Lie equations, Cartan bundles, Tanaka theory and differential invariants (I)

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Jet formalism

Let $J^\ell M \to M$ be the bundle, whose points are ℓ-jets of functions $u : M \to \mathbb{R}$. (More generally for a bundle $\pi : \mathcal{V} \to M$ get $J^k\pi$.) Coordinates x^i on M lead to coordinates (x^i, u_α) on $J^\ell M$, with α being a multi-index of length $|\alpha| \leq \ell$. It is important to note that $\pi_{\ell,\ell-1} : J^\ell M \to J^{\ell-1}M$ is an affine bundle modelled on $S^\ell T^* M$.

The infinite jet bundle $J^\infty M$ is a projective limit of $J^\ell M$, and the space of functions on it is the injective limit of $C^\infty(J^\ell M)$. The bundle $J^\infty M$ has a canonical flat connection, the so-called Cartan distribution, for which the horizontal lift

$$\mathcal{D}(M) \ni X \dashrightarrow D_X \in \mathcal{D}(J^\infty M)$$

is characterized by

$$(D_X f) \circ j^\infty u = X(f \circ j^\infty u), \quad \forall f \in C^\infty(J^\infty M), u \in C^\infty(M).$$

In local coordinates, if $X = a^i \partial_i$, then $D_X = a^i D_i$, where $D_i = \partial_i + u_{i\alpha}\partial_{u_\alpha}$ is the operator of total derivative.

Equations and Prolongations

A (scalar) differential operator of order $\leq \ell$ on M is a function $F \in C^\infty(J^\ell M) \subset C^\infty(J^\infty M)$. It defines a PDE (system)

$$\mathcal{E} = \{F = 0\} \subset J^\ell M.$$

Its prolongations are given by the formulae

$$\mathcal{E}^{(k)} = \{D_\alpha F = 0 : |\alpha| \leq k\} \subset J^{k+\ell} M$$

and the projective limit is $\mathcal{E}^{(\infty)} = \{D_\alpha F = 0\} \subset J^\infty M$.

Denote $\mathcal{E}_k = \mathcal{E}^{(k-\ell)}$ for $k \geq \ell$ and $\mathcal{E}_k = J^k$ for $k < \ell$. Equation $\mathcal{E}$ is called compatible (or formally integrable) if $\pi_{k,k-1} : \mathcal{E}_k \to \mathcal{E}_{k-1}$ is a submersion for all k, and consistent (or formally solvable) if $\pi_\infty : \mathcal{E}^{(\infty)} \to M$ is a submersion.

If $\mathcal{E}^{(\infty)}$ is finite-dimensional or analytic then by CK theorem (Cauchy-Kovalevsky or Cartan-Kahler) there are solutions to $\mathcal{E}$ (also true for some kind of elliptic systems).

Finite type: reduction to ODEs

Assume compatibility of $\mathcal{E}$.
If $\mathcal{E}_f \simeq \mathcal{E}_{f+1}$ for some $f \geq \ell$ then the prolongation stablized $\mathcal{E}_\infty \simeq \mathcal{E}_f$ and the equation is of finite type.

The Cartan distribution $\mathscr{C}$ on J^∞ induces the distribution $\mathscr{C}_\mathcal{E}$ (horizontal, of the same rank $m = \dim M$) on $\mathcal{E}_\infty$ and hence also on $\mathcal{E}_f$. This is flat, and the solutions are parallel sections:

$$\mathrm{Sol}(\mathcal{E}) \simeq \mathcal{E}/\mathscr{C}_\mathcal{E}.$$

Here we identify the quotient of $\mathcal{E}$ by the leaves foliation of $\mathscr{C}_\mathcal{E}$ with the space of Cauchy data.

Hence finding solutions to a PDE system $\mathcal{E}$ is reduced to ODEs. The symmetry group of $\mathcal{E}$ is the symmetry of this ODE and hence if it has solvable subroup acting transitively in transversal to $\mathscr{C}_\mathcal{E}$, the system is integrable in quadratures.

Tractor type connection for finite type systems

Let us now by-pass the compatibility assumption, but let us still assume that $\pi_{k,k-1} : \mathcal{E}_k \to \mathcal{E}_{k-1}$ is a submersion for $\ell < k \leq f$. In other words, there are no compatibility conditions up to order f.

Then the Cartan distribution $\mathscr{C}_\mathcal{E}$ on $\mathcal{E}_f$ is still horizontal of rank m, hence it is a connection (linear if $\mathcal{E}$ is linear, otherwise general). This connection is invarianrt wrt symmetry/structure/gauge group available, and any other connection is obtained by tensor perturbation ("curvature corrections"). We have:

$$\dim \mathrm{Sol}(\mathcal{E}) \leq \dim \mathcal{E}/\mathscr{C}_\mathcal{E}$$

with the equality iff $\mathcal{E}$ is compatible (Frobenius condition).

