# On genus–one fibered Calabi–Yau threefolds

## The Unreasonable Effectiveness of Toric Geometry, ESI Wien

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July 8, 2026

based on work (in progress) with J. Knapp and T. Schimannek

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# Introduction

# Calabi–Yau threefolds

Classification of Calabi–Yau threefolds is still open. About 99 % of the known examples are obtained as complete intersections in toric varieties (Kreuzer, Skarke 2000)

There are well–known relations among distinct Calabi–Yau threefolds that lead (partially) to the same physics, e.g.

- Mirror symmetry
- Geometric transitions (Higgs transitions)
- Homological equivalence (the same set of D-branes)

# Invariants of Calabi–Yau threefolds

Invariants of a simply–connected Calabi–Yau threefold $X$

- under diffeomorphisms: $\mathrm{rk}\,\mathrm{H}^2(X,\mathbb{Z})$, $\mathrm{rk}\,\mathrm{H}^3(X,\mathbb{Z})$, i.e. the Hodge numbers $h^{1,1}(X)$ and $h^{2,1}(X)$, and

$$\kappa : \mathrm{H}^2(X,\mathbb{Z})^3 \to \mathbb{Z}, \quad (D_i, D_j, D_k) \mapsto D_i \cdot D_j \cdot D_k$$
$$\mathrm{c}_2 : \mathrm{H}^2(X,\mathbb{Z}) \to \mathbb{Z}, \qquad D \mapsto \mathrm{c}_2(X) \cdot D$$

- under change of Kähler structure (e.g. birational maps)

$$D^b(\mathrm{Coh}\,X)$$

- under deformation: Gromov–Witten invariants

$$N_{g,\beta}, \qquad g \geq 0, \beta \in \mathrm{H}_2(X,\mathbb{Z}) \text{ effective}$$

Goal: Understand the relations among these invariants

Due to limitations of time we will not say much about Gromov–Witten invariants today

# Example 1: Pfaffian–Grassmannian equivalence

Let $V$ be a $\mathbb{C}$-vector space, $\dim V = 7$, $\mathrm{Gr}_2(V) \subset \mathbb{P}(\wedge^2 V)$ be the Plücker embedding, $H$ a hyperplane in $\mathbb{P}(\wedge^2 V)$,

$$X = \mathrm{Gr}_2(V) \cap H^7, \quad K_X = \mathscr{O}_X$$
$$h^{1,1}(X) = 1, \quad h^{2,1}(X) = 50, \quad H^3 = 42, \quad \mathrm{c}_2 \cdot H = 84$$

Let $\varphi : \mathbb{P}(V) \to \mathbb{P}(V^\vee), \varphi = -\varphi^\vee$ and $D_4(\varphi) = \{x \in \mathbb{P}(V) \mid \mathrm{rk}\, \varphi(x) \leq 4\}$ be a Pfaffian variety.

$$Y = D_4(\varphi), \quad K_Y = \mathscr{O}_Y,$$
$$h^{1,1}(Y) = 1, \quad h^{2,1}(Y) = 50, \quad H^3 = 14 \quad \mathrm{c}_2 \cdot H = 56$$

$X$ and $Y$ are not birationally equivalent, but
Theorem (Borisov, Căldăraru 2009, Kuznetsov 2006):

$$D^b(\mathrm{Coh}\, X) \cong D^b(\mathrm{Coh}\, Y)$$

Conjectured by Rødland 1998 from mirror symmetry.

# Example 2: Reye congruence

Let $V$ be a $\mathbb{C}$-vector space, $\dim V = 5$. Let $\mathscr{X}$ be the Chow variety of two points on $\mathbb{P}(V)$ embedded into $\mathbb{P}(\mathrm{Sym}^2 V^\vee)$. Let $P \subset \mathbb{P}(\mathrm{Sym}^2 V)$ be a 4-plane regarded as a linear system of quadrics $P = |Q_1, Q_2, \ldots, Q_5|$

$$X = \mathscr{X} \cap P^\perp, \quad K_X = \mathscr{O}_X,$$
$$h^{1,1}(X) = 1, \quad h^{2,1}(X) = 26, \quad H^3 = 35, \quad \mathrm{c}_2 \cdot H = 50$$

Let $\mathscr{H} = \{\varphi : \mathbb{P}(V) \to \mathbb{P}(V^\vee) \mid \varphi = \varphi^\vee, \mathrm{rk}\,\varphi \leq 4\} \subset \mathbb{P}(\mathrm{Sym}^2 V^\vee)$ be the locus of singular quadrics in $\mathbb{P}(V)$. Let $\mathscr{Y} \to \mathscr{H}$ be the double cover branched along $\{\varphi = \varphi^\vee, \mathrm{rk}\,\varphi \leq 3\}$.

$$Y = \mathscr{Y} \cap P, \quad K_Y = \mathscr{O}_Y,$$
$$h^{1,1}(Y) = 1, \quad h^{2,1}(Y) = 26, \quad H^3 = 10, \quad \mathrm{c}_2 \cdot H = 40$$

$X$ and $Y$ are not birationally equivalent, but
Theorem (Hosono, Takagi 2012–2016, Rennemo 2020):

$$D^b(\mathrm{Coh}\, X) \cong D^b(\mathrm{Coh}\, Y)$$

# Example 3: Octic double solids

Let $X$ be a smooth complete intersection of 4 quadrics in $\mathbb{P}^7$

$$h^{1,1}(X) = 1, \quad h^{2,1}(X) = 65, \quad H^3 = 16, \quad \mathrm{c}_2 \cdot H = 64$$

Let $Y \xrightarrow{2:1} \mathbb{P}^3$ be a double cover branched along
$B = \{x \in \mathbb{P}^3 \mid \det A(x) = 0\}$, $A(x) \in \mathrm{Mat}(8 \times 8, \mathbb{C}[x])$, linear in $x$
$Y$ is a singular Calabi-Yau threefold with 84 nodal singularities.
$Y$ admits a noncommutative crepant resolution $(\mathbb{P}^3, \mathcal{B})$ for a sheaf of Azumaya algebras $\mathcal{B}$
Theorem (Kuznetsov 2007):

$$D^b(\mathrm{Coh}\, X) \cong D^b(\mathrm{Coh}\, \mathbb{P}^3, \mathcal{B})$$

$Y$ admits a non–algebraic (only analytic) crepant small resolution $\widehat{Y}$ with nontrivial Brauer group $\mathrm{Br}(\widehat{Y}) \cong \mathbb{Z}_2$.
Theorem (Addington 2009):

$$D^b(\mathrm{Coh}\, X) \cong D^b(\mathrm{Coh}\, \widehat{Y}, \alpha), \quad \alpha \in \mathbb{Z}_2 \setminus \{e\}$$

Conjecture (Katz, Klemm, Schimannek, Sharpe 2022)

$$h^{1,1}(\widehat{Y}, \alpha) = 1, \quad h^{2,1}(\widehat{Y}, \alpha) = 65, \quad H^3 = 2, \quad \mathrm{c}_2 \cdot H = 44$$

# General framework

These equivalences can all be understood in terms of the the gauged linear sigma model (Witten 1993).

