# Quotient branching laws for $p$-adic general linear groups and affine Hecke algebras of type A

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ESI workshop: Minimal representations and theta correspondence

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# Some Joint work with Gordan Savin

- Bernstein-Zelevinsky derivatives: a Hecke algebra approach, IMRN, (2019)
- Iwahori component of the Gelfand-Graev representation, Math. Z. (2018)

An earlier work on Hecke algebra by Gordan:

Math. Ann. 280, 185–190 (1988)

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# Quotient branching law

Let $G_n = \mathrm{GL}_n(F)$, where $F$ is non-Archimedean. View $G_n$ as subgp. of $G_{n+1}$ via $g \mapsto \begin{pmatrix} g & \\ & 1 \end{pmatrix}$.

**Theorem (Aizenbud-Gourevitch-Rallis-Schiffman 10, Sun-Zhu 12)**

*Let* $\pi_1 \in \mathrm{Irr}(G_{n+1})$ *and let* $\pi_2 \in \mathrm{Irr}(G_n)$. *Then*

$$\dim \mathrm{Hom}_{G_n}(\pi_1, \pi_2) \leq 1.$$

Question: when $\mathrm{Hom}_{G_n}(\pi_1, \pi_2) \neq 0$? We will give an answer at the end. We first state the following improved result:

**Theorem (C.)**

*Let* $\lambda_1$ *and* $\lambda_2$ *be standard representations (in the sense of Langlands) of* $G_{n+1}$ *and* $G_n$ *respectively. Then*

$$\mathrm{Hom}_{G_n}(\lambda_1, \lambda_2^\vee) \cong \mathbb{C}, \quad \mathrm{Ext}^i(\lambda_1, \lambda_2^\vee) = 0 \quad i \geq 1$$

# Some previous known cases for quotient branching laws

1. Generic representations (Jacquet–Piateski-Shapiro–Shalika 83)
2. Distinguished case i.e. $\pi_2$ is trivial repn of $G_n$ and $\pi_1$ is arbitrary (Prasad 93 for $n = 2$, Venketasubramanian 13 for general $n$)
3. $\pi_1$ is a generalized Steinberg repn of $G_{n+1}$ and $\pi_2$ is arbitrary (C.-Savin 21)
4. Arthur-type representations, non-tempered Gan-Gross-Prasad conj. 20 (C. 22, earlier important work of Gurevich 22)

# Bernstein-Zelevinsky derivatives

Now we turn to Bernstein-Zelevinsky derivatives. It is based on the following preprint:

- Construction of simple quotients of Bernstein-Zelevinsky derivatives and highest derivative multisegments, arXiv:2111.13286

# Bernstein-Zelevinsky derivatives

- Let $R_i$ contain all unipotent matrices of the form

$$\begin{pmatrix} I_{n-i} & * \\ & u \end{pmatrix}, \quad u \in U_i \text{ (upper triangular matrices).}$$

- Let $\psi_i : U_i \to \mathbb{C}$ be a non-degenerate character e.g.

$$\psi_i(u) = \bar{\psi}(u_{1,2} + \ldots + u_{n-1,n})$$

  for a non-trivial character $\bar{\psi} : F \to \mathbb{C}$. Extend trivially to a character $\psi : R_i \to \mathbb{C}$.

- For a $G_n$-representation $\pi$, define

$$\pi^{(i)} := \pi_{R_i,\psi}$$

  to be the maximal quotient which $R_i$ acts by $\psi$ invariantly, so-called $\psi$-twisted Jacquet functor, or *$i$-th Bernstein-Zelevinsky derivative* i.e.

$$\pi^{(i)} = \delta_{R_i}^{-1/2} \cdot \frac{\pi}{\langle u.x - \psi(u)x : u \in R_i, x \in \pi \rangle}.$$

# Some main properties of BZ derivatives

- $\pi \mapsto \pi^{(i)}$ is an exact functor.
- $\pi \mapsto \pi^{(i)}$ preserves finite lengthness.
- For $\pi \in$ Irr, the socle (max. semisimple submodule) and cosocle (max. semisimple quotient) of $\pi^{(i)}$ are multiplicity-free (following from BZ theory and AGRS).
- $\mathrm{soc}(\pi^{(i)}) \cong \mathrm{cosoc}(\pi^{(i)})$

