Bulk fields in conformal field theory

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based on work with Jürgen Fuchs and Gregor Schaumann

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Bulk fields in 2d CFT

Long history:

- Associativity constraints for OPE (1980s)
- ADE-Classifications based on modular data (1980-1990s) and their failure
- TFT construction of RCFT correlators:
  from semisimple modular tensor category
  and a semisimple indecomposable module category over it (2000)

Focus today:

- Beyond semisimplicity (Logarithmic conformal field theory)
- still keeping finiteness properties.

A tour,
starting with some classical representation theory of finite-dimensional algebras,
then turning to monoidal categories and module categories over them
and ending with bulk and defect fields.

Overview

1. Eilenberg-Watts calculus and Nakayama functors
   - Finite tensor categories
   - Coends
   - Eilenberg-Watts equivalences and Nakayama functors
2. Module categories, relative Serre functors
   - Radford's S^4-theorem for bimodules
   - Relative Serre functors and pivotal module categories
3. The field content of two-dimensional local conformal field theories
   - Fields in two-dimensional conformal field theories
   - Frobenius bulk algebras from pivotal module categories
   - Outlook

Chapter 1

Eilenberg-Watts calculus and Nakayama functors

Finite tensor categories

Let k be a field.

Definition (Finite category)

A k-linear category $\mathcal{C}$ is finite, if

1. $\mathcal{C}$ has finite-dimensional spaces of morphisms.
2. Every object of $\mathcal{C}$ has finite length.
3. $\mathcal{C}$ has enough projectives.
4. There are finitely many isomorphism classes of simple objects.

Remark

A linear category is finite, if and only if it is equivalent to the category A-mod of finite-dimensional A-modules over a finite-dimensional k-algebra.

Definition (Finite tensor category)

A finite tensor category is a finite rigid monoidal linear category.

In particular, the tensor product is exact in each argument.

Eilenberg-Watts calculus

Classical result about finite categories:

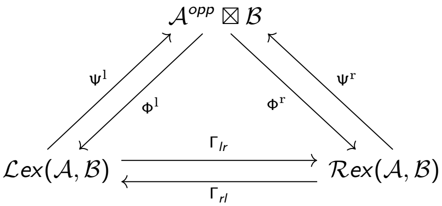
Proposition

Let A-mod and B-mod be finite tensor categories. Let

$$G : A\text{-mod} \to B\text{-mod}$$

be a right exact functor. Then $G \cong G({}_AA_A) \otimes_A -$.
The B-A-bimodule $G({}_AA_A)$ is a right A-module via the image of right multiplication $r_A : A \to A$ under $\mathrm{End}_A(A) \xrightarrow{G} \mathrm{End}_B(G(A))$.
A similar statement allows to express left exact functors in terms of bimodules.

Morita-invariant formulation: triangle of explicit adjoint equivalences, based on the Deligne product and (co)ends.



Coends

Implement the "sum over all states":
- Do not sum over all irreps up to isomorphism.
- Sum over *all* representations up to *all* morphisms.

Coend:

$$\bigoplus_{X \xrightarrow{f} Y} (Y^\vee \otimes X)_f \rightrightarrows \bigoplus_{X \in \mathcal{C}} X^\vee \otimes X \to \int^{X \in \mathcal{C}} X^\vee \otimes X \to 0$$

"Direct sum over all objects, with all morphisms taken into account."

The components of the two maps are for $X \xrightarrow{f} Y$

$$(Y^\vee \otimes X)_f \xrightarrow{f^\vee \otimes \mathrm{id}_X} X^\vee \otimes X \quad \text{and} \quad (Y^\vee \otimes X)_f \xrightarrow{\mathrm{id}_{Y^\vee} \otimes f} Y^\vee \otimes Y$$

Universal property. Coends are generalizations of direct sums. Direct sums are characterized by the fact that maps out of direct sums are families of maps:

$$\mathrm{Hom}(\oplus_i X_i, Y) \cong \prod_i \mathrm{Hom}(X_i, Y)$$

Ends are defined by reversing arrows.