Example 1: Hori, Tong 2007
Example 2: Hori 2013
Example 3: Caldararu, Distler, Hellerman, Pantev, Sharpe 2007
Example 4: Orlov equivalence Herbst, Hori, Page 2008

Examples 1–3 have a one–dimensional Kähler parameter space, i.e. $h^{1,1}(X) = 1$.

To get a better understanding, we want to study examples with higher–dimensional parameter spaces. The gauged linear sigma model provides a natural framework.

It turns out that Calabi–Yau threefolds $X$ admitting genus one fibrations provide an interesting and manageable class of examples with $h^{1,1}(X) = 2$

In particular, we find noncommutative versions of genus one fibrations for which the analogues of topological and Gromov–Witten invariants can be determined.

# Gauged linear sigma models

# The gauged linear sigma model I

Recall from Johanna's talk:

The data for a gauged linear sigma model (GLSM) is $(G, \rho, R, W, t)$ with

$G$ a compact Lie group,
$\rho : G \to GL(V)$ faithful unitary rep.
$R : U(1) \to GL(V)$ faithful unitary rep.
$W \in S^G$ of $R$–weight 2, where $S = \mathrm{Sym}^* V^\vee$
$t \in \mathcal{T} = \left(\frac{\mathfrak{t}_{\mathbb{C}}^\vee}{2\pi i P(\mathfrak{g})}\right)^{\mathcal{W}(T,G)}$, the socalled FI-theta parameter

Let $\mu : V \to \mathfrak{g}^\vee$ be the moment map on $V$ by $\mu(\phi)(X) = \langle X \cdot \phi, \phi \rangle$ for $\phi \in V$ and $X \in \mathfrak{g}$. For a fixed, regular value $\zeta \in Re\,\mathcal{T}$, set

$$X_\zeta = \mu^{-1}(\zeta)/G \cap dW^{-1}(0).$$

This is the Higgs branch.

We will assume that $\rho$ factors through $SL(V)$, so that $X_\zeta$ is Calabi–Yau. Then, in favorable cases, $\dim \mathcal{T} = h^{1,1}(X)$.

# The gauged linear sigma model II

An important role is played by the loci in $\mathcal{T}$ where the GLSM is singular.

Coulomb branch: For $\sigma \in Z(\mathfrak{g}_{\mathbb{C}}) \subset \mathfrak{t}_{\mathbb{C}}$ set

$$\widetilde{W}_{eff}(\sigma, t) = -t(\sigma) + 2\pi i \rho_{\mathrm{W}}(\sigma) - \sum_{Q_j \in \mathfrak{h}^\vee} Q_j(\sigma)\left(\log(Q_j(\sigma)) - 1\right)$$

where $Q_j$ denotes the weights of the representation $\rho = \oplus_j \rho_j$.

There can be mixed Coulomb–Higgs branches for certain subgroups $H \subset G$, yielding similar $\widetilde{W}^H_{eff}$.

We define the stringy Kähler moduli space

$$\mathcal{M}_K = \mathcal{T} \setminus \Delta$$

where $\Delta$ is the union of the critical loci of $\widetilde{W}^H_{eff}$ over all subgroups $H$.

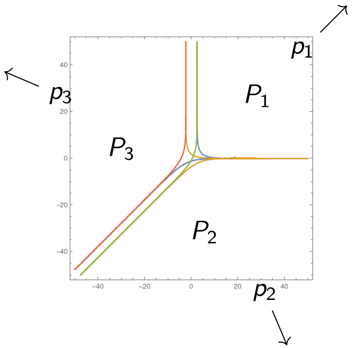
# The gauged linear sigma model III

$Re\,\mathcal{M}_K$ consists of finitely many connected components $P_i$, $i \in I$, called phases. By the equivalence of symplectic quotients with GIT quotients, this corresponds to the possible choices of linearizations.

Each phase $P_i$ has a limiting point $p_i \in \partial\,\mathrm{Re}\,\mathcal{T}_i \cap P_i$, $i \in I$.

Varying $\mathrm{Im}\,\mathcal{T}$ allows for smooth interpolation between $P_i$ and $P_j$.

$X_{\zeta_1}$ and $X_{\zeta_2}$ are diffeomorphic for $\zeta_1, \zeta_2$ in the same connected component



# D-branes in the gauged linear sigma model I

To a GLSM $(G, \rho, R, W, t)$ there is an associated triangulated category $\mathcal{D}$ of D–branes. Roughly speaking, objects are quintuples $\mathcal{B} = (M, Q, \rho_M, \mathbf{r}_*, \gamma)$

- $M$ a $\mathbb{Z}_2$–graded free $S$–module
- $Q \in \mathrm{End}^1_S(M)$
- $\rho_M : G \to \mathrm{GL}(M)$ complex repr.
- $\mathbf{r}_* : \mathfrak{u}(1) \to \mathrm{GL}(M)$ complex repr.
- $\gamma \subset \mathfrak{t}_{\mathbb{C}}$ middle–dimensional cycle

satisfying:

- $Q$ is a matrix factorization of $W$, i.e. $Q^2 = W \mathrm{id}_M$
- $Q$ is $G$–equivariant, i.e. $Q(\rho(x)) \circ \rho_M = \rho_M \circ Q(x)$
- $Q$ has weight 1 with respect to $\mathbf{r}_*$

# D-branes in the gauged linear sigma model II

To each limiting point $p_i \in P_i$ there is a limiting triangulated category $\mathcal{D}_i$ which is associated to $X_\zeta$ for $\zeta \in P_i$.

