2-vector bundles, with applications to twisted K-theory and spin geometry

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Outline

I.) 2-vector spaces (:= algebras, bimodules, intertwiners)

II.) 2-vector bundles (⊇ algebra bundles, bundle gerbes)

III.) Applications:

a) Twisted K-theory (twist = 2-line bundles)

b) Spin geometry (spin lifting gerbe = Clifford bundle)

c) String geometry (the stringor bundle)

Bicategory of 2-vector spaces
:= algebras, bimodules, intertwiners

Remarks:
- Everything is over a field ($K = \mathbb{R}, \mathbb{C}$)
- Everything is graded: graded algebras, graded bimodules, even intertwiners
- For now, everything is finite-dimensional

Notation: 2-Vect_K

Note: 1-isomorphisms in 2-Vect_K are precisely the Morita equivalences.

2-vector spaces

The group of 1-isomorphism classes of GCSAs is called the Brauer-Wall group of the field K.

Real case

$h_0(GCSA_{\mathbb{R}}) \cong \mathbb{Z}_8$

$Cl^n \longleftrightarrow n$

Complex case

$h_0(GCSA_{\mathbb{C}}) \cong \mathbb{Z}_2$

$\mathbb{C}l^n \longleftrightarrow n$

Bott periodicity!

Wall '64, Donovan-Karoubi '70, Lam '04

Fact: 2-Vect_K is a symmetric monoidal bicategory.

Note: • the tensor unit is the ground field, K
• $\mathrm{End}_{2\text{-Vect}_K}(K) \cong K\text{-}K\text{-Bimod} \cong \mathrm{Vect}_K$

1.) Dualizability

Lemma: Every 2-vector space is dualizable.

- The dual is given by the opposed algebra, A^{opp}.
- evaluation $A \otimes A^{opp} \xrightarrow{A} K$
- coevaluation $K \xrightarrow{A} A^{opp} \otimes A$

2.) Full dualizability:

Theorem (ungraded: Bartlett-Douglas-Schommer-Pries-Vicary)

A 2-vector space is fully dualizable iff it is a semisimple algebra.

3.) Invertibility:

Classical:
A 2-vector space is invertible iff it is a graded-central-simple algebra (GCSA).

Because for a GCSA evaluation $A \otimes A^{op} \xrightarrow{A} K$ and coevaluation $K \xrightarrow{A} A^{op} \otimes A$ are Morita equivalences, i.e., 1-isomorphisms.

Note: GCSAs are the „2-Lines", and form the „Picard" sub-bicategory

$GCSA_K = (2\text{-Vect}_K)^{\times} \subseteq 2\text{-Vect}_K$

2-vector bundles

We work over smooth manifolds

First run: bicategory $Alg_{\mathbb{k}}$-$Bdl(X)$

- objects: graded algebra bundles over X
- 1-morphisms: graded bimodule bundles over X
- 2-morphisms: even intertwining bundle morphisms

We also consider the full sub-bicategory over the $GCSA_{\mathbb{k}}$-Bundles over X, $GCSA_{\mathbb{k}}$-$Bdl(X)$

Nice: varying M, these form a presheaf of bicategories over the category of smooth manifolds

SAD: they do not form a sheaf of bicategories (a.k.a. 2-stack)

← This is one of the main points of this talk: 2-vector bundles are not algebra bundles.

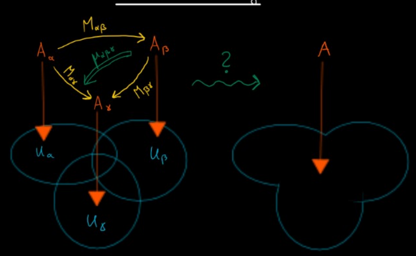
The group of isomorphism classes of objects in $GCSA_{\mathbb{k}}$-$Bdl(X)$ has been computed by Donovan–Karoubi and used as twists for K-theory („K-theory with local coefficients").

Theorem (Donovan-Karoubi '70)

$$h_0(GCSA_{\mathbb{R}}\text{-}Bdl(X)) \cong H^0(X,\mathbb{Z}_8) \times H^1(X,\mathbb{Z}_2) \times H^2(X,\mathbb{Z}_2)$$

$$h_0(GCSA_{\mathbb{C}}\text{-}Bdl(X)) \cong H^0(X,\mathbb{Z}_2) \times H^1(X,\mathbb{Z}_2) \times \mathrm{Tor}(H^3(X,\mathbb{Z}))$$

Sheaf = Gluing



To see whether or not $Alg_{\mathbb{k}}$-Bdl or $GCSA_{\mathbb{k}}$-Bdl form sheaves is a subtle point.