# Some main properties of BZ derivatives

- (Zelevinsky 80) The level of $\pi$ is the largest integer $i$ such that $\pi^{(i)} \neq 0$. When $i$ is the level of $\pi$, highest derivative $\pi^{(i)}$ is irreducible. Hence one can repeat the process, and this gives a partition corresponding to its wavefront set.
- Descriptions of BZ-derivatives:
  1. sq.-integrable representations (Zelevinsky 80)
  2. ladder repns, including Speh repns (Lapid-Mínguez 14, earlier work of Tadić 87)
- One also defines a left BZ derivative using transpose $R_i^t$:

$$^{(i)}\pi = \pi_{R_i^t, \psi^t}$$

- Asymmetry property of simple quotients of Left-Right BZ derivatives (C. 21)

# How BZ derivatives relate to the branching problems of $p$-adic groups and affine Hecke algebras?

Short answers:

1. Bernstein-Zelevinsky filtration gives: for $\pi_1 \in \mathrm{Irr}(G_{n+1})$ and $\pi_2 \in \mathrm{Irr}(G_n)$,

$$\mathrm{Hom}_{G_n}(\pi_1, \pi_2) \neq 0 \Rightarrow \mathrm{Hom}_{G_{n-i}}(\nu^{1/2} \cdot \pi_1^{(i)}, {}^{(i-1)}\pi_2) \neq 0,$$

   for some $i$, where $\nu(g) = |\mathrm{det} g|_F$ is a character of $G_{n-i}$.
2. Affine Hecke algebra: Categorical equivalence between smooth repn category of $p$-adic groups and affine Hecke algebra module category

# St-derivatives

Let us first recall the classification of ess. sq. integrable repns.

- A segment is a data $[a,b]_\rho$ for $a,b \in \mathbb{Z}$ with $b-a \in \mathbb{Z}_{\geq 0}$ and a cuspidal repn $\rho$, and we shall regard as a set:

$$\left\{ \nu^a\rho, \nu^{a+1}\rho, \ldots, \nu^b\rho \right\}.$$

- (Zelevinsky 80) Essentially sq.-integrable repns $\mathrm{St}([a,b]_\rho)$ are parametrized by segments $[a,b]_\rho$.
- Explicit construction: $\mathrm{St}([a,b]_\rho)$ is the unique irr. quotient of

$$\nu^a\rho \times \ldots \times \nu^b\rho.$$

(product means corresponding parabolic induction)

# St-derivatives

**Theorem (Jantzen 07 and Mínguez 09 for $\Delta = \rho$, Lapid-Mínguez 16 for general, etc)**

*Let* $\mathrm{St}(\Delta)$ *be a essentially sq.-integrable repn of* $G_r$*. There is at most one (up to isomorphism)* $\tau \in \mathrm{Irr}(G_{n-r})$ *such that, as* $G_{n-r} \times G_r$*-module,*

$$\tau \boxtimes \mathrm{St}(\Delta) \hookrightarrow \pi_{N_r}$$

If such $\tau$ exists, we shall denote such $\tau$ by $D_\Delta(\pi)$, otherwise set to $D_\Delta(\pi) = 0$.

More general result for $\square$-irr. repns is by Kang-Kashiwara-Kim-Oh.

**Remark**

Derivatives are also used for

- theta correspondence (e.g. Mínguez, Gan-Takeda, Bakić-Hanzer)
- Aubert-Zelevinsky duals (e.g. Mœglin-Waldspurger, Atobe-Mínguez)
- Arthur packets (e.g. Mœglin, Xu, Atobe, Hazeltine-Liu-Lo)

# Simple quotients of Bernstein-Zelevinsky derivatives

For a multisegment $\mathfrak{m} = \{\Delta_1, \ldots, \Delta_r\}$, we arrange $\Delta_1 = [a_1, b_1]_\rho, \ldots, \Delta_r = [a_r, b_r]_\rho$ in an ascending order: $a_1 \leq \ldots \leq a_r$. Then, for $\pi \in \mathrm{Irr}_\rho$,

$$D_{\mathfrak{n}}(\pi) = D_{\Delta_r} \circ \ldots \circ D_{\Delta_1}(\pi).$$

## Proposition

Let $\pi \in \mathrm{Irr}_\rho$. Then

$$D_{\Delta_r} \circ \ldots \circ D_{\Delta_1}(\pi)$$

is a simple quotient (and submodule) of $\pi^{(i)}$, where $i$ is sum of lengths of segments.

