A local approach to Anosov groups

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Anosov representations: motivation

2006 Labourie $\rho\colon \pi_1(S_g) \to \mathrm{PSL}_n\mathbb{R}$ $(g \geq 2)$.
Gives a geometric interpretation for representations in the Hitchin component
$H_n(S_g) = \mathrm{hom}_0(\pi_1(S_g) \to \mathrm{PSL}_n\mathbb{R})/\mathrm{PSL}_n\mathbb{R}$.
$H_2(S_g) = \mathrm{Teich}(S_g)$, $H_3(S_g) = \mathrm{Proj}(S_g)$, $H_n(S_g) = ?$

2012 Guichard-Wienhard $\rho\colon \Gamma \to G$, Γ Gromov hyperbolic, G semisimple Lie group

- Anosov is the higher rank analog of convex cocompact

- Anosov group: image of Anosov representation
- Symmetric space approach: $X = G/K$ symmetric space of non-compact type
  - Kapovich-Leeb-P
  - Guéritaud-Guichard-Kassel-Wienhard
  - ...

Goal: Give a characterization of Anosov from finitely many elements of Γ.

Parabolic subgroups and flag manifolds

Def: G semisimple Lie group. $P < G$ is parabolic if G/P is a projective variety

Example: $G = \mathrm{SL}_n\mathbb{R}$ and P= {upper triangular matrices}. G/P is the flag manifold

$$\mathrm{Flag}(\mathbb{P}^{n-1}) = \{f_0 < f_1 < \cdots < f_{n-2} \subset \mathbb{P}^{n-1} \mid f_i \text{ linear and } \dim f_i = i\}$$

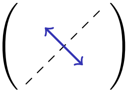
Example: $G = \mathrm{SL}_3\mathbb{R}$ has 3 conjugacy classes of parabolic subgroups

$$P_1 = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix},\ P_2 = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix},\ B = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

$$G/P_1 = \mathbb{P}^2,\ G/P_2 = \check{\mathbb{P}}^2,\ G/B = \{(p, l) \in \mathbb{P}^2 \times \check{\mathbb{P}}^2 \mid p \in l\}$$

- P_1 is opposite to P_2, $B^{opp} = B$
- Opposition: is an involution on the space of conjugacy classes of parabolic subgroups (duality in flag manifolds/Cartan involution in the symmetric space)

- In $\mathrm{SL}_n\mathbb{R}$, opposition is reflection with respect to the antidiagonal



More examples of parabolic subgroups and flag manifolds

Ex: $G = \mathrm{SL}_4\mathbb{R}$ has 7 conjugacy classes of non-trivial parabolic subgroups.
3 of them are self-opposite:

$$P_1 = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \qquad P_2 = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \qquad B = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$$

The flag manifolds are:

- $G/P_1 = \{(p,h) \in \mathbb{P}^3 \times \check{\mathbb{P}}^3 \mid p \in h\}$
- $G/P_2 = L(\mathbb{P}^3) = \{l \mid l \text{ proj line in } \mathbb{P}^3\}$
- $G/B = \{(p,l,h) \in \mathbb{P}^3 \times L(\mathbb{P}^3) \times \check{\mathbb{P}}^3 \mid p \in l \subset h\}$

G/B is the full flag manifold, G/P_i are partial flag manifolds

Ex: $G = \mathrm{SL}_2\mathbb{R} \times \mathrm{SL}_2\mathbb{R}$, $X = G/K = \mathbb{H}^2 \times \mathbb{H}^2$.

- G has 3 conjugacy classes of non-trivial parabolic subgroups:
  $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \times \mathrm{SL}_2\mathbb{R}, \qquad \mathrm{SL}_2\mathbb{R} \times \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \qquad \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \times \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$
- They are all self-opposite.
- Flag manifolds: $\partial_\infty \mathbb{H}^2 \times \{*\}$, $\{*\} \times \partial_\infty \mathbb{H}^2$ and $\partial_\infty \mathbb{H}^2 \times \partial_\infty \mathbb{H}^2$.

Definition of Anosov representation

- G semisimple, $P \subset G$ self-opposite conjugacy class of parabolic subgroups, and Γ word-hyperbolic group.

Def A representation $\rho \colon \Gamma \to G$ is P-Anosov if:

a) $\exists \beta \colon \partial_\infty \Gamma \to G/P$ antipodal Γ-equivariant embedding

b) $\forall r \colon \mathbb{N} \to \Gamma$ normalized geodesic ray, $r(+\infty) = \xi \in \partial_\infty \Gamma$,

$$\lim_{n \to +\infty} |d_{\beta(\xi)}(\rho(r(n)))| = 0$$

(i.e. $\rho(r(n)) \in G$ contracts at $\beta(\xi) \in G/P$ by factor $\to 0$)

- *Antipodal:* for $\xi \neq \xi'$, $\beta(\xi)$ and $\beta(\xi')$ are antipodal flags (generic)
- *normalized geod ray:* $r(0) = e \in \Gamma$ and $d_\Gamma(r(m), r(n)) = |m - n|$

Remark: Why called Anosov? Labourie considers the geodesic flow on $T^1 S_g$ and Guichard-Wienhard, the geodesic flow on Γ

- Anosov representations are discrete and have finite kernel.