Lemma (PK-ML-DM-KW)

$GCSA_{\mathbb{R}}$-Bdl is a sheaf.

$GCSA_{\mathbb{C}}$-Bdl is not a sheaf.

$Alg_{\mathbb{C}}$-Bdl is not a sheaf.

We do not know whether or not $Alg_{\mathbb{R}}$-Bdl is a sheaf.

If something is not a sheaf, sheafify!

For presheaves of bicategories $\mathcal{F}$, Nikolaus-Schweigert invented the plus construction $\mathcal{F}^+$ and proved that $\mathcal{F}^+$ is a sheaf of bicategories.

Main Definition of this talk:

$$2\text{-}Vect\text{-}Bdl_{\mathbb{k}} := (Alg_{\mathbb{k}}\text{-}Bdl)^+$$

Thus, a 2-vector bundle over X is a tuple (Y, π, A, M, μ):

$$\begin{array}{ccccccc} & & A & & M & & \\ & & \downarrow & & \downarrow & & \\ X & \xleftarrow{\pi} & Y & \Leftarrow & Y^{[2]} & \Lleftarrow & Y^{[3]} \quad \Lleftarrow \quad Y^{[4]} \end{array}$$

- algebra bundle → A
- invertible bimodule bundle: $M_{y_1 y_2}$ is an A_{y_1}–A_{y_2}-bimodule
- $M_{y_2 y_3} \otimes_{A_{y_2}} M_{y_1 y_2} \overset{\mu}{\cong} M_{y_1 y_3}$ — invertible intertwining bundle morphism over $Y^{[3]}$, associative over $Y^{[4]}$
- surjective submersion, e.g. $Y = \coprod_\alpha U_\alpha$

Two particular examples of 2-vector bundles

1.) Algebra bundles are 2-vector bundles ($\mathcal{F} \subset \mathcal{F}^+$) ← fully faithful

$$\begin{array}{ccccccc} A & & A & & & & \\ \downarrow & & \downarrow & & & & \\ X & \xleftarrow{id} & X & \Leftarrow & X & \Lleftarrow & X \quad \Lleftarrow \quad X \end{array}$$

, remaining structure trivial

2.) Bundle gerbes are 2-vector bundles ($BVect_{\mathbb{k}} \hookrightarrow 2\text{-}Vect_{\mathbb{k}}$)

$$\begin{array}{ccccccc} & & Y\times\mathbb{C} & & L & & \mu \\ & & \downarrow & & \downarrow & & \\ X & \xleftarrow{\pi} & Y & \Leftarrow & Y^{[2]} & \Lleftarrow & Y^{[3]} \quad \Lleftarrow \quad Y^{[4]} \end{array}$$

3.) Freed-Hopkins-Teleman's twistings are precisely the complex 2-line bundles with vanishing class in $H^0(X,\mathbb{Z}_2)$.

Classifikation of 2-line bundles (i.e., A is a bundle of GCSAs)

Theorem (DM)

$$2\text{-}Line_{\mathbb{R}}\text{-}Bdl(X) \cong H^0(X,\mathbb{Z}_8) \times H^1(X,\mathbb{Z}_2) \times H^2(X,\mathbb{Z}_2)$$

$$2\text{-}Line_{\mathbb{C}}\text{-}Bdl(X) \cong H^0(X,\mathbb{Z}_2) \times H^1(X,\mathbb{Z}_2) \times H^3(X,\mathbb{Z})$$

This proves that $GCSA_{\mathbb{C}}$-Bdl is not a sheaf — sheafification added objects!

complex 2-line bundles represent geometrically the full group of twistings for K-theory! Including the grading!

Application to Twisted K-Theory

Darvin Mertsch's PhD thesis, submitted July 2020

Can one realize the twisted K-theory groups in this geometric way?

Yes, in case of a 2-line bundle $\mathcal{L}$ with torsion class. Then, one can mimik Karoubi's „grading description" of ordinary K-theory.

$$K^{\mathcal{L}}(X) := \left\{ (M, \eta_1, \eta_2) \right\} \big/ \eta_1 \sim \eta_2$$

- $M: \mathcal{L} \longrightarrow \mathbb{C}_X$ ungraded 1-morphism
- η_1, η_2: two gradings on M
- $\eta_1 \sim \eta_2$: homotopic in the space of gradings

Theorem (DM)

1.) $\mathbb{C}l^n$ the trivial Clifford algebra bundle:
$K^{\mathbb{C}l^n}(X)$ is the degree n K-theory of Atiyah-Hirzebruch (1961)