Ends and coends

Remarks

- Examples of coends and ends: trace and natural transformations

$$\int^{v\in\mathrm{vect}_k} v\otimes v^* = k \qquad \text{and} \qquad \mathrm{Nat}(F,G) = \int_{c\in\mathbb{C}} \mathrm{Hom}_{\mathcal{D}}(F(c),G(c))$$

- (Co-)Yoneda lemma: $G:\mathcal{D}\to\mathcal{C}$ linear, then

$$\int^{Y\in\mathcal{D}} G(y)\otimes \mathrm{Hom}_{\mathcal{D}}(y,-)\cong G(-)$$

and

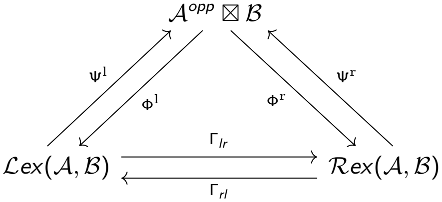
$$\int_{Y\in\mathcal{D}} G(y)\otimes \mathrm{Hom}_{\mathcal{D}}(-,y)^*\cong G(-)$$

Theorem (Fuchs, Schaumann, CS)

Peter-Weyl theorem: as A-bimodules

$$\int_{m\in A\text{-mod}} m\otimes_k m^* = A \qquad \textit{and} \qquad \int^{m\in A\text{-mod}} m\otimes_k m^* = A^*$$

Eilenberg-Watts calculus



$$\Phi^{\mathrm{l}} \equiv \Phi^{\mathrm{l}}_{\mathcal{A},\mathcal{B}} : \quad \begin{array}{l} \mathcal{A}^{opp} \boxtimes \mathcal{B} \xrightarrow{\simeq} \mathcal{L}ex(\mathcal{A},\mathcal{B}) \,, \\ \overline{a} \boxtimes b \longmapsto \mathrm{Hom}_{\mathcal{A}}(a,-) \otimes b \,, \end{array}$$

$$\Psi^{\mathrm{l}} \equiv \Psi^{\mathrm{l}}_{\mathcal{A},\mathcal{B}} : \quad \begin{array}{l} \mathcal{L}ex(\mathcal{A},\mathcal{B}) \xrightarrow{\simeq} \mathcal{A}^{opp} \boxtimes \mathcal{B} \,, \\ F \longmapsto \int^{a\in\mathcal{A}} \overline{a} \boxtimes F(a) \,, \end{array}$$

$$\Phi^{\mathrm{r}} \equiv \Phi^{\mathrm{r}}_{\mathcal{A},\mathcal{B}} : \quad \begin{array}{l} \mathcal{A}^{opp} \boxtimes \mathcal{B} \xrightarrow{\simeq} \mathcal{R}ex(\mathcal{A},\mathcal{B}) \,, \\ \overline{a} \boxtimes b \longmapsto \mathrm{Hom}_{\mathcal{A}}(-,a)^{*} \otimes b \,, \end{array}$$

$$\Psi^{\mathrm{r}} \equiv \Psi^{\mathrm{r}}_{\mathcal{A},\mathcal{B}} : \quad \begin{array}{l} \mathcal{R}ex(\mathcal{A},\mathcal{B}) \xrightarrow{\simeq} \mathcal{A}^{opp} \boxtimes \mathcal{B} \,, \\ G \longmapsto \int_{a\in\mathcal{A}} \overline{a} \boxtimes G(b) \end{array}$$

In particular, $\mathrm{id}_{\mathcal{A}} \in \mathcal{L}ex(\mathcal{A},\mathcal{A})$ is mapped to the right exact functor

Nakayama functors

$$N^r_{\mathcal{A}} := \int^{a\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(-,a)^* \otimes a \quad \text{and} \quad N^l_{\mathcal{A}} := \int_{a\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(a,-) \otimes a$$

For $\mathcal{A} = A\text{-mod}$:

$$N^r_{\mathcal{A}} = A^* \otimes_A - \cong \mathrm{Hom}_A(-,A)^* \quad \text{and} \quad N^l_{\mathcal{A}} = \mathrm{Hom}_A(A^*,-) \; .$$

For this reason, we call $N^r_{\mathcal{A}}$ and $N^l_{\mathcal{A}}$ Nakayama functors.