Depending on the structure of $X_\zeta$, $\mathcal{D}_i$ is expected to be equivalent to $D^b(\mathrm{Coh}\, X_\zeta)$, $MF_\Gamma(W')$ for some finite subgroup $\Gamma \subset G$ and restriction $W'$ of $W$, a proper noncommutative scheme, and so on.

Conjecture: Herbst, Hori, Page 2008 Let $G$ be abelian

- For all $i \in I$ there exist essential surjections $\pi_i : \mathcal{D} \to \mathcal{D}_i$,
- For all $i, j \in I$ there exist natural equivalences

$$\mathcal{D}_i \xrightarrow{\sim} \mathcal{D}_j$$

induced by paths in $\mathcal{M}_K$ from $p_i$ to $p_j$.

If $G$ is nonabelian, we may need to restrict to a certain subcategory $\mathcal{S} \subset \mathcal{D}$ Eager, Hori, Knapp, Romo

To make use of this we need an appropriate function depending on $\mathcal{M}_K$ and on objects in $\mathcal{D}$.

# Hemisphere partition function I

The hemisphere partition function is Hori, Romo 2013

$$Z_{D^2} : K_0(\mathcal{D}) \times \mathcal{M}_K \to \mathbb{C}$$

$$Z_{D^2}([\mathcal{B}], t) = C \int_{\gamma \subset \mathfrak{t}_{\mathbb{C}}} \mathrm{d}^r\sigma \prod_{\alpha \in \Pi^+} \alpha(\sigma) \sinh \pi\alpha(\sigma) \prod_{j=1}^{m} \Gamma\left(iQ_j(\sigma) + \frac{R_j}{2}\right) \mathrm{e}^{it\sigma} f_{\mathcal{B}}(\sigma).$$

where $r = \mathrm{rk}(G)$, $C$ is a normalization constant, $\Pi^+$ is the set of positive roots associated to $(\mathfrak{g}, \mathfrak{t})$ and

$$f_{\mathcal{B}}(\sigma) = \mathrm{tr}_M\left(\mathrm{e}^{i\pi\mathbf{r}_*}\mathrm{e}^{2\pi\sigma}\right).$$

The contour $\gamma \subset \mathfrak{t}_{\mathbb{C}}$ must be such that the integral converges.

$Z_{D^2}([\mathcal{B}], t)$ is a flat section of a flat connection $\nabla$ on $\mathcal{M}_K$, i.e. a solution of a linear partial differential equation in $z = \mathrm{e}^t$

# Hemisphere partition function II

To each limiting point $p_i$ there is (conjecturally) an associated CohFT:

- A state space $H_i = \mathrm{HH}_*(\mathcal{D}_i)$ with pairing $\langle -, - \rangle : H_i \times H_i \to \mathbb{C}$
- Givental's formalism associates the so-called I– and J–functions $J_i \in H_i[[t]][[e^t]]$, $t$ local coordinate near $p_i \in \partial\mathcal{M}_K$.
- Iritani's Gamma class $\widehat{\Gamma}_i \in \mathrm{End}(H_i)$.

Conjecture: Knapp, Romo, S (2016): For every $i \in I$,

$$Z_{D^2}([\mathcal{B}], t) = \langle \mathrm{ch}(\pi_i(\mathcal{B})), \widehat{\Gamma}_i \circ J_i(t) \rangle$$

for $t \in P_i + (i\mathbb{R}/\mathbb{Z})^{\dim \mathcal{M}_K}$.

The LHS is defined everywhere in $\mathcal{M}_K$ while the RHS is only defined in a phase. $Z_{D^2}([\mathcal{B}], t)$ being a flat section, the various RHS are related to each other by analytic continuation

This data allows for a reconstruction of the genus 0 prepotential $F_0(t)$ and the determination of the enumerative invariants

# Genus one fibrations

# Elliptic normal curves

An elliptic normal curve of degree $N$, $N \geq 3$, is a smooth projective curve $E$ of genus 1 with an immersion in a projective space $\mathbb{P}^{N-1}$ as a degree $N$ curve that is contained in no hyperplane.

| $N$ | equations for $E$ |
|---|---|
| 3 | Cubic in $\mathbb{P}^2$ |
| 4 | intersection of two quadrics in $\mathbb{P}^3$ |
| 5 | Pfaffian variety $D_2(\varphi) \subset \mathbb{P}^4$ |
| | intersection of five hyperplanes in $\mathrm{Gr}(2,5)$ |
| 6 | $\sigma(\mathbb{P}^2 \times \mathbb{P}^2) \cap H \cap H \cap H \subset \mathbb{P}^8$ |
| $\geq 7$ | intersection of $N(N-3)/2$ quadrics in $\mathbb{P}^{N-1}$ |

$\sigma : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^8$ Segre embedding, $H$ hyperplane in $\mathbb{P}^8$

# Pfaffian varieties

Let $A$ be a $2r+1 \times 2r+1$ skew–symmetric matrix with entries that are linear homogeneous polynomials in $x_0, \ldots, x_n$.

Let $M_i$, $i = 1, \ldots, 2r+1$, be the $2r \times 2r$ minor obtained by deleting the $i$–th row and $i$–th column of $A$, $\mathrm{pf}(M_i)$ the corresponding Pfaffian.

The Pfaffian variety $D_r(A)$ associated to $A$ is given by the vanishing locus of the ideal $\mathrm{Pf}_r(A)$ generated by the Pfaffians $\mathrm{pf}(M_i)$.

The variety $D_r(A)$ is in general not a complete intersection, but shares many of their properties. Also note: $\mathrm{codim}\, D_r(A) = 3$.

Let $I_{2r}(A)$ be the ideal generated by the $2r \times 2r$ minors of $A$. Then $I_{2r}(A)^2 = \mathrm{Pf}_r(A)^2$.

More generally, let $E \to B$ be a holomorphic vector bundle of rank $2r+1$, $\varphi : E \to E^\vee$ a skew–symmetric bundle map, then we obtain a Pfaffian subvariety $D_r(\varphi) \subset \mathbb{P}(E)$.
We often write $\mathrm{Pf}(r, E)$ instead of $D_r(\varphi)$.