Proof: Frobenius reciprocity gives:

$$D_{\Delta_r} \circ \ldots \circ D_{\Delta_1}(\pi) \boxtimes \mathrm{St}(\Delta_1) \times \ldots \times \mathrm{St}(\Delta_r) \to \pi_{N_i}$$

and use the unique simple quotient of $\mathrm{St}(\Delta_1) \times \ldots \times \mathrm{St}(\Delta_r)$ is generic (Jacquet-Shalika 83, related to the injectivity conj. of Casselman-Shalika)

# Multisegments and simple quotients

Let $\pi \in \mathrm{Irr}_\rho$. Let $\tau$ be a simple quotient of $\pi^{(i)}$ for some $i$. Define

$$\mathcal{S}(\pi, \tau) := \{\mathfrak{n} \in \mathrm{Mult}_\rho : D_\mathfrak{n}(\pi) \cong \tau\}$$

Questions on $\mathcal{S}(\pi, \tau)$:

1. Combinatorial structure?
2. Is it non-empty?

# Combinatorial structure of $\mathcal{S}(\pi, \tau)$

A multisegment $\mathfrak{m}$ is obtained from $\mathfrak{n}$ by an elementary intersection-union operatn if $\exists$ a pair of (linked) segments $\Delta_1, \Delta_2$ in $\mathfrak{n}$ such that

$$\mathfrak{m} = \mathfrak{n} - \{\Delta_1, \Delta_2\} + \{\Delta_1 \cup \Delta_2, \Delta_1 \cap \Delta_2\}.$$

If $\mathfrak{m}$ is obtained from $\mathfrak{n}$ by a sequence of elementary intersection-union processes, then we write

$$\mathfrak{m} \leq_Z \mathfrak{n}.$$

**Theorem (C.)**

*(Closedness property) Suppose $\mathfrak{m}, \mathfrak{n} \in \mathcal{S}(\pi, \tau)$ with $\mathfrak{m} \leq_Z \mathfrak{n}$. If $\mathfrak{n}'$ satisfies $\mathfrak{m} \leq_Z \mathfrak{n}' \leq_Z \mathfrak{n}$, then $\mathfrak{n}' \in \mathcal{S}(\pi, \tau)$.*

**Theorem (C. )**

*(Uniqueness of minimality) If $\mathcal{S}(\pi, \tau) \neq \emptyset$, then $\mathcal{S}(\pi, \tau)$ has a unique minimal element w.r.t. $\leq_Z$.*

Remark: No such uniqueness for maximality in general

# Highest derivative multisegments: a special case of minimality

For $\mathfrak{m} \in \mathrm{Mult}_\rho$, $\mathfrak{m}$ is said to be a multisegment at a point $c$ if any segment $\Delta$ in $\mathfrak{m}$ takes the form $[c, b]_\rho$ for some $b$.

**Lemma**

*For any integer $c$, there exists a unique maximal multisegment $\mathfrak{m}_c$ at the point $c$ such that $D_{\mathfrak{m}_c}(\pi) \neq 0$.*

Define the highest derivative multisegment of $\pi$ as:

$$\mathfrak{hd}(\pi) = \sum_c \mathfrak{m}_c$$

**Theorem (C. )**

*$\mathfrak{hd}(\pi)$ is the unique $\leq_Z$-minimal element in $\mathcal{S}(\pi, \pi^-)$, where $\pi^- = \pi^{(i^*)}$ for the level $i^*$ of $\pi$.*

# Representation-theoretic interpretation of minimaltiy

We have a unique non-zero map:

$$D_{\Delta_r} \circ \ldots \circ D_{\Delta_1}(\pi) \boxtimes (\mathrm{St}(\Delta_1) \times \ldots \times \mathrm{St}(\Delta_r)) \to \pi_N$$

**Conjecture**

If the ascending sequence $\Delta_1, \ldots, \Delta_r$ forms a $\leq_Z$-minimal multisegment for $\pi$,

$$D_{\Delta_r} \circ \ldots \circ D_{\Delta_1}(\pi) \boxtimes (\mathrm{St}(\Delta_1) \times \ldots \times \mathrm{St}(\Delta_r)) \hookrightarrow \pi_N$$

is an injection.