Proposition

1. *The Nakayama functors are adjoints,* $N^l_{\mathcal{A}} \dashv N^r_{\mathcal{A}}$.
2. $N^l_{\mathcal{A}}$ *equivalence* $\Leftrightarrow$ $N^r_{\mathcal{A}}$ *equivalence.* $\Leftrightarrow$ $\mathcal{A}$ *is selfinjective.*
3. $N^l_{\mathcal{A}} \cong \mathrm{id}_{\mathcal{A}}$ *and* $N^r_{\mathcal{A}} \cong \mathrm{id}_{\mathcal{A}}$ $\Leftrightarrow$ $\mathcal{A}$ *is symmetric Frobenius.*

Corollary

There is a canonical isomorphism

$$\int^{a\in\mathcal{A}} \overline{a} \boxtimes a = \Psi^l(\mathrm{id}_{\mathcal{A}}) \cong \Psi^r \Phi^r \Psi^l(\mathrm{id}_{\mathcal{A}}) = \Psi^r(N^r_{\mathcal{A}}) = \int_{a\in\mathcal{A}} \overline{a} \boxtimes N^r_{\mathcal{A}}(a)$$

Chapter 2

Monoidal categories,module categories and relative Serre functors

Module categories

Definition (Module categories)

Let $\mathcal{A}$ and $\mathcal{B}$ be linear monoidal categories.

1. A left $\mathcal{A}$-module category is a linear category $\mathcal{M}$ with a bilinear functor $\otimes : \mathcal{A} \times \mathcal{M} \to \mathcal{M}$ and natural isomorphisms

$$\alpha : \otimes \circ (\otimes \times \mathrm{id}_{\mathcal{M}}) \xrightarrow{\sim} \otimes \circ (\mathrm{id}_{\mathcal{A}} \times \otimes) \qquad \lambda : \otimes \circ (\mathrm{id}_{\mathcal{A}} \times -) \xrightarrow{\sim} \mathrm{id}_{\mathcal{M}}$$

   satisfying obvious pentagon and triangle axioms. We write $a.m := a \otimes m$.
2. Right module categories are defined analogously.
3. An $\mathcal{A}$-$\mathcal{B}$ bimodule category is a linear category $\mathcal{D}$, with the structure of a left $\mathcal{A}$ and right $\mathcal{D}$-module category and a natural associator isomorphism $(a.d).b \cong c.(d.b)$.
4. Module functors, module natural transformations defined in obvious way.

Example

Any monoidal category $\mathcal{A}$ is a bimodule category over itself.

Internal homs

Definition (Finite module categories)

Let $\mathcal{A}$ be a finite tensor category over k. A left $\mathcal{A}$-module category is finite, if the underlying category is a finite abelian category over k and the action is k-linear in each variable and right exact in the first variable.

Definition (Internal Hom)

Let $\mathcal{M}$ be a $\mathcal{C}$-module category and $m, m' \in \mathcal{M}$.
Then the internal Hom $\underline{\mathrm{Hom}}(m, m') \in \mathcal{C}$ is the object such that $\mathrm{Hom}_{\mathcal{C}}(c, \underline{\mathrm{Hom}}(m, m')) \cong \mathrm{Hom}_{\mathcal{M}}(c.m, m')$ for all $c \in \mathcal{C}$.

Examples

- $\mathcal{C}$ super vector spaces. Homs are grade preserving linear maps. Internal Homs are super vector spaces and have an odd component.
- For $\mathcal{M} = \mathcal{C}$, we have $\underline{\mathrm{Hom}}(c, c') = c' \otimes c^{\vee}$.

Internal Homs admit an associative composition:

$$\underline{\mathrm{Hom}}(m', m'') \otimes \underline{\mathrm{Hom}}(m, m') \to \underline{\mathrm{Hom}}(m, m'')$$

Radford's S^4-theorem

For linear functors, we have

Theorem (Fuchs, Schaumann, CS)

Let $\mathcal{A}, \mathcal{B}$ be finite categories. Let $F \in \mathcal{L}ex(\mathcal{A}, \mathcal{B})$ such that F^{la} is left exact so that F^{lla} exists. Assume that F^{lla} is left exact as well.
Then there is a natural isomorphism

$$\varphi^l_F : \quad N^l_{\mathcal{B}} \circ F \cong F^{lla} \circ N^l_{\mathcal{A}}$$

that is coherent with respect to composition of functors.

