\mathbb{R}^4$ with Minkowski spacetime $\mathbb{R}^{1,3}$ instead of $\mathbb{R}^{2,2}$

Wick rotation: Replace our earlier 2×4 matrices **A** and **B** with

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & p_0 - p_3 & p_1 + ip_2 \\ 0 & 1 & p_1 - ip_2 & p_0 + p_3 \end{pmatrix} \text{ and } \mathbf{Q} = \begin{pmatrix} 1 & 0 & q_0 - q_3 & q_1 + iq_2 \\ 0 & 1 & q_1 - iq_2 & q_0 + q_3 \end{pmatrix}$$

Incidence equation

$$\widetilde{PQ} \;=\; \det\begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix} \;=\; (p_0 - q_0)^2 - (p_1 - q_1)^2 - (p_2 - q_2)^2 - (p_3 - q_3)^2$$

Lemma

The Grassmannian $\mathrm{Gr}(2,4)$ *is a natural compactification of spacetime* $\mathbb{R}^{1,3}$.
Two particles are light-like separated if and only if the lines P *and* Q *intersect.*

Fact: The boundary of the lightcone is a Schubert divisor in $\mathrm{Gr}(2,4)$.

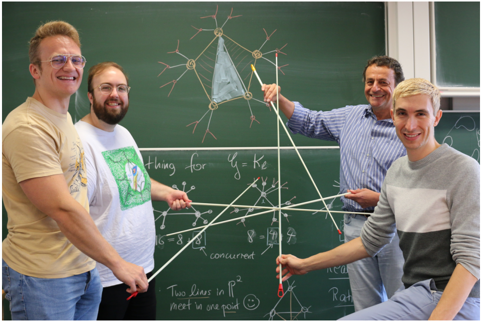
Corollary 8.2. *The incidence variety V_G parametrizes configurations of ℓ particles in $\mathbb{R}^{1,3}$, where light-like separation is prescribed for pairs of particles that are edges in the graph G.*

We now take a step back and take a quick look at the larger scientific landscape. Particle physics studies fundamental interactions among elementary particles in nature. A *scattering experiment* consists in making n particles with momenta $P, Q, \ldots \in \mathbb{R}^{1,3}$ collide in a particle accelerator, such as the Large Hadron Collider at CERN. By repeating the experiment many times, experimental physicists measure the joint probability for different outcomes as a function of the momenta. This probability can be obtained as the square modulus of its probability amplitude, called *scattering amplitude*. Theoretical physicists use the framework of Quantum Field Theory (QFT) to compute such scattering amplitudes, both to compare with scattering experiments and to advance the mathematical understanding of QFT itself.

Scattering amplitudes are computed by summing *Feynman integrals* [19]. These are functions of the momenta $P, Q, \ldots \in \mathbb{R}^{1,3}$ and are expressed as integrals of rational functions in 4ℓ variables. Their arguments are ℓ *loop momenta* $K, L, \ldots \in \mathbb{R}^{1,3}$. That rational function in 4ℓ variables is singular when some linear combinations of momenta and loop momenta is light-like separated. This is where Lemma 8.1 comes in. In the geometric approach we pursue, the n momenta $P, Q, \ldots$ and the ℓ loop momenta $K, L, \ldots$ are lines in 3-space, given by points in the Grassmannian $\mathrm{Gr}(2, 4)$. Light-like separation means that two lines intersect.

The rational functions to be integrated have denominators that are products of quadrics like $\widetilde{PQ}$ in (29). The relation between the singular locus of these integrands and the singularities of Feynman integrals and scattering amplitudes is the subject of the *Landau analysis*. Recent work in [6] connects Landau analysis to computational algebraic geometry. However, their geometry looks disturbingly different from ours. This is a feature, not a bug. By [19, Section 2.5], there many different ways to write a Feynman integral. Ours closely relates to the momentum representation, while Fevola et al. [6] use the Lee-Pomeransky representation.

The End



The original motivation for this project comes from particle physics. The varieties V_G and V_{G_u} are essential in the Landau analysis of scattering amplitudes that are expressed in Grassmannian coordinates. This is the topic of our forthcoming companion paper [12], which will be aimed at a physics audience. Section 8 offers a friendly invitation for mathematicians.

B. Hollering, E. Mazzucchelli, M. Parisi, BSt:
Landau analysis in the Grassmannian
2026 (coming soon)

Thanks for listening