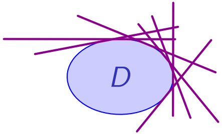
Examples

Ex: Let $\Gamma < G = \mathrm{Isom}(\mathbb{H}^n) \cong \mathrm{PO}(n,1)$ discrete with limit set $\Lambda \subset \partial_\infty \mathbb{H}^n$.

- $\mathrm{convex\ hull}(\Lambda) \subset \mathbb{H}^n$ is the smallest convex with ideal boundary Λ.
- Γ is convex cocompact if $\mathrm{convex\ hull}(\Lambda)/\Gamma$ is compact.
- Γ cvx cocompact iff Γ is P-Anosov (for $P = \mathrm{Stab}_G(\xi)$, $\xi \in \partial_\infty \mathbb{H}^n$, so $G/P = \partial_\infty \mathbb{H}^n$).
- Γ cvx cocompact iff the orbit map $\Gamma \mapsto \Gamma x \subset \mathbb{H}^n$ quasi-isometric embedding.

Labourie Representations in the Hitchin component of $\hom(\pi_1(S_g) \to \mathrm{PSL}_{n+1}\mathbb{R})$ are B-Anosov, with $B < \mathrm{PSL}_{n+1}(\mathbb{R})$ upper triangular matrices.

Benoist: For D/Γ strictly convex closed projective manifold, $\Gamma < \mathrm{PGL}_{n+1}(\mathbb{R})$ is P-Anosov, where P=stabilizer of a partial flag in $\{(p,H) \in \mathbb{P}^n \times \check{\mathbb{P}}^n \mid p \in H\}$



$$M = D/\Gamma, \qquad D \Subset \mathbb{R}^n = \mathbb{RP}^n - \mathbb{RP}^{n-1}$$
$$\partial_\infty \Gamma \cong \{(p,H) \mid p \in \partial \overline{D},\ H = T_p \partial \overline{D}\}$$

Symmetric spaces of non-compact type

- $X = G/K$ is a symmetric space of non-compact type
  - $X \cong X_1 \times \cdots \times X_n$, with X_i irreducible, non-compact, and $X_i \ncong \mathbb{R}^k$
  - $G = \mathrm{Isom}_0(X)$ is a semisimple Lie group and $K < G$ a maximal compact subgroup

Examples $\mathbb{H}^n = \mathrm{PSO}(1, n)/\mathrm{SO}(n)$
$X = \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$
$\mathbb{H}^2 \times \mathbb{H}^2 = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2) \times \mathrm{SO}(2)$

Goal Characterize discrete $\Gamma < G$ that are Anosov according to the action on $X = G/K$

Def A flat is a totally geodesic $F \subset X$ isometric to $\mathbb{R}^k$.

Def The rank of X is the dimension of any maximal flat.

Remark:
- G acts transitively on the set of maximal flats
- $\sec(X) \leq 0$ and $\sec(X) < 0$ iff $\mathrm{rank}(X) = 1$
- Higher rank: $r = \mathrm{rank}(X) \geq 2$, X contains flats of dim≥ 2
- Anosov: negative curvature behavior in higher rank.

Maximal flats, Weyl group, and Weyl chambers

Examples

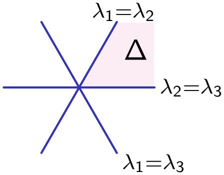
- $X = \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$, $\mathrm{rank}(X) = n-1$

  Maximal flat: $\exp(\mathfrak{a})$ where $\mathfrak{a} = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \mid \lambda_1 + \cdots + \lambda_n = 0 \right\}$
- $\mathrm{rank}(\mathbb{H}^n) = 1$, $\mathrm{rank}(\mathbb{H}^m \times \mathbb{H}^n) = 2$

Def Weyl group: stabilizer of a pair (Maximal flat, point).
The Weyl group W acts as a Coxeter group (with reflection walls and a fundamental domain Δ called Weyl chamber).