2.) $\mathcal{A}$ a bundle of GCSA's over X:
$K^{\mathcal{A}}(X)$ is Donovan-Karoubi's K-theory with local coefficients (1970)

3.) $\mathcal{G}$ a bundle gerbe over X:
$K^{\mathcal{G}}(X)$ is bundle gerbe K-theory of Bouwknegt, Carey, Mathai, Murray, Stevenson (2003)

4.) $\mathcal{L}$ a complex 2-line-bundle with vanishing H^0-part and torsion H^3-part
$K^{\mathcal{L}}(X)$ is Freed-Hopkins-Teleman's K-theory twisted by the graded central extension corresponding to $\mathcal{L}$.

Further results in this direction:

- Relative twisted K-theory can be treated
- Thom isomorphism can be realized explicitly within this model for twisted K-theory
- A pushforward map can be constructed along arbitrary smooth maps, taking into account a change of twist.
- Non-torsion twists can be treated by a nice „twisted geometric cycle" model à la Baum-Douglas.

Applications to Spin geometry

What is a spin structure on an oriented Riemannian manifold X?

It is a lift of the structure group of the frame bundle along the central extension

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(d) \to SO(d) \to 1$$

$\Updownarrow$

It is a trivialization of the lifting bundle gerbe $\mathcal{L}_X$.

Another definition (Stolz-Teichner ??) is:

A spin structure on X is a

$Cl(TX)$ – Cl^d – bimodule bundle.

Clifford algebra bundle of X; trivial bundle over X with fibre Cl^d.

How are these two definitions of spin structures related?

Thm (PK-ML-DR-KW)
For every oriented Riemannian manifold X, there is a canonical isomorphism

$$Cl(TX) \otimes (Cl^d)^{op} \cong \mathcal{L}_X$$

as 2-vector bundles over X.

Thus: trivialization of $\mathcal{L}_X$

$= 1$-isomorphism $\mathcal{L}_X \to \underline{\mathbb{C}}$

$= 1$-isomorphism $Cl(TX) \otimes (Cl^d)^{op} \to \underline{\mathbb{C}}$

$= Cl(TX)$ - $\underline{Cl^d}$ - bimodule bundle.

We also have Spin^c and $\mathrm{Pin}, \mathrm{Pin}^c$ versions of this relation!

In draft paper circulating since 2006, Stolz-Teichner raised the question, if – in analogy – a string manifold has a „stringor bundle".

$\frac{1}{2}p_1(X) = 0$; string manifolds are the spacetimes of anomaly-free supersymmetric sigma models.

Thm (PK, PK-ML-KW, following a proposal of Stolz-Teichner)
The stringor bundle can be realized as an infinite-dimensional 2-vector bundle.

The construction is outlined in the following.

1.) The surjektive submersion is the path fibration:

$$P_x X \xrightarrow{ev_1} X$$

2.) It has a rigged Clifford-von Neumann algebra bundle with typical fibre

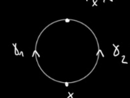
$$\mathcal{A} \to P_x X$$

$Cl(L^2([0,1], \mathbb{C}^d))$, completed under a Fock space representation to a hyperfinite type III_1-factor.

3.) Bimodule bundle: obtained by pullback

$$\mathcal{F} \downarrow LX$$

$$P_x X^{[2]} \longrightarrow LX$$



$$(\gamma_1, \gamma_2) \longmapsto \gamma_1 \cup \gamma_2 := \overline{\gamma_2} * \gamma_1$$

Main point here: we exhibit the fibre $\mathcal{F}_{\gamma_1 \cup \gamma_2}$ as an $\mathcal{A}_{\gamma_1}$ - $\mathcal{A}_{\gamma_2}$ - bimodule.

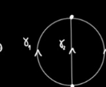
spinor bundle on loop space

It is a rigged Hilbert space bundle with typical fibre the fermionic Fock space for the Hilbert space

$$V := L^2(S^1, \mathbb{S} \otimes \mathbb{C}^d)$$

of odd spinors on the circle.

4.) The intertwiner on $P_x X^{[3]} \ni$



$$\mathcal{F}_{\gamma_2 \cup \gamma_3} \boxtimes_{\mathcal{A}_{\gamma_2}} \mathcal{F}_{\gamma_1 \cup \gamma_2} \cong \mathcal{F}_{\gamma_1 \cup \gamma_3}$$

Connes fusion

Obtained by combining algebraic (Connes) fusion of Fock spaces with geometric (Stolz-Teichner, KW) fusion of loop spaces.