Apply this to bimodule categories over finite tensor categories:

Theorem (Fuchs, Schaumann, CS)

Let $\mathcal{A}, \mathcal{B}$ be finite tensor categories and $\mathcal{M}$ an $\mathcal{A}$-$\mathcal{B}$ bimodule.
Then the Nakayama functor has the structure of a twisted bimodule functor:

$$N^l_{\mathcal{M}}(a.m.b) \cong a^{\vee\vee}.N^l_{\mathcal{M}}(m).{}^{\vee\vee}b$$

Recovering Radford's S^4-theorem

$$N^l_{\mathcal{M}}(a.m.b) \cong a^{\vee\vee}.N^l_{\mathcal{M}}(m).{}^{\vee\vee}b$$

Observe

- The finite tensor category $\mathcal{A}$ is a bimodule over itself.
- $$N^l_{\mathcal{A}}(1) = \int_{a\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(a,1)\otimes a = D_{\mathcal{A}}$$
  is the distinguished invertible object of $\mathcal{A}$.
- Compute
  $$N^l_{\mathcal{A}}(a) = N^l_{\mathcal{A}}(a\otimes 1) = a^{\vee\vee}\otimes N^l_{\mathcal{A}}(1) = a^{\vee\vee}\otimes D_{\mathcal{A}}$$
  and
  $$N^l_{\mathcal{A}}(a) = N^l_{\mathcal{A}}(1\otimes a) = N^l_{\mathcal{A}}(1)\otimes {}^{\vee\vee}a = D_{\mathcal{A}}\otimes {}^{\vee\vee}a$$
- We recover Radford's S^4-theorem in its categorical form $D_{\mathcal{A}}\otimes a\otimes D_{\mathcal{A}}^{-1} = a^{\vee\vee\vee\vee}$ [ENO, 2004]

Relative Serre functors

Definition (Fuchs, Schaumann, CS)

Let $\mathcal{M}$ be a $\mathcal{C}$-module. A right/left relative Serre functor is an endofunctor $\mathrm{S}^{\mathrm{r}}_{\mathcal{M}}$ / $\mathrm{S}^{\mathrm{l}}_{\mathcal{M}}$ of $\mathcal{M}$ together with a family

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(m,n)^{\vee} & \xrightarrow{\cong} & \underline{\mathrm{Hom}}(n,\mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(m)) \\ {}^{\vee}\underline{\mathrm{Hom}}(m,n) & \xrightarrow{\cong} & \underline{\mathrm{Hom}}(\mathrm{S}^{\mathrm{l}}_{\mathcal{M}}(n),m) \end{array}$$

of isomorphisms natural in $m,n \in \mathcal{M}$.

- Relative Serre functors exist, iff $\mathcal{M}$ is an exact module category (i.e. $p.m$ is projective, if $p \in \mathcal{C}$ is projective).
- Serre functors are equivalences of categories.
- Serre functors are twisted module functors:

$$\phi_{c,m}:\ \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(c.m) \longrightarrow c^{\vee\vee}.\,\mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(m) \quad \text{and} \quad \tilde{\phi}_{c,m}:\ \mathrm{S}^{\mathrm{l}}_{\mathcal{M}}(c.m) \longrightarrow {}^{\vee\vee}c.\,\mathrm{S}^{\mathrm{l}}_{\mathcal{M}}(m)$$

Theorem

Let $\mathcal{M}$ be an exact $\mathcal{A}$-module. Then

$$N^{l}_{\mathcal{M}} \cong D_{\mathcal{A}}.\mathrm{S}^{\mathrm{l}}_{\mathcal{M}} \quad \textit{and} \quad N^{r}_{\mathcal{M}} \cong D_{\mathcal{A}}^{-1}.\mathrm{S}^{\mathrm{r}}_{\mathcal{M}}$$

Pivotal module categories

Serre functors are twisted module functors:

$$\phi_{c,m} : \quad \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(c.m) \longrightarrow c^{\vee\vee}.\,\mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(m) \quad \text{and} \quad \tilde{\phi}_{c,m} : \quad \mathrm{S}^{\mathrm{l}}_{\mathcal{M}}(c.m) \longrightarrow {}^{\vee\vee}c.\,\mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(m)\,.$$

Definition (Schaumann 2015, Shimizu 2019)