In fact, solutions of $\mathcal{E}$ are bijective to parallel sections of this connection $\nabla_\mathcal{E}$ on $\mathcal{E}_f \to M$.

Example: Killing tensors

Let $g = (g_{ij})$ be a Riemannian metric on M. It defines the Hamiltonian $H \in C^\infty(T^*M)$, $H(x,p) = \frac{1}{2} g^{ij}(x) p_i p_j$.
A Killing tensor of order d is a homogeneous polynomial $K = k^{i_1 \dots i_d}(x) p_{i_1} \dots p_{i_d}$ on T^*M that Poisson commutes with H:

$$\{H, K\} = 0.$$

This is a overdetermined PDE system on $\binom{m+d-1}{d}$ functions of m arguments, consisting $\binom{m+d}{d+1}$ first order differential equations.

This system $\mathcal{E}$ can be prolonged to order $d+1$ (no compatibility for the first d prolongations), where it closes to a Frobenius system. If $\mathcal{E}$ is compatible then $\mathrm{Sol}(\mathcal{E})$ can be identified as A_m-irrep

$$S^d \Lambda^2 (T^* \oplus \mathbb{R})_\circ = \begin{array}{|c|c|c|c|c|c|}\hline & & & \cdots & & \\ \hline & & & \cdots & & \\ \hline \end{array}$$

Note that $\mathcal{E}$ is projectively invariant, hence we use the larger group $SL(m+1) \supset SO(m+1) \ \vee \ SO(m) \ltimes \mathbb{R}^m \ \vee \ SO(1,m)$ from the isometry groups of space forms (projectively equivalent).

Killing example: coordinate details

Denote the space of Killing d-tensors ("higher spins") by

$$Q_d(g) = \{K \in C^\infty(T^*M) : \{H, K\} = 0, \deg(K) = d\}.$$

Then $Q(g) = \oplus Q_d(g)$ is a graded Poisson algebra, and we have:

$$\dim Q_d \leq \frac{(m+d-1)!(m+d-2)!}{d!(d+1)!(m-1)!(m-2)!}$$

with the equality iff (M, g) is a space form.

For instance, in the flat case $g = \sum dx_i^2$ we have:

$Q_1 = \langle p_i, r_{ij} = x_i p_j - x_j p_i \rangle$, $Q_2 = \langle p_i p_j, L_{ijk} = r_{ij} p_k, R_{ijkl} = r_{ij} r_{kl} \rangle$

with the relations $L_{ij} = -L_{ji}$, $L_{ijk} + L_{jki} + L_{kij} = 0$ and the Riemann curvature tensor identities for R_{ijkl}:

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> plethysm([2],[0,1,0,0,0,0],A6)
 1X[0,0,0,1,0,0] +1X[0,2,0,0,0,0]
> plethysm([3],[0,1,0,0,0,0],A6)
 1X[0,0,0,0,0,1,0]+1X[0,1,0,1,0,0,0]+1X[0,3,0,0,0,0,0]
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Much less is known even about Q_2 in the non-flat case...

Symbols

For $F \in C^\infty(J^\ell M)$ the vertical part of the 1-form $dF \in \Omega^1(J^\ell M)$, i.e. restriction to $\pi^{-1}_{\ell,\ell-1}(\cdot)$, is a homogeneous polynomial on $\pi_\ell^* T^* M$:

$$\sigma_F = \sum_{|\alpha|=\ell} (\partial_{u_\alpha} F)\partial_\alpha \in \Gamma(\pi_\ell^* S^\ell TM).$$

This is called the symbol of F, at the points of $\mathcal{E}$ it is coordinate-independent.

Note $\sigma(D_\alpha F) = \sigma_F \circ \partial_\alpha$. Let $F^{(k)} = \cup_{|\alpha|=k} D_\alpha F$. Define the symbols of $\mathcal{E}$

$$g_k = \mathrm{Ker}(\sigma_{F^{(k-\ell)}}) \subset S^k T^* M \text{ for } k \geq \ell$$

and $g_k = S^k T^* M$ for $k < \ell$ (for system use $\otimes \mathcal{V}$ in the rhs).

Remark: Dualization over $\mathbb{R}$ (resp $\mathbb{C}$) makes $g^* = \oplus g_k^*$ into a module over commutative algebra $R = \oplus S^k T$. This gives rise to duality of Castelnuovo-Mumford and Cartan-Spencer theories.

Involution: prolongation-projection

How to check compatibility? Spencer δ-complex is the restriction of the de Rham complex

$$\dots \longrightarrow g_{i+1} \otimes \Lambda^{j-1}T^* \longrightarrow g_i \otimes \Lambda^j T^* \longrightarrow g_{i-1} \otimes \Lambda^{j+1}T^* \longrightarrow \dots$$

The cohomology $H^{i,j}(g)$ of the term $g_i \otimes \Lambda^j T^*$ is called the Spencer δ-cohomology. Particular meanings:
$H^{0,0}(g)$ - dependent vars (fields), $H^{\bullet,1}(g)$ - equations (constraints), $H^{\bullet,2}(g)$ - compatibility conds ("cross-diff").