# Homological projective duality for elliptic normal curves of degree 5

$V$: $\mathbb{C}$–vector space of dimension 5
$L \subset \wedge^2 V^\vee$: linear subspace of dimension 5
$L^\perp$: orthogonal complement of $L$ in $\wedge^2 V$

$$\begin{tikzcd} & \mathbb{P}(\wedge^2 V) & \\ \mathbb{P}(L^\perp) \arrow[ur, hook] & & \mathrm{Gr}(2,V) \arrow[ul, hook'] \\ & X_0 \arrow[ul, hook'] \arrow[ur, hook] & \end{tikzcd}
\qquad
\begin{tikzcd} & \mathbb{P}(\wedge^2 V^\vee) & \\ \mathrm{Pf}(2,V^\vee) \arrow[ur, hook] & & \mathbb{P}(L) \arrow[ul, hook'] \\ & Y_0 \arrow[ul, hook'] \arrow[ur, hook] & \end{tikzcd}$$

$\mathrm{Pf}(2, V^\vee)$ classical projective dual of $\mathrm{Gr}(2, V)$
Homological projective duality: $D^b(\mathrm{Coh}\, X_0) \cong D^b(\mathrm{Coh}\, Y_0)$
Kuznetsov 2006, Shinder, Zhang 2020
This implies the known result: $X_0 \cong Y_0$. Hille, van den Bergh 2007
In general, the homological projective dual is only a triangulated category with no geometric space associated to it

# GLSM realization of the elliptic normal quintic

Hori, Knapp 2013 Consider $G = U(2)$ with representations

| | $p^1, \ldots, p^5$ | $x_1, \ldots, x_5$ | FI |
|---|---|---|---|
| $U(2)$ | $\det^{-1}$ | $\square$ | $\zeta$ |

$$W = \sum_{i,j,k=1}^{5} \sum_{a,b=1}^{2} A_k^{ij} p^k \varepsilon_{ab} x_i^a x_j^b$$

$A^{ij}(p) = \sum_{k=1}^{5} A_k^{ij} p^k$ is a generic antisymmetric $5 \times 5$ matrix

$X_{\zeta \gg 0} = X_0$

$\zeta \ll 0$: Strongly coupled gauge theory with $G = SU(2)$. Further analysis yields $X_{\zeta \ll 0} = \{p \in \mathbb{P}^4 \mid \mathrm{rk}\, A^{ij}(p) = 2\} = Y_0$.

$$\begin{aligned}\widetilde{W}_{eff}(\sigma, t) = {} & -t(\sigma_1 + \sigma_2) + 5(\sigma_1 + \sigma_2)[\log(-\sigma_1 - \sigma_2) - 1] \\ & - 5\sigma_1[\log(\sigma_1) - 1] - 5\sigma_2[\log(\sigma_2) - 1] + i\pi(\sigma_1 - \sigma_2)\end{aligned}$$

$\mathcal{M}_K = \mathbb{C}^* \setminus \{e^{-t} = \frac{1}{2}\left(11 \pm 5\sqrt{5}\right)\}$

General philosophy: Homological projective dual pairs appear as two different phases of the same GLSM

# Genus one fibrations I

A smooth projective Calabi–Yau 3–fold is genus one fibered if there is a flat proper holomorphic surjection $\pi : X \to B$ whose fibers are curves of genus one.

The map $\pi$ is not required to have a section.

But $\exists$ minimal $N > 0$ such that $\pi$ has an $N$–section, i.e. there is a divisor $D$ on $X$ such that $\pi|_D : D \xrightarrow{N:1} B$ is an $N$–fold cover.

If $N = 1$ then $\pi$ has a section and is called an elliptic fibration.

Equivalently, there exists a line bundle $L \to X$ s.t. $\deg(L|_C) = N$ for every fiber $C$, i.e. the intersection number is $D \cdot C = N$.

Theorem (Oguiso 1993): Let $X$ be a smooth projective Calabi–Yau threefold, $D$ an effective divisor on $X$ with $D^3 = 0$, $D^2 \not\equiv 0$, and $D \cdot c_2(X) > 0$, then there exists a genus one fibration on $X$ determined by $D$ with $N = D^2 \cdot D'$ for some appropriate divisor $D'$.

# Genus one fibrations II

| $N$ | fiber $C$ |
|---|---|
| 1 | degree 6 hypersurface in $\mathbb{P}(1,2,3)$ |
| 2 | degree 4 hypersurface in $\mathbb{P}(1,1,2)$ |
| $\geq 3$ | degree $N$ elliptic normal curve in $\mathbb{P}^{N-1}$ |

$N \leq 4$:

- $X \to B$ realized as complete intersection in a toric variety.
- $X$ and its stringy Kähler moduli space can be studied using a GLSM with abelian group $G$.
- Genus one fibrations are important in F-theory: [Morrison, Park 2012] ...
- and in topological string theory: ... [Cota, Klemm, Schimannek 2019]

$N = 5$: [Knapp, Schimannek, S 2021]

$N = 6$: [Knapp, Schimannek, S work in progress]

# Genus one fibered CY threefolds with 5–sections

# Genus one fibered Calabi–Yau threefolds with 5–sections

[Knapp, Schimannek, S 2021]

Let $\pi : X \to B$ a smooth projective Calabi–Yau threefold with a genus one fibration (without reducible fibers) and a 5–section.

- $X$ can be realized as a (determinantal) subvariety of a Grassmann bundle.
- $X$ and its stringy Kähler moduli space can be studied using a GLSM with $G = U(M) \times U(1)^n$.
- Such genus one fibrations come in pairs $(X, Y)$. Conjecturally, these are homologically projective dual.
- The topological string partition function of $X$ or $Y$ are obtained from modular forms for $\Gamma_1(5)$. The relative Fourier–Mukai transform $X \leftrightarrow Y$ extends this group to $\Gamma_0(5)$.

# Construction

$F \to B$: vector bundle of rank 5 over a rational surface $B$
$E^\vee \subset \wedge^2 F^\vee$: subbundle of rank 5
$(E^\vee)^\perp$: orthogonal complement of $E^\vee$ in $\wedge^2 F$

$$\begin{tikzcd} & \mathbb{P}(\wedge^2 F) & \\ \mathbb{P}((E^\vee)^\perp) \arrow[ur, hook] & & \mathrm{Gr}(2,F) \arrow[ul, hook'] \\ & X \arrow[ul, hook'] \arrow[ur, hook] & \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{P}(\wedge^2 F^\vee) & \\ \mathrm{Pf}(2,F^\vee) \arrow[ur, hook] & & \mathbb{P}(E^\vee) \arrow[ul, hook'] \\ & Y \arrow[ul, hook'] \arrow[ur, hook] & \end{tikzcd}$$

$$X = \mathrm{Gr}(2,F) \times_{\mathbb{P}(\wedge^2 F)} \mathbb{P}((E^\vee)^\perp) \quad Y = \mathrm{Pf}(2,F^\vee) \times_{\mathbb{P}(\wedge^2 F^\vee)} \mathbb{P}(E^\vee)$$

CY condition:

$$\omega_B \otimes \det(F^\vee)^{\otimes 2} \otimes \det E = \mathscr{O}_B$$

# Example $(X_1, Y_1)$

Choose $B = \mathbb{P}^2$, $F = \mathcal{O}_{\mathbb{P}^2}^{\oplus 5}$, $E = \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}$.