# Properties of minimal sequences

**Theorem (C.)**

*(Subsequent property) Let* $\pi \in \mathrm{Irr}_\rho$. *Let* $\mathfrak{n} \in \mathrm{Mult}_\rho$ *be minimal to* $\pi$. *Then, for any submultisegment* $\mathfrak{n}'$ *of* $\mathfrak{n}$, $\mathfrak{n}'$ *is still minimal to* $\pi$.

**Theorem (C.)**

*(Commutativity and preserving minimality) Let* $\pi \in \mathrm{Irr}_\rho$. *Let* $\mathfrak{n} \in \mathrm{Mult}_\rho$ *be minimal to* $\pi$. *Let* $\Delta \in \mathfrak{n}$. *Then*

$$D_{\mathfrak{n}-\Delta} \circ D_\Delta(\pi) \cong D_{\mathfrak{n}}(\pi),$$

*and* $\mathfrak{n} - \Delta$ *is minimal to* $D_\Delta(\pi)$.

A special case is that $\mathfrak{n}$ contains two linked segments. In such case, it can be rephrased as: for $\Delta_1 < \Delta_2$, if $D_{\Delta_2} \circ D_{\Delta_1}(\pi) \not\cong D_{\Delta_2 \cup \Delta_1} \circ D_{\Delta_2 \cap \Delta_1}(\pi)$, then

$$D_{\Delta_2} \circ D_{\Delta_1}(\pi) \cong D_{\Delta_1} \circ D_{\Delta_2}(\pi).$$

# $\epsilon$-invariant

We have shown several properties of $\mathcal{S}(\pi, \tau)$ and properties of the minimal sequence. The key is to transfer the problem to combinatorics and we now introduce $\epsilon$-invariant.

For a segment $\Delta = [a, b]_\rho$ and an irr. repn $\pi$, let $\epsilon_\Delta(\pi)$ be the largest (non-negative) integer such that

$$\overbrace{D_\Delta \circ \ldots \circ D_\Delta}^{k \text{ times}}(\pi) \neq 0.$$

Roughly speaking, the highest derivative multisegment $\mathfrak{hd}(\pi)$ contains all the information of those $\epsilon_\Delta$-invariants. For a segment $\Delta$ and $\pi \in \mathrm{Irr}$, one defines a multisegment $\mathfrak{r}(\Delta, \pi)$ that record the change of $\mathfrak{hd}(\pi)$ to $\mathfrak{hd}(D_\Delta(\pi))$. (We won't give a precise defn for $\mathfrak{r}$.)

# Criteria for elements in $\mathcal{S}(\pi, \tau)$

Let $\mathfrak{n} \in \mathrm{Mult}_\rho$ with segments $\Delta_1, \ldots, \Delta_r$ in ascending order. Define

$$\mathfrak{r}(\mathfrak{n}, \pi) := \mathfrak{r}(\Delta_r, \ldots \mathfrak{r}(\Delta_1, \mathfrak{hd}(\pi))..)$$

**Theorem (C.)**

*Let* $\mathfrak{n}_1, \mathfrak{n}_2 \in \mathrm{Mult}_\rho$. *Then* $\mathfrak{n}_1, \mathfrak{n}_2 \in \mathcal{S}(\pi, \tau)$ *i.e.* $D_{\mathfrak{n}_1}(\pi) \cong D_{\mathfrak{n}_2}(\pi)$ *if and only if* $\mathfrak{r}(\mathfrak{n}_1, \pi) = \mathfrak{r}(\mathfrak{n}_2, \pi)$.

We have the following double derivatives result:

**Theorem (C.)**

*Let* $\mathfrak{n} \in \mathrm{Mult}_\rho$. *Suppose* $D_{\mathfrak{n}}(\pi) \neq 0$. *Then*

$$D_{\mathfrak{r}(\mathfrak{n},\pi)} \circ D_{\mathfrak{n}}(\pi) = \pi^-.$$

# $\eta$-invariant

For a segment $[a,b]_\rho$, define a finer invariant:

$$\eta_\Delta(\pi) = (\epsilon_{[a,b]_\rho}(\pi), \epsilon_{[a-1,b]_\rho}(\pi), \ldots, \epsilon_{[b]_\rho}(\pi)).$$

A key property is the following:

**Proposition**

For $\pi \in \mathrm{Irr}$, let

$$\mathfrak{n} = \{ \overbrace{[a,b]_\rho}^{\epsilon_{[a,b]_\rho}(\pi) \text{ times}}, \ldots, \overbrace{[b]_\rho}^{\epsilon_{[b]_\rho}(\pi) \text{ times}} \}.$$

Then $D_{\mathfrak{n}}(\pi) \boxtimes \mathrm{St}(\mathfrak{n})$ is a direct summand of $\pi_N$.