A pivotal structure on an exact module category $\mathcal{M}$ over a pivotal finite tensor category $(\mathcal{C}, \pi)$ is an isomorphism of functors $\tilde{\pi} : \mathrm{id}_{\mathcal{M}} \rightarrow \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}$ such that the following diagram commutes for all $c \in \mathcal{C}$ and $m \in \mathcal{M}$:

$$\begin{array}{ccc} c.m & \xrightarrow{\pi_c.\tilde{\pi}_m} & c^{\vee\vee}.\mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(m) \\ & {\scriptstyle \tilde{\pi}_{c.m}} \searrow \quad \nearrow {\scriptstyle \phi_{c,m}} & \\ & \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(c.m) & \end{array}$$

- For indecomposable exact module categories, the pivotal structure is unique up to scalar.
- The algebras $\underline{\mathrm{Hom}}(m, m) \in \mathcal{C}$ for m in a pivotal module category have the structure of symmetric Frobenius algebras.

Chapter 3

The field content of two-dimensional local conformal field theories

Reminder about chiral conformal field theory

Definition (Modular tensor category)

A modular tensor category $\mathcal{C}$ is a finite ribbon category such that the braiding is maximally non-degenerate.
Various formulations exist and are equivalent [Shimizu 2016]:

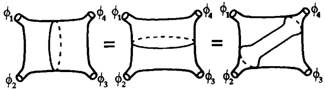
- Braided equivalence $\mathcal{C} \boxtimes \mathcal{C}^{rev} \simeq \mathcal{Z}(\mathcal{C})$
- Coend $L := \int^{\mathcal{C}} U^{\vee} \otimes U$ has non-degenerate Hopf pairing $\omega_{\mathcal{C}}$
- Map $\mathrm{Hom}(1, L) \to \mathrm{Hom}(L, 1)$ induced by $\omega_{\mathcal{C}}$ is isomorphism.
- $\mathcal{C}$ has no transparent objects.

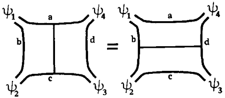
Remarks

- The representation category of suitable vertex algebras or nets of observable algebras has naturally the structure of a modular tensor category:
  The chiral data of a (finite) conformal field theory are described by a modular tensor category.
- From a modular tensor category, one can construct a modular functor (Lyubashenko, $\sim$ 1995)

Fields in two-dimensional local conformal field theory

- Fields + OPE $\rightsquigarrow$ (symmetric Frobenius) algebras.
- Symmetric Frobenius algebras in the appropriate monoidal category

(a)


(c)


Additional datum to specify local CFT given a modular tensor category: Suitable module category $\mathcal{M}$ over the modular tensor category $\mathcal{C}$.

Boundary

Boundary condition:	Object of $\mathcal{M}$
Boundary fields from bc m to n	$\underline{\mathrm{Hom}}(m,n) \in \mathcal{C}$
OPE	composition of inner Homs

- Modular tensor category $\mathcal{C}$ is pivotal.
- Require $\mathcal{M}$ to be a pivotal module category
- Then $\underline{\mathrm{Hom}}(m,m)$ is a symmetric Frobenius algebra for each $m \in \mathcal{M}$.

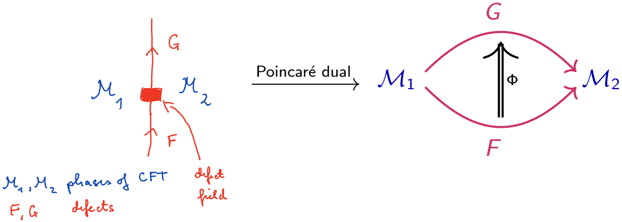
Bulk algebra: commutative symmetric Frobenius algebra in $\mathcal{C} \boxtimes \mathcal{C}^{rev} \simeq \mathcal{Z}(\mathcal{C})$.

Tasks:

1. Obtain bulk Frobenius algebras from boundary data
2. Obtain also descriptions of defect fields and disorder fields.

Bulk fields and defect fields for a fixed modular tensor category $\mathcal{C}$

Include defects and defect fields:



Defects are labelled by right exact $\mathcal{C}$-module functors $F, G : \mathcal{M}_1 \to \mathcal{M}_2$

For defect field, need an object $\mathbb{D}^{F,G} \in \mathcal{Z}(\mathcal{C}) \simeq \mathcal{C}^{rev} \boxtimes \mathcal{C}$:
Fact:
$\mathcal{R}ex_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ is a $\mathcal{Z}(\mathcal{C})$-module by $(c.F)(m_1) := c.F(m_1)$
and module functor structure given by half-braiding.
Consider the internal homs $\underline{\mathrm{Nat}}(F, G) \in \mathcal{Z}(\mathcal{C})$ of this module category.