Consider $\pi : \mathrm{Gr}_2(F) \to \mathbb{P}^2$, $X = Z(s)$, $s \in H^0(G, \mathcal{O}_{\mathrm{Gr}_2(F)/\mathbb{P}^2}(1) \otimes \pi^* E)$.

$$X_1 = \left\{ \begin{matrix} 0 = s_k = \sum_{i,j=1}^5 \sum_{\ell=1}^5 \sum_{a,b=1}^2 A_k^{ij\ell} \varepsilon_{ab} x_i^a x_j^b b_\ell & k = 1,2,3 \\ 0 = s_k = \sum_{i,j=1}^5 \sum_{a,b=1}^2 A_k^{ij} \varepsilon_{ab} x_i^a x_j^b & k = 4,5 \end{matrix} \right\} \subset \mathrm{Gr}_2(F)$$

$b_\ell$ homogeneous coord. on $\mathbb{P}^2$, $x_i^a$ homogeneous coord. on $\mathrm{Gr}_2(F_b)$.

Consider $\pi' : \mathbb{P}(E^\vee) \to \mathbb{P}^2$, $\varphi : \pi'^* F \to \pi'^* F^\vee \otimes \mathcal{O}_{\mathbb{P}(E^\vee)/\mathbb{P}^2}(1), \varphi = -\varphi^\vee$.

$$Y_1 = \mathrm{Pf}(2, \varphi) \subset \mathbb{P}(E^\vee)$$

(complicated equations)

# Towards a classification

| $X, Y$ | $n$ | $E$ | $F$ | $h^{11}$ | $h^{21}$ | $\chi$ | $J_1^2 J_2$ | $J_1^3$ | $c_2 J_1$ |
|---|---|---|---|---|---|---|---|---|---|
| | $1_a$ | $\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 3}$ | $\mathcal{O}_{\mathbb{P}^2}^{\oplus 5}$ | 2 | 47 | −90 | 15 | 10 | 64 |
| | $1_b$ | | | 2 | 47 | −90 | 5 | 5 | 38 |
| $X_1$ | $2_a$ | $\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}$ | $\mathcal{O}_{\mathbb{P}^2}^{\oplus 5}$ | 2 | 47 | −90 | 15 | 15 | 66 |
| $Y_1$ | $2_b$ | | | 2 | 47 | −90 | 5 | 0 | 36 |
| | $3_a$ | $T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}$ | $\mathcal{O}_{\mathbb{P}^2}^{\oplus 5}$ | 2 | 47 | −90 | 15 | 20 | 68 |
| | $3_b$ | | | 2 | 47 | −90 | 15 | 25 | 70 |
| | $4_a$ | $\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 3}(1) \oplus \mathcal{O}_{\mathbb{P}^2}$ | $\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 4}$ | 2 | 49 | −94 | 11 | 3 | 54 |
| | $4_b$ | | | 2 | 49 | −94 | 9 | 10 | 52 |
| | $5_a$ | $\mathcal{O}_{\mathbb{P}^2}(3) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 4}(2)$ | $\mathcal{O}_{\mathbb{P}^2}^{\oplus 4}(1) \oplus \mathcal{O}_{\mathbb{P}^2}$ | 2 | 49 | −94 | 9 | 0 | 48 |
| | $5_b$ | | | 2 | 49 | −94 | 11 | 13 | 58 |
| | $6_a$ | $\mathcal{O}_{\mathbb{P}^2}^{\oplus 5}(1)$ | $\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 4}$ | 2 | 49 | −94 | 11 | 8 | 56 |
| | $6_b$ | | | 2 | 49 | −94 | 9 | 5 | 50 |
| | $7_a$ | $\mathcal{O}_{\mathbb{P}^2}(3) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 4}(1)$ | $\mathcal{O}_{\mathbb{P}^2}^{\oplus 2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 3}$ | 2 | 50 | −96 | 17 | 28 | 76 |
| | $7_b$ | | | 2 | 50 | −96 | 13 | 26 | 68 |
| | $8_a$ | $\mathcal{O}_{\mathbb{P}^2}(3) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}(1)$ | $\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}$ | 2 | 50 | −96 | 13 | 11 | 62 |
| | $8_b$ | | | 2 | 50 | −96 | 7 | 7 | 46 |
| | $9_a$ | $\mathcal{O}_{\mathbb{P}^2}^{\oplus 2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 3}(1)$ | $\mathcal{O}_{\mathbb{P}^2}^{\oplus 2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 3}$ | 2 | 50 | −96 | 17 | 33 | 78 |
| | $9_b$ | | | 2 | 50 | −96 | 13 | 21 | 66 |
| | $10_a$ | $\mathcal{O}_{\mathbb{P}^2}^{\oplus 4}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$ | $\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}$ | 2 | 50 | −96 | 13 | 16 | 64 |
| | $10_b$ | | | 2 | 50 | −96 | 7 | 2 | 44 |
| | $11_a$ | $\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}(3) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}(2)$ | $\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 3}(1) \oplus \mathcal{O}_{\mathbb{P}^2}$ | 2 | 52 | −100 | 15 | 29 | 74 |
| | $11_b$ | | | 2 | 52 | −100 | 15 | 29 | 74 |
| | $12_a$ | $\mathcal{O}_{\mathbb{P}^2}^{\oplus 2}(3) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$ | $\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}$ | 2 | 54 | −104 | 9 | 9 | 54 |
| | $12_b$ | | | 2 | 54 | −104 | 11 | 17 | 62 |
| | $13_a$ | $\mathcal{O}_{\mathbb{P}^2}(3) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 4}$ | $\mathcal{O}_{\mathbb{P}^2}^{\oplus 5}$ | 6 | 51 | −90 | 15 | 0 | 60 |
| | $13_b$ | | | 6 | 51 | −90 | 5 | 15 | 42 |

$(X, Y)$ smooth, $E, F$ globally generated homogeneous vector bundles over $B = \mathbb{P}^2$. Furthermore, $J_2^3 = 0$, $J_1 J_2^2 = 5$, and $c_2 J_2 = 36 = 12\chi(\mathcal{O}_B)$. By Oguiso's theorem, these are genus one fibrations with 5-sections.