It may happen that some $\pi_{k,l} : \mathcal{E}_k \to \mathcal{E}_l$ are not surjective. Then redefine $\mathcal{E}$ to be this new smaller equation (as a submanifold in jets; larger in terms of defining relations) and restart computation.

In the analytic context (and often in regular smooth) the procedure stops in a finite number of steps, by the CK (Cartan-Kuranishi) theorem. This is equivalent to differential closure of the D-module given by F and results in an involutive system $\bar{\mathcal{E}}$.

Characteristic variety

A general PDE system $\mathcal{E}$ of order ℓ together with its prolongation is a locus of a function $F : J^{k+\ell}(M,\mathcal{V}) \to J^k(M,\mathcal{W})$ for $k \geq 0$, where $\mathcal{V},\mathcal{W}$ are some (vector) bundles over M.

The symbol σ_F of F is a homogeneous degree l polynomial on $\pi_\infty^* T^*M$ with values in $\mathrm{Hom}(\mathcal{V},\mathcal{W})$. The characteristic variety of $\mathcal{E}$ is

$$\mathrm{Char}(\mathcal{E}) = \{[\theta] \in \mathbb{P}(\pi_\infty^* T^*M) : \sigma_F(\theta) \text{ is not injective}\}.$$

If $\mathcal{V},\mathcal{W}$ have the same rank ("determined system"), then $[\theta]$ is characteristic iff $\sigma_F(\theta)$ is not surjective.

For a solution $u \in \mathrm{Sol}(\mathcal{E})$ we identify $M_u \simeq (j_\infty u)(M) \subset J^\infty M$. Thus the characteristic variety is a bundle over M_u whose fiber at x is the projective variety

$$\mathrm{Char}(\mathcal{E},u)_x = \{[\theta] \in \mathbb{P}(T_x^* M_u) : \sigma_F(\theta) = 0\}.$$

Variations: $\mathrm{Char}_{\mathsf{aff}}(\mathcal{E})$, $\mathrm{Char}^{\mathbb{C}}(\mathcal{E})$, etc.

Solution space: Dimensional count

For compatible PDE system (pass from $\mathcal{E}$ to $\bar{\mathcal{E}}$) we observe that by the Hilbert-Serre theorem

$$P(k) = \sum_{i \le k} \dim g_i = c\, k^d + \dots$$

is a polynomial for large k. The numbers $d = \deg(P)$ and $c = \frac{1}{d!} P^{(d)}$ are Cartan genre and Cartan integer.

A formal solution $u \in \mathrm{Sol}(\mathcal{E})$ depends on c functions of d variables (and some number of functions with fewer variables). So d can be called functional dimension and c - functional rank.

Invariantly this data can be computed as follows: let $\kappa = \dim \mathrm{Ker}\, \sigma_F(p)$, $p \in \mathrm{Char}(\mathcal{E})$ (assume for simplicity the characteristic variety to be irreducible). Then

$$d = \dim \mathrm{Char}^{\mathbb{C}}_{\mathsf{aff}}(\mathcal{E}), \quad c = \kappa \deg \mathrm{Char}^{\mathbb{C}}(\mathcal{E}).$$

Example: Heat equation

For the heat equation

$$u_t = u_{xx}$$

we have: $\mathrm{Char}(\mathcal{E}) = \{p_x^2 = 0\}$. Therefore $d = 1$, $c = 2$. So the general analytic solution depends on 2 functions of 1 argument: they come via the Cauchy data $u|_{x=0} = \phi(t)$, $u_x|_{x=0} = \psi(t)$.

In analysis the characteristic initial data is $u|_{t=0} = \varphi(x)$ yields $d = 1$, $c = 1$. This approach works well in the smooth setup, but it defines only the semi-flow on the space of functions $\varphi(x)$ and breaks down analytic solutions.

For instance, with initial condition $\varphi(x) = (1-x)^{-1}$ the solution

$$u(t,x) \doteq \frac{1}{1-x} + \frac{2}{1}\frac{t}{(1-x)^3} + \frac{4!}{2!}\frac{t^2}{(1-x)^5} + \dots + \frac{(2n)!}{n!}\frac{t^n}{(1-x)^{2n+1}} + \dots$$

is divergent everywhere outside $t = 0$.