It is based on the proof of the case that $\Delta$ is a singleton $[b]_\rho$ by Grojnowski-Vazirani, Jantzen, Mínguez.

# Right-derivative and Left-integral

For $\pi \in \mathrm{Irr}$ and a segment $\Delta$, denote, by $I_\Delta(\pi)$ (or $I^L_\Delta(\pi)$), the unique submodule of

$$\mathrm{St}(\Delta) \times \pi.$$

One may think that the integral is an 'inverse' of (left version) of derivatives i.e.

$$D^L_\Delta \circ I^L_\Delta(\pi) \cong \pi.$$

# Combinatorially RdLi-commutative triple

## Definition

Let $\pi \in \mathrm{Irr}$. Let $\Delta_1$ and $\Delta_2$ be two segments. We say that $(\Delta_1, \Delta_2, \pi)$ is a combinatorially Riderivative-Leintegral (RdLi) commutative triple if

$$\eta_{\Delta_1}(I^L_{\Delta_2}(\pi)) = \eta_{\Delta_1}(\pi).$$

(Recall that $\eta$ is defined for the derivatives on the 'right'.)

The notion of combinatorial commutation suggests from the following result:

## Theorem (C.)

*If* $(\Delta_1, \Delta_2, \pi)$ *is a combinatorially RdLi-commutative triple, then*

$$I^L_{\Delta_2} \circ D^R_{\Delta_1}(\pi) \cong D^R_{\Delta_1} \circ I^L_{\Delta_2}(\pi).$$

The converse of above is not true in general.

# Strong commutation for multisegments

## Definition

Let $\pi \in \mathrm{Irr}$. Let $\mathfrak{m}, \mathfrak{n}$ be multisegments. Write the segments for $\mathfrak{m}$ in an ascending order:

$$\Delta_1, \ldots, \Delta_r$$

and for $\mathfrak{n}$ in ascending order:

$$\Delta'_1, \ldots, \Delta'_s.$$

We say that $(\mathfrak{m}, \mathfrak{n}, \pi)$ is a strongly RdLi-commutative triple if for each $1 \leq k \leq r$ and $1 \leq l \leq s$,

$$(\Delta_i, \Delta'_j, I^L_{\{\Delta'_1, \ldots, \Delta'_{j-1}\}} \circ D^R_{\{\Delta_1, \ldots, \Delta_{i-1}\}}(\pi))$$

is a combinatorially RdLi-commutative triple.

# (Generalized) Relevant pairs

**Definition**

Let $\pi \in \mathrm{Irr}$ and let $\pi' \in \mathrm{Irr}$. We say that $(\pi, \pi')$ is relevant if there exist multisegments $\mathfrak{m}$ and $\mathfrak{n}$ such that $(\mathfrak{m}, \mathfrak{n}, \nu^{1/2} \cdot \pi)$ is a strongly RdLi-commutative triple and

$$I^L_{\mathfrak{n}} \circ D^R_{\mathfrak{m}}(\nu^{1/2} \cdot \pi) = \pi'.$$

- One may think $D^R_{\mathfrak{m}}$ is the BZ derivatives while $I^L_{\mathfrak{n}}$ is the BZ-induction (the adjoint functor of BZ derivatives).
- Gan-Gross-Prasad (2020) define a notion of relevant pairs for Arthur type repns (for classical groups). Checking combinatorially GGP relevance $\Rightarrow$ above relevance is easier.

Indeed, the notion is symmetric (and so compatible with left and right branching laws):

**Theorem (C.)**

*$(\pi, \pi')$ is relevant if and only if $(\pi', \pi)$ is relevant.*

# Quotient branching law

**Theorem A (C.)**

Let $\pi \in \mathrm{Irr}(G_{n+1})$ and let $\pi' \in \mathrm{Irr}(G_n)$. Then $(\pi, \pi')$ is relevant if and only if $\mathrm{Hom}_{G_n}(\pi, \pi') \neq 0$.