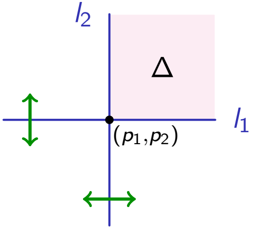
- For $X = \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$
  - Weyl group; W =permutation group of the $\lambda_1, \ldots, \lambda_n$
  - Weyl chamber: $\Delta = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}$

$(n = 3)$ $\quad \lambda_1 + \lambda_2 + \lambda_3 = 0$



Weyl group (contd)

Ex: $\mathbb{H}^2 \times \mathbb{H}^2$. Maximal flats are products of lines $l_1 \times l_2$.
Let $(p_1, p_2) \in l_1 \times l_2$. The Weyl group W is generated by π-rotations on a factor $\mathbb{H}^2$ around p_i (inversions on l_i)
$W \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and Δ is the product of two rays.



Remark: A line in $l_1 \times l_2$ through (p_1, p_2) is contained in more than one maximal flat iff it is a wall (constant in one factor):

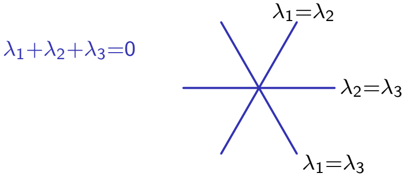
- $l_1 \times \{p_2\} \subset l_1 \times l_2'$ for any line $l_2' \subset \mathbb{H}^2$ containing p_2.
- $\{p_1\} \times l_2 \subset l_1' \times l_2$ for any line $l_1' \subset \mathbb{H}^2$ containing p_1.

Singular and regular directions

Def: A geodesic is regular if contained in a *unique* maximal flat, and singular if contained in *more that one* maximal flat.

Lemma: Singular geodesics through x_0 are those contained in walls.

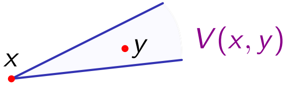
Ex: $X = \mathrm{SL}_3(\mathbb{R})/\mathrm{SO}(3)$, $\mathfrak{a} = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \mid \lambda_1 + \lambda_2 + \lambda_3 = 0 \right\}$



$\{\lambda_1 = \lambda_2\}$ contained in $\mathfrak{a} \cap g\mathfrak{a}g^{-1}$ for $g = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Regularity

Def Given any two points $x \neq y \in X$ there exist always a maximal flat containing them, and Weyl chamber $V(x,y) \subset X$ with tip x and containing y



$V(x,y) \subset X$ is unique if the segment $\overline{xy}$ is regular (not in a wall)

- if $\operatorname{rank} X = 1$, $V(x,y)$ is just a ray.

Def For $\varepsilon > 0$, the segment $\overline{xy}$ is ε-regular if $\dfrac{d(y, \partial V(x,y))}{d(x,y)} > \varepsilon$

Ex $X = \mathrm{SL}_3(\mathbb{R})/\mathrm{SO}(3)$, $x = \mathrm{Identity}$, $y = \mathrm{diag}(e^{\lambda_1}, e^{\lambda_2}, e^{\lambda_3})$, $\lambda_1 \geq \lambda_2 \geq \lambda_3$

$$\overline{xy} \text{ is } \varepsilon\text{-regular iff } \frac{\min(\lambda_1 - \lambda_2, \lambda_2 - \lambda_3)}{(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^{1/2}} > \varepsilon$$

For $x = \mathrm{Identity}$ and any $y \in X$, look at singular values $\sqrt{\mathrm{eigenvalues\ of\ } y^t y}$

Remark: A regular direction and a direction perpendicular to the Weyl chamber span a tangent plane with negative curvature

Uniform regularity, undistortedness and Anosov

Def $\Gamma < G$ is *uniformly regular* if, for any $\gamma_1, \gamma_2 \in \Gamma$, with $d(\gamma_1 x, \gamma_2 x) > N$

$$\overline{\gamma_1 x, \gamma_2 x} \text{ is } \varepsilon\text{-uniformly regular,}$$

for a given $x \in X$ and some uniform $N, \varepsilon > 0$.

Def $\Gamma < G$ is *undistorted* if Γ is finitely generated and the orbit map $\begin{cases} \Gamma & \to & X \\ \gamma & \mapsto & \gamma x \end{cases}$ is a quasi-isometric embedding.

Thm (Kapovich-Leeb-P 2017)

$$\Gamma < G \text{ is } B\text{-Anosov iff it is uniformly regular and undistorted}$$

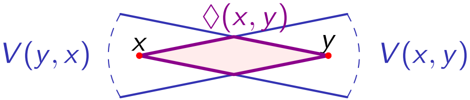
Remark:

- In particular uniformly regular and undistorted implies word hyperbolic
- B is the smallest possible parabolic subgroup (Borel subgroup)
  For other parabolic subgroups, adapt the definition of regularity: allow to approach certain walls (in terms of matrices allow some singular eigenvalues be equal)
- In rank one: Γ is convex cocompact iff it is undistorted.