Bulk and defect fields II

Theorem

$$\underline{\mathrm{Nat}}(F,G) \in \mathcal{Z}(\mathcal{C}) = \int_{m_1\in\mathcal{M}_1} \underline{\mathrm{Hom}}(F(m_1),G(m_1)) \;\in \mathcal{Z}(\mathcal{C})$$

Remarks

- Recall natural transformations:

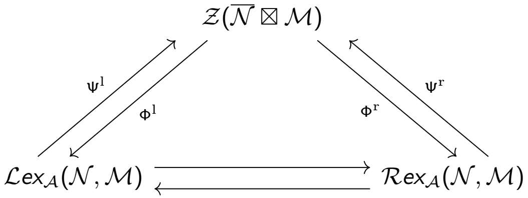
$$\mathrm{Nat}(F,G) \;=\; \int_{m_1\in\mathcal{M}_1} \mathrm{Hom}(F(m_1),G(m_1)) \;\subset\; \prod_{m_1\in\mathcal{M}_1} \mathrm{Hom}(F(m_1),G(m_1))$$

  For $\mathcal{C}=\mathcal{M}=A$-mod, get $Z(A)=\mathrm{Nat}(\mathrm{id},\mathrm{id}) \;=\; \int_{m_1\in\mathcal{M}_1} \mathrm{Hom}(m_1,m_1)$

- Defect fields = "internalized" natural transformations.
  In particular, bulk algebra $= \int_{m\in\mathcal{M}} \underline{\mathrm{Hom}}(m,m) =$ "internalized center".
- We have horizontal and vertical compositions of relative natural transformations, obeying the usual relations, including Eckmann-Hilton.

Symmetric Frobenius algebras

For CFT, we need symmetric Frobenius algebras.
A module Eilenberg-Watts calculus which for a pivotal tensor category $\mathcal{C}$ yields



Theorem

$\mathcal{C}$ be a pivotal finite tensor category and $\mathcal{M}$ and $\mathcal{N}$ exact $\mathcal{C}$-modules.

1. *The functor category $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ is an exact module category over $\mathcal{Z}(\mathcal{C})$ with relative Serre functor $N^r_{\mathcal{N}} \circ (D.-) \circ N^r_{\mathcal{M}}$.*
2. *If $\mathcal{C}$ is unimodular pivotal and $\mathcal{M}$ and $\mathcal{N}$ are pivotal $\mathcal{C}$-modules, then $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ is a pivotal $\mathcal{Z}(\mathcal{C})$-module category.*
3. *In particular, then $\underline{\mathrm{Nat}}(F,F)$ is a symmetric Frobenius algebra in the Drinfeld center $\mathcal{Z}(\mathcal{C})$ and $\underline{\mathrm{Nat}}(\mathrm{id}_{\mathcal{M}},\mathrm{id}_{\mathcal{M}})$ has a natural structure of a commutative symmetric Frobenius algebra.*

Sewing constraints

(a) (b) (c)

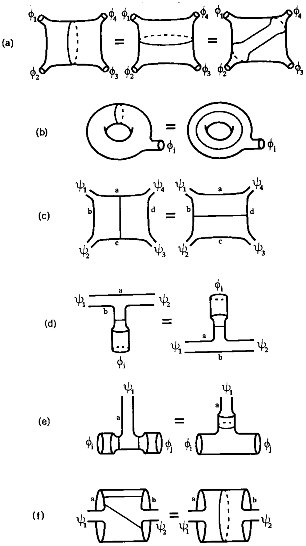
(Lewellen, 1992)

Structure morphisms:

– Multiplications and comultiplications
– Component maps $\underline{\mathrm{Nat}}(\mathrm{id}, \mathrm{id}) \to \underline{\mathrm{Hom}}(m, m)$

Relations:

– (a), (c): bulk and boundary are Frobenius
– (e): component map is morphism of algebras
– (d) dinaturality of the (co)end component morphisms
– (b) and (f)=Cardy relation are genus 1



Outlook

Outlook

1. Genus one constraints.
2. Description of correlators via modular functors.
3. Beyond rigid categories.