Let $\pi \in \mathrm{Irr}(G_{n+1})$. If $\tau$ is a simple quotient of $\pi^{(i)}$ for some $i$, then we can form an irreducible repn $\tau \times \sigma \in \mathrm{Irr}(G_n)$ for some cuspidal $\sigma$ such that

$$\mathrm{Hom}_{G_n}(\pi, \tau \times \sigma) \neq 0$$

Thus, this relates to the following theorem although indeed one also needs Theorem B to prove Theorem A:

**Theorem B (C.)**

Let $\tau$ be an irreducible simple quotient of $\pi$. Then there exists a multisegment $\mathfrak{n}$ such that $D_{\mathfrak{n}}(\pi) \cong \tau$ i.e.

$$\mathcal{S}(\pi, \tau) \neq \emptyset.$$

# BZ derivatives for affine Hecke algebras

**Definition (Bernstein, Lusztig etc)**

The affine Hecke algebra $\mathcal{H}_n(q)$ has generators $\theta_1, \ldots, \theta_n$ and $T_1, \ldots, T_{n-1}$ satisfying:

1. $\theta_i\theta_j = \theta_j\theta_i$ for any $i, j$;
2. $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$;
3. $\theta_iT_j - T_j\theta_j = 0$ if $|i-j| > 1$;
4. $T_i\theta_i - \theta_{i+1}T_i = (q-1)\theta_i$.

It contains two distinguished subalgebras:

(1) The commutative subalgebra $\mathcal{A} = \langle \theta_1, \ldots, \theta_n \rangle$;

(2) The finite Hecke algebra $\mathcal{H}_{S_n}$ generated by $T_1, \ldots, T_{n-1}$.

# BZ derivatives for affine Hecke algebras

- Define

$$\mathrm{sgn}_i = \sum_{w \in S_i} (-1)^{l(w)} q^{l(w)} T_w \in 1 \otimes \mathcal{H}_{S_i} \subset \mathcal{H}_{n-i} \otimes \mathcal{H}_i.$$

- (C.-Savin 19) Define $\mathbf{BZ}_i(\pi)$ to be a $\mathcal{H}_{n-i}$-module with space

$$\{\mathrm{sgn}_i \cdot v : v \in \pi\} \subset \pi.$$

This setups a quotient branching problem for $(\mathcal{H}_n, \mathcal{H}_{n-i})$ for higher unequal rank i.e.

$$\mathrm{Hom}_{\mathcal{H}_{n-i}}(\mathbf{BZ}_i(\pi), \pi') = ?$$

When $i = 1$. it is solved by Grojnowski-Vazirani (01).

# Categorical equivalence

There exists a categorical equivalence (Bernstein, Borel, Casselman):

$$\mathrm{GL}_k(F)\text{-repns generated by Iwahori fixed vectors} \leftrightarrow \mathcal{H}_k\text{-modules}$$

The upshot is to transfer between the two categories e.g work of Aubert, Barbasch, Ciubotaru, Heiermann, Lusztig, Moy, Opdam, Reeder, Savin, Solleveld, ..., jointly and independently. Most of time, one uses Hecke algebra to transfer to information on $p$-adic groups, but we shall turn the table around to use branching laws to get some information on Hecke algebras.

# Connection between $\mathcal{H}_n$ and $\mathrm{GL}_n(F)$ repns

**Theorem (C.-Savin 2019)**

*Under the above equivalences, the i-th Bernstein-Zelevinsky derivative corresponds to* $\mathbf{BZ}_i$.

Indeed, we prove for all the Bernstein components.

**Corollary**

*Let q be not of root-of-unity. For* $\pi \in \mathrm{Irr}(\mathcal{H}_n)$*, the socle and cosocle of* $\mathbf{BZ}_i(\pi)$ *are multiplicity-free, as* $\mathcal{H}_{n-i}$*-module.*

The exhaustion theorem of BZ derivatives by St-derivatives solves the problem

$$\mathrm{Hom}_{\mathcal{H}_{n-i}}(\mathbf{BZ}_i(\pi), \pi') \neq 0$$

for $\pi \in \mathrm{Irr}(\mathcal{H}_n)$ and $\pi' \in \mathrm{Irr}(\mathcal{H}_{n-i})$.

Thank you,

Happy Birthday Gordan!