Goal: find sufficient conditions for finitely many elements in Γ so that it is Anosov

Morse property

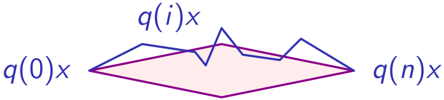
Def If the segment $\overline{xy} \subset X$ is regular, the Diamond $\Diamond(x, y) = V(x, y) \cap V(y, x)$



Def $\Gamma < G$ is Morse if for every $q \colon [0, n] \cap \mathbb{Z} \to \Gamma$ geodesic segment of length $n \geq N$

- $\overline{q(0)x, q(n)x}$ is ε-regular
- The orbit $i \mapsto q(i)x$ is (L, A)-quasi-geodesic
- $d(q(i)x, \Diamond(q(0)x, q(n)x)) < D$

(for some uniform L, A, N, ε, D)



Thm (KLP 2017) $\Gamma < G$ is B-Anosov iff it is unif. regular and undistorted iff it is Morse.

Remark:

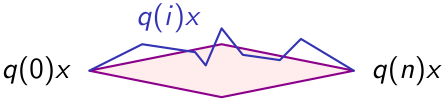
- The definition of Morse is stronger that uniform regularity
- The proof requires a higher rank Morse lemma
- Morse property can be localized (segments up to some length)

Local Morse

- Let Γ be a word hyperbolic group and $\rho\colon \Gamma \to G$ a representation.

Def A representation $\rho\colon \Gamma \to G$ is local Morse with constants (L, A, ε, D) at scale S if, for every geodesic segment $q\colon [0, S] \cap \mathbb{Z} \to \Gamma$ with $q(0) = e$:

- the orbit $i \mapsto q(i)x$ is (L, A) quasi-geodesic
- the segment $\overline{q(0)x, q(S)x}$ is ε-regular, and
- $d(q(i)x, \Diamond(q(0)x, q(n)x) < D$



Remark: Anosov implies local Morse for some constants (L, A, D, ε) and some scale S.

Thm (KLP) Local to global:
Given $X = G/K$ and $\varepsilon, L, A, D > 0$, there exist a scale S such that:
if Γ is word-hyperbolic and $\rho\colon \Gamma \to G$ is a representation local Morse at scale S with constants (L, A, D, ε), then ρ is global Morse (hence Anosov).

Consequence: Algorithmic semi-decidability

- Γ word hyperbolic and $\rho\colon \Gamma \to G$ a representation

Corollary There exists an algorithm that stops iff ρ is Anosov

Algorithm For each n, set $\varepsilon = \frac{1}{n}$, $L = A = D = n$ and find scale S_n provided by the theorem.

- If ρ is local Anosov at scale S with constants (L, A, D, ε), then stop.
- Otherwise proceed to $n + 1$.
  - If it stops, then ρ is Anosov by the theorem
  - If it is Anosov, then it is local Anosov for some constants and *every* scale large enough.
    If it is Anosov at step n, it is so at $n + 1$.
- $S = S(X, L, A, D, \varepsilon)$ is computable (M. Riestenberg)

Remark:

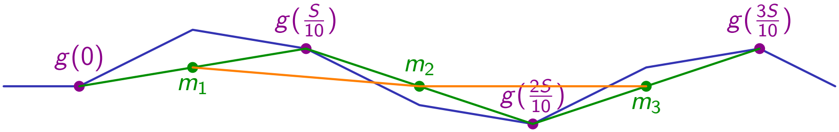
- Only SEMI-decidability because it may not stop.
- Not known before even in rank one.
- Discreteness for two-generator groups in $\mathrm{Isom}(\mathbb{H}^3)$ is undecidable in the Blum–Shub–Smale (BSS) computability model (M. Kapovich 2016).

Idea of the proof

Thm Given $X = G/K$ and $\varepsilon, L, A, D > 0$, there exist a scale S such that: if Γ is word-hyperbolic and $\rho\colon \Gamma \to G$ is a representation local Morse at scale S with constants (L, A, D, ε), then ρ is Anosov.

Idea: If $g\colon \mathbb{R} \to \mathbb{H}^n$ restricted to a any interval of length S is (L, A)-quasi-geodesic, then g is globally (L', A')-quasi-geodesic (for $S = S(L, A)$ sufficiently large)

- Sequence $g(\frac{S}{10}\mathbb{Z})$. Take midpoints and join them.



- By comparison, the angle at m_2 between $m_2 m_3$ and $m_2 m_1$ is close to π if S is sufficiently large.
- In $\mathbb{H}^n$ this yields that the path of mid-points is quasi-geodesic

- Apply the same argument but use regularity to guarantee "good negative curvature properties"

Thanks for your attention!