Example: Self-dual conformal structures

The DFK master-equation $\mathcal{E}$ for SD is

$$\partial_x Q(u) - \partial_y Q(v) = 0,$$
$$(\partial_w - u_y\partial_x + v_y\partial_y)Q(v) + (\partial_z + u_x\partial_x - v_x\partial_y)Q(u) = 0;$$

$$Q = \partial_x\partial_w + \partial_y\partial_z - u_y\partial_x^2 + (u_x + v_y)\partial_x\partial_y - v_x\partial_y^2.$$

It has the following symbol in variables $p = (p_x, p_y, p_z, p_w)$:

$$\sigma_F(p) = \begin{pmatrix} p_x\sigma_Q(p) & -p_y\sigma_Q(p) \\ (p_z + u_xp_x - v_xp_y)\sigma_Q(p) & (p_w - u_yp_x + v_yp_y)\sigma_Q(p) \end{pmatrix},$$

$$\sigma_Q(p) = p_xp_w + p_yp_z - u_yp_x^2 + (u_x + v_y)p_xp_y - v_xp_y^2.$$

Its characteristic variety $\mathrm{Char}(\mathcal{E}) = \{\sigma_Q(p) = 0\}$ is a nondegenerate quadric of multiplicity 3. Hence locally self-dual metrics are parametrized by $c = 6$ functions of $d = 3$ arguments.

Lie equations

Let q be a geometric structure (tensor or another natural object). Infinitesimal Lie equation on symmetries in $E = TM$ is

$$\mathcal{E} = \mathrm{Lie}(q) = \{X \in \mathcal{D}(M) : L_X(q) = 0\}.$$

The above methods apply to this PDE system. For first order system with symbol $\mathfrak{g} = g_1 \subset \mathrm{End}(TM)$ we get

$$\mathrm{Char}_{\mathsf{aff}}(\mathcal{E}) = \{p \in T^* : \exists v \in T, p \otimes v \in \mathfrak{g}\}.$$

The Sternberg prolongation $g_\bullet = \mathrm{pr}(\mathfrak{g}) \subset S^\bullet T^* \otimes T$ is finite-dimensional iff $\mathrm{Char}^{\mathbb{C}} = \emptyset$.

Similarly arises the nonlinear Lie equation for automorphisms, for which $\mathrm{Lie}(q)$ is the linearization $\ell_{\mathcal{E}}$. Goldschmidt-Spencer characterized its integrability via Spencer D-cohomology.

Compatibility of $\mathrm{Lie}(q)$ expreses through curvatures of q, and the maximal symmetry dimension corresponds to "flat" q. For many finite type structures, the dimension gap $\mathfrak{S}_{\mathsf{max}} \rightsquigarrow \mathfrak{S}_{\mathsf{sub.max}}$ has been computed (BK, D.The, H.Winther, V.Matveev, ...).

Examples of the symmetry dimension bounds

For finite type systems $\dim \mathrm{Sol}(\mathcal{E}) = \sum_{i=0}^{\infty} \dim g_i$. In non-flat case, passing to $\bar{\mathcal{E}}$ this is modified: $\dim \mathrm{Sol}(\bar{\mathcal{E}}) \leq \dim \mathrm{Sol}(\mathcal{E})$.

$\mathbb{E}\mathbf{x}$ 1: Conformal structures $c \in \Gamma(E)$ on $M^{p,q}$, $p+q=n$.
Here $E = S^2T^*M/\mathbb{R}_+$ and we get for $\mathrm{Lie}(c)$: $g_{0,1,2} \neq 0$, $g_3 = 0$;
$\mathfrak{S}_{\mathsf{max}} = \binom{n+2}{2}$ and $\mathfrak{S}_{\mathsf{sub.max}} = \binom{n-1}{2} + 6$.

$\mathbb{E}\mathbf{x}$ 2: Einstein metrics $\eta \in \mathrm{Sol}(\mathcal{E})$ on $M^{p,q}$, $p+q=n$.
Here $E = S^2T^*M$, $\mathcal{E} \subset J^2(E)$ and for $\mathrm{Lie}(\eta)$: $g_{0,1} \neq 0$, $g_2 = 0$;
$\mathfrak{S}_{\mathsf{max}} = \binom{n+1}{2}$ and $\mathfrak{S}_{\mathsf{sub.max}} = \binom{n-1}{2} + 5$.

$\mathbb{E}\mathbf{x}$ 3: Killing 2-tensors $k \in \mathrm{Sol}(\mathcal{E})$ on $M^{p,q}$, $p+q=n$.
Here $E = S^2T^*M$, $\mathcal{E} \subset J^1(E)$ and for $\mathrm{Sol}(\mathcal{E})$: $g_{0,1} \neq 0$, $g_2 = 0$
$\mathfrak{S}_{\mathsf{max}} = \frac{n(n+1)^2(n+2)}{12}$ and $\mathfrak{S}_{\mathsf{sub.max}} \geq \binom{n+1}{2} + \frac{n^2(n^2-1)}{12}$.

Thanks for your attention!