# Relative homological projective duality

**Conjecture**: (Knapp-Schimannek-S 2021) Each of the pairs $(X, Y)$ in the table satisfies

$$D^b(\mathrm{Coh}\, X) \cong D^b(\mathrm{Coh}\, Y)$$

as a consequence of relative homological proj. duality Kuznetsov 2007

Evidence:
The pair $(X_1, Y_1)$ has an isomorphic representation $(X_1', Y_1')$ where $X_1'$ and $Y_1'$ are complete intersections of a rank 8 vector bundle in resolved joins $\mathrm{Join}(\mathrm{Gr}_2(V_5), \mathbb{P}^2 \times \mathbb{P}^2)$ and $\mathrm{Join}(\mathrm{Gr}_2(V_5^\vee), \mathbb{P}^2 \times \mathbb{P}^2)$, respectively.
Theorem (Inoue 2022): $D^b(\mathrm{Coh}\, X_1') \cong D^b(\mathrm{Coh}\, Y_1')$ as a consequence of homological projective duality for categorical joins. Kuznetsov, Perry 2021. Moreover, $X_1'$ and $Y_1'$ are not birationally equivalent.

Proof of conjecture : Morimura 2023

# The GLSM for $(X_1, Y_1)$ I

Consider the example $(X_1, Y_1)$: Gauge group $G = U(2) \times U(1)$

| | $p^1, p^2, p^3$ | $p^4, p^5$ | $x_1, \ldots, x_5$ | $b_1, b_2, b_3$ | FI |
|---|---|---|---|---|---|
| $U(2)$ | $\det^{-1}$ | $\det^{-1}$ | $\square$ | $\mathbb{1}$ | $\zeta$ |
| $U(1)$ | $-1$ | $0$ | $0$ | $1$ | $\zeta_1$ |

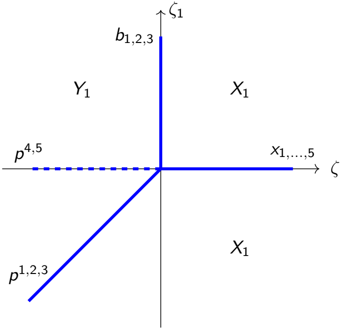
$$W = \sum_{i,j=1}^{5} \sum_{a,b=1}^{2} \left( \sum_{k,\ell=1}^{3} A_k^{ij\ell} b_\ell + \sum_{k=4}^{5} A_k^{ij} \right) p^k \varepsilon_{ab} x_i^a x_j^b$$

$\frac{\partial W}{\partial p^k} = s_k, k = 1, \ldots 5$, with $s = (s_1, \ldots, s_5) \in H^0(G, \mathscr{O}_{G/\mathbb{P}^2}(1) \otimes \pi^* E)$

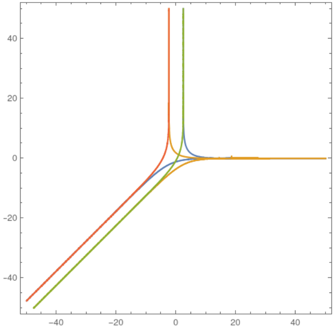
$$\begin{aligned}\widetilde{W}_{eff} = & -t(\sigma_1 + \sigma_2) - t_1\sigma_3 - 3(-\sigma_1 - \sigma_2 - \sigma_3)\left[\log(-\sigma_1 - \sigma_2 - \sigma_3) - 1\right] \\ & - 2(-\sigma_1 - \sigma_2)\left[\log(-\sigma_1 - \sigma_2) - 1\right] - 5\sigma_1\left[\log \sigma_1 - 1\right] \\ & - 5\sigma_2\left[\log \sigma_2 - 1\right] - 3\sigma_3\left[\log \sigma_3 - 1\right] + i\pi(\sigma_1 - \sigma_2)\,.\end{aligned}$$

# The GLSM for $(X_1, Y_1)$ II

Extended Kähler cone




Stringy Kähler moduli space $\mathcal{M}_K$




The two cones labeled by $X_1$ are birationally isomorphic. $Y_1$ is separated from the other cones by double lines, indicating that $Y_1$ is not birationally equivalent to $X_1$. By the general conjecture of Eager, Hori, Knapp, Romo:

$$D^b(\mathrm{Coh}\, X_1) \cong D^b(\mathrm{Coh}\, Y_1)$$

# Toric degenerations and conifold transitions

$\mathrm{Gr}(2,5)$ admits a degeneration to a toric variety $P(2,5)$ Sturmfels (1996)

$P(2,5)$ admits a small crepant resolution $\widehat{P}(2,5)$. A Calabi–Yau complete intersection $X \subset \mathrm{Gr}(2,5)$ admits a conifold transition to a Calabi–Yau complete intersection $X' \subset \widehat{P}(2,5)$ Batyrev, Ciocan–Fontanine, Kim, van Straten (1998)

We classify reflexive polytopes $\Delta^\circ$ yielding a fibration of $\widehat{P}(2,5)$ over $\mathbb{P}^2$ and codimension 5 nef partitions compatible with the fibration

Similarly we classify reflexive polytopes $\Delta^\circ$ yielding a fibration of $\mathbb{P}(\mathcal{O}^{\oplus 3} \oplus \mathcal{O}(1))$ over $\mathbb{P}^2$ and codimension 3 nef partitions compatible with the fibration

This reproduces the table of pairs $(X, Y)$ and shows that they all admit conifold transitions to Calabi–Yau complete intersection in toric varieties

# Genus one fibered CY threefolds with 6–sections

# Elliptic normal curves

Elliptic normal curves of degree 6 can be realized in many different ways:

- $\sigma(\mathbb{P}^2 \times \mathbb{P}^2) \cap H \cap H \cap H \subset \mathbb{P}^8$
- $\sigma(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) \cap H \cap H \subset \mathbb{P}^7$
- $v_2(3H \subset \mathbb{P}^2) \subset \mathbb{P}^5$
- $D_1(\varphi) = \{x \in \mathbb{P}^5 \mid \operatorname{rk} \varphi(x) \leq 1\}$, $\varphi \in \operatorname{Hom}(E, F)$, $\operatorname{rk} E = \operatorname{rk} F = 3$
- $Z(s) \subset \operatorname{Gr}(3,6)$ for $s \in \mathrm{H}^0(\operatorname{Gr}(3,6), \wedge^2 S^{\vee \oplus 2} \oplus \wedge^3 S^{\vee \oplus 2})$.

Each of these yields a different homological projective dual, in general a noncommutative resolution of singularities of some algebraic variety $Y$.

A noncommutative resolution of singularities of an algebraic variety $Y$ is a coherent sheaf $\mathcal{A}$ of $\mathcal{O}_Y$–algebras which is a matrix algebra at the generic point of $Y$ and has finite homological dimension.

# A GLSM for a CY 3–fold with a 6-section I

Example: Gauge group $G = U(3) \times U(1)$

| | $p^1, p^2$ | $p^3, p^4$ | $x_1, \ldots, x_6$ | $b_1, b_2, b_3$ | FI |
|---|---|---|---|---|---|
| $U(3)$ | $\overline{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}$ | $\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$ | $\square$ | $\mathbb{1}$ | $\zeta$ |
| $U(1)$ | $0$ | $-1 \quad 0$ | $0$ | $1$ | $\zeta_1$ |

$$W = \sum_{k=1}^{4} p^k s_k$$

with

$$\begin{aligned} s_1, s_2 &\in \mathrm{H}^0(\mathrm{Gr}(3,6) \times \mathbb{P}^2, \det S^\vee) \\ s_3 &\in \mathrm{H}^0(\mathrm{Gr}(3,6) \times \mathbb{P}^2, \wedge^2 S^\vee) \\ s_4 &\in \mathrm{H}^0(\mathrm{Gr}(3,6) \times \mathbb{P}^2, \wedge^2 S^\vee \otimes \pi^* \mathscr{O}_{\mathbb{P}^2}(-1)) \end{aligned}$$

where $\pi : \mathrm{Gr}(3,6) \times \mathbb{P}^2 \to \mathbb{P}^2$, $S$ the tautological bundle on $\mathrm{Gr}(3,6)$.

# A GLSM for a CY 3–fold with a 6-section II

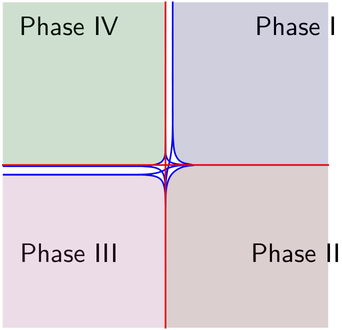



Figure: The phase structure of $\mathrm{Re}\,\mathcal{M}_K$.

# A GLSM for a CY 3–fold with a 6-section III

Phase I: A smooth genus one fibration $X$ with 6–section.

$$h^{1,1}(X) = 2, \qquad h^{2,1}(X) = 36$$

$$D_1^3 = 32, \qquad D_1^2 D_2 = 24, \qquad D_1 D_2^2 = 6, \quad D_2^3 = 0,$$

$$c_2(X) D_1 = 80, \quad c_2(X) D_2 = 36$$

Phase IV: A singular Calabi-Yau threefold $\overline{Y}$ that is a double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ with 92 isolated nodal *terminal* singularities.
The triangulated category corresponding to $\overline{Y}$ is expected to be a noncommutative projective Calabi–Yau threefold.

# Noncommutative resolutions and deformations

We have the following general picture (cf. Example 3)

$$\begin{tikzcd}
Y' & \overline{Y} \arrow[l, squiggly] & \\
 & Y_{\mathrm{nc},\alpha} \arrow[u] \arrow[r, leftrightarrow] & X
\end{tikzcd}$$

Phase IV Phase I

where $Y_{\mathrm{nc},\alpha}$ stands for the noncommutative "space" $D^b(\mathrm{Coh}\,\widehat{Y},\alpha)$ with $\widehat{Y}$ a non–Kähler resolution of $\overline{Y}$ and $\alpha \in \mathrm{Br}(\widehat{Y})$.
$Y'$ is the smooth deformation of $\overline{Y}$ (important for what follows)
The singularities of $\overline{Y}$ are stabilized by a flat but topologically non-trivial B-field in the GLSM. $\alpha$ is the cohomology class of this B-field. (Katz, Klemm, Schimannek, Sharpe (2023))
The GLSM predicts

$$D^b(\mathrm{Coh}\,\widehat{Y},\alpha) \cong D^b(\mathrm{Coh}\,X)$$

# A GLSM for with a 6-section IV

A noncommutative resolution of a singular double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ similar to $Y_{\mathrm{nc},\alpha}$ was constructed by Calabrese, Thomas (2016).

From the GLSM the invariants of the smooth deformation $Y'$ of $\overline{Y}$ can be determined

$$h^{1,1}(Y') = 2, \qquad h^{2,1}(Y') = 128$$

$$D_1^3 = 0, \qquad D_1^2 D_2 = 2, \qquad D_1 D_2^2 = 0, \qquad D_2^3 = 0,$$

$$c_2(Y')D_1 = 36, \quad c_2(Y')D_2 = 24$$

$Y'$ can be realized as a smooth hypersurface in a toric variety.

The prediction for the invariants of $D^b(\mathrm{Coh}\,\widehat{Y}, \alpha)$ are

$$h^{1,1}(\widehat{Y}, \alpha) = 2, \qquad h^{2,1}(\widehat{Y}, \alpha) = 36$$

$$D_1^3 = 0, \qquad D_1^2 D_2 = 2, \qquad D_1 D_2^2 = 0, \qquad D_2^3 = 0,$$

$$c_2(\widehat{Y}, \alpha)D_1 = 36, \quad c_2(\widehat{Y}, \alpha)D_2 = 24$$

This is a genus one fibration with a 2–section and nontrivial Brauer class $\alpha \in \mathbb{Z}_3$.

# Outlook

# Questions

Given a proper noncommutative Calabi–Yau threefold, how does one define and determine the analogues of the topological invariants $b_2$, $b_3$, $\kappa$, $c_2$ directly from the category ?

How is the Brauer group $\mathrm{Br}(\widehat{X})$ encoded in the GLSM ?

How can one obtain the sheaf $\mathcal{B}$ of noncommutative (dg–)algebras representing a non–trivial element $\alpha \in \mathrm{Br}(\widehat{X})$ (or the other way round) within the framework of the GLSM ?

Can one construct the Fourier–Mukai transforms between the phases directly in the category of matrix factorizations of the GLSM ?
Eager, Hori, Knapp, Romo: work in progess

Thank you!

# Topological invariants I

To compute invariants such as Hodge numbers and intersection numbers, one needs the class of the normal bundle $N_{X/G}$ resp. $N_{Y/P}$ in the respective Chow groups.
For $X \subset G$ we have $N_{X/G} = E|_X$. But for $\iota : Y \to P$ we need

$$0 \to \mathscr{I}_Y^2 \to \mathscr{I}_Y \to \iota_* N_{Y/P}^\vee \to 0$$

For a complete intersection $\mathscr{I}_X$ has a minimal free resolution in terms of the Koszul complex, for a Pfaffian variety $\mathscr{I}_Y$ has a minimal free resolution of length 3 in terms of the Buchsbaum–Eisenbud complex, $\mathscr{I}_Y^2$ has a minimal free resolution of length 4 in terms of the Boffi–Sanchez complex.
If $E$ and $F$ are homogeneous vector bundles over a homogeneous variety $B$, these complexes can be expressed in terms of Schur functors applied to $E$ and $F$, e.g.

$$0 \to \mathbb{S}_{(2^r)} F \otimes L^{-2k-1} \to \mathbb{S}_{(2,1^{r-1})} F \otimes L^{-k-1} \to \mathbb{S}_{(1^{r-1})} F \otimes L^{-k} \to \mathscr{I}_Y \to 0$$

where $r = 2k+1 = \operatorname{rk} F$, $L \in \operatorname{Pic}(P)$.

# Topological invariants II

To obtain $\mathrm{ch}(N_{Y/P}) \in CH^*(Y)$ we apply the Riemann–Roch theorem for a closed embedding $\iota : Y \hookrightarrow P$ of non–singular varieties: For a vector bundle $V$ on $Y$ we have

$$\mathrm{ch}(\iota_* V) = \iota_*(\mathrm{ch}(V)\,\mathrm{td}(N_{Y/P})^{-1}) \in CH^*(P)$$

Choosing $V = N^\vee_{Y/P}$, we have $\iota_* 1 = [Y]$ and $\mathrm{ch}(\iota_* V) = [Y]\alpha$ from the conormal bundle sequence. If $CH^*(Y)$ is generated in degree 2, the Chern classes $\mathrm{c}_i(N_{Y/P})$ are obtained by solving the equation $\alpha = \mathrm{ch}(N^\vee_{Y/P})\,\mathrm{td}(N_{Y/P})^{-1}$ in the ring $CH^*(P)$ using Gröbner bases.

The Hodge numbers $h^i(X, \Omega^1_X)$ are computed from the hypercohomology spectral sequences for the resolutions of $\mathscr{I}$ and $\mathscr{I}^2$, the Leray spectral sequence for the fibrations $\mathrm{Gr}_2(F) \to B$ and $\mathbb{P}(E^\vee) \to B$, and the Borel-Weil-Bott theorem.

## Torsion refined Gopakumar–Vafa invariants

Assume that $\mathrm{Br}(\widehat{X}) \cong \mathrm{H}_2(\overline{X}, \mathbb{Z})_{\mathrm{tors}} \cong \mathbb{Z}_N$. Then the torsion refined Gopakumar–Vafa invariants $n_{0,d,p}$ at genus 0 can be defined by the generating functions for $\alpha = k = 0, \ldots, N-1$

$$F_{0,k}(t) = \frac{1}{3!} D_i D_j D_k t^i t^j t^k + \frac{1}{2} A_{ij} t^i t^j + \frac{1}{24} \mathrm{c}_2(\widehat{X}, k) D_i t^i + \frac{1}{2}\chi(\widehat{X}, k) \frac{\zeta(3)}{(2\pi i)^3} + \sum_{d \geq 1} \sum_{p=0}^{N-1} n_{0,d,p} \mathrm{Li}_3(q^d \exp(2\pi i \tfrac{kp}{N}))$$

with

$$\mathrm{c}_2(\widehat{X}, k) = \mathrm{c}_2(X'), \quad \chi(\widehat{X}, k) = \chi(X') + r(k) \cdot \#\text{nodes}, \, r(k) \in \mathbb{Q}$$

Braun, Kreuzer, Ovrut, S (2007), Schimannek (2021) Katz, Klemm, Schimannek, Sharpe (2023). McGovern, Knapp (2025)

In our case $\mathrm{Br}(\widehat{X}) \cong \mathbb{Z}_3$. $F_{0,0}$ is determined by the hemisphere partition function of $Y$, while $F_{0,\pm}$ is determined by the hemisphere partition function of $\widehat{X}$ by analytic continuation.

## Torsion refined Gopakumar–Vafa invariants for $\widehat{X}$

| $n_{g=0,d_1,d_2,p=0}$ | $d_2 = 0$ | 1 | 2 | 3 |
|---|---|---|---|---|
| $d_1 = 0$ | | 880 | 74332 | 12882160 |
| 1 | 112 | 72032 | 50681264 | 33340279712 |
| 2 | 68 | 2131216 | 6546189640 | 11610720130560 |
| 3 | 112 | 34995904 | 393483773040 | 1654179109531904 |
| 4 | 68 | 403119552 | 14399669203616 | 133685979448605792 |
| 5 | 112 | 3639310496 | 369088980923696 | 7169735868108075232 |
| 6 | 68 | 27426473760 | 7221764858585392 | 280236931072747481392 |
| 7 | 112 | 179500302208 | 114206981203048656 | 8498841107173098879808 |

| $n_{g=0,d_1,d_2,p=\pm 1}$ | $d_2 = 0$ | 1 | 2 | 3 |
|---|---|---|---|---|
| $d_1 = 0$ | | 808 | 74710 | 12877672 |
| 1 | 88 | 72272 | 50675240 | 33340382768 |
| 2 | 92 | 2130040 | 6546228604 | 11610718718208 |
| 3 | 88 | 34999456 | 393483534792 | 1654179120738944 |
| 4 | 92 | 403108896 | 14399670267728 | 133685979371288016 |
| 5 | 88 | 3639336368 | 369088976530280 | 7169735868532957072 |
| 6 | 92 | 27426412656 | 7221764874150376 | 280236931070633558632 |
| 7 | 88 | 179500431424 | 114206981151361944 | 8498841107182367400544 |