<u>Last time</u>: $\cdot$ Twistor coordinates & $\tilde{\tilde{Z}}: Gr_{k,n}^{\geq 0} \xrightarrow{\tilde{Z}} Gr_{k,k+m} \xrightarrow{\sim\ \iota_Z} Gr_m(W) \subseteq Gr_{m,n}$.

$\uparrow$ colspan Z, $(k+m)$ dim'l

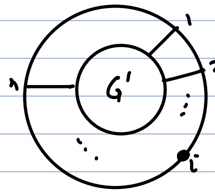
<u>Lem</u>: $Y \in Gr_{m,n}$. $Y = \tilde{\tilde{Z}}(A) \iff Y \perp A$ & $Y \subseteq \text{colsp}(Z) = W$

- Tiles for $m=1,2$: semi-alg. descriptions in $Gr_{k,k+m}$ specifying signs of twistors
  equiv, " $Gr_m(W)$ " Plückers
- Tilings for $m=1,2$ use semi-alg descriptions heavily to e.g. check tiles are disjoint.

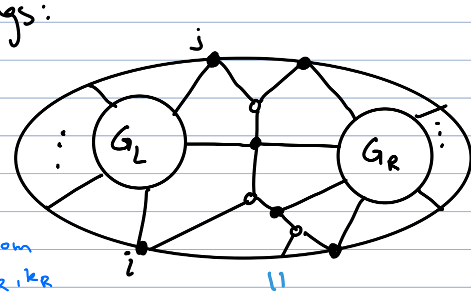
For $m=4$, want to show BCFW recursion gives tilings:

Pick i.

$$A_{n,k} = \sum_{G'}$$



$$+ \sum_{j, G_L, G_R}$$



G': from $A_{n-1,k}$

G_L: from A_{n_L,k_L}

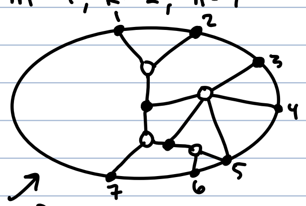
G_R: from A_{n_R,k_R}

$k_L + k_R = k-1$

$\shortparallel$ $G_L \bowtie G_R$

- $m=4$, $k=2$, $n=7$:

$G=$



BCFW graph

Z_G is a tile, but inverting $\tilde{Z}|_{C_G}$ & writing semi-alg. description of Z°_G requires quadratic poly. in twistors.

e.g. $\langle 1247\rangle\langle 3567\rangle - \langle 1237\rangle\langle 4567\rangle < 0$

- Not obvious why if Z_{G_L} & Z_{G_R} are tiles (in 2 dif. amplituhedra), $Z_{G_L \bowtie G_R}$ is a tile.

§7. <u>Vector-reln configurations</u>: Inspired by [Affolter-Glick-Pylyavskyy-Ramassamy]

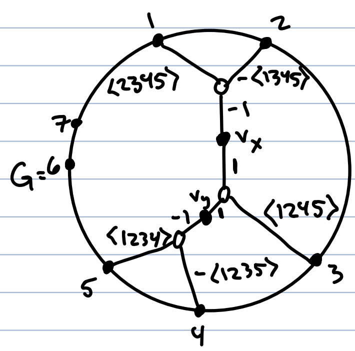
<u>Defn</u>: Fix $m \geq 1$. An <u>m-vector-relation configuration</u> (m-VRC) on G is $(\underline{v}, \underline{R})$ where

$$\underline{v} = \{v_b\}_{b \in B} \subseteq \mathbb{R}^m \quad \& \quad \underline{R} = \{r_e\}_{e \in E} \subseteq \mathbb{R}^* \quad \text{s.t.}$$

1) <u>Bdry</u> of $(\underline{v}, \underline{R})$ $\partial(\underline{v}, \underline{R}) = [v_1 \cdots v_n]$ is full rk.

2) For ea. $w \in W$, we have $\sum_{bw \in E} r_{bw} v_b = 0$.

e.g. $m=4$. Choose $v_1, \ldots, v_7 \in \mathbb{R}^4$. If $v_1, \ldots, v_5$ generic,



$\text{span}(v_1, v_2) \cap \text{span}(v_3, v_4, v_5) =$ line spanned by

$$\underbrace{\langle 2345\rangle v_1 - \langle 1345\rangle v_2}_{v_x} = -\big[\underbrace{\langle 1245\rangle v_3 - \langle 1235\rangle v_4 + \langle 1234\rangle v_5}_{v_y}\big]$$

(This is ! 4-VRC w/ bdry $[v_1 \cdots v_7]$ up to rescaling)

If not generic, $\nexists$ 4-VRC w/ bdry $[v_1 \cdots v_7]$.

- $VRC_G^m = \{m\text{-VRC on } G\} / GL_m$ & gauge

GL_m: acts on all v_b by L-mult

gauge: action of $(\mathbb{R}_{>0})^{W \sqcup B_{int}}$ by rescaling

Write $[\underline{v}, \underline{R}] \in VRC_G^m$. $\partial[\underline{v}, \underline{R}] = \text{rowsp}[v_1 \cdots v_n] \in Gr_{m,n}(\mathbb{R})$

Prop: [Even-Zohar–Parisi–S–Tessler–Williams] Fix G plabic gr, (σ_e) (orthogonal) Kasteleyn signs for G. comb. descr. of these.

certain Zariski-open set

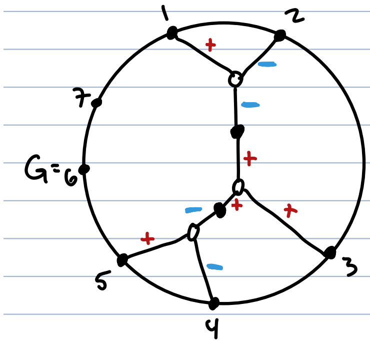
For $Y \in Gr^\circ_{m,n}$

$$\{[\underline{v}, \underline{R}] : \partial = Y \ \& \ \sigma_e r_e > 0 \ \forall e\} \xleftrightarrow{1-1} \{A \in C_G : Y \perp A\}.$$

$$[\underline{v}, \underline{R}] \longleftrightarrow A = D^G(\sigma_e r_e)$$

e.g. for G above, signs are & $Gr^\circ_{4,7} = Gr_{4,7}$. So above shows for $Y \in Gr_{4,7}$, either

$$Y \perp A \text{ for } ! \, A \in C_G \quad \text{or} \quad Y \not\perp A \ \ \forall A \in C_G.$$



$\Rightarrow$ For any $Z \in Mat^{>0}_{7,1+4}$, $\tilde{\tilde{Z}}$ is injective on C_G. Also,

$$\tilde{\tilde{Z}}(C_G) = \left\{ Y \in Gr_4(W) : \begin{matrix} \langle 23457\rangle, \langle 12457\rangle, \langle 12347\rangle > 0 \\ \langle 1345\rangle, \langle 1235\rangle < 0 \end{matrix} \right\}$$

$$Z^\circ_G = \left\{ X \in Gr_{2,6} : \begin{matrix} \langle 23457\rangle, \langle 12457\rangle, \langle 12347\rangle > 0 \\ \langle 1345\rangle, \langle 1235\rangle < 0 \end{matrix} \right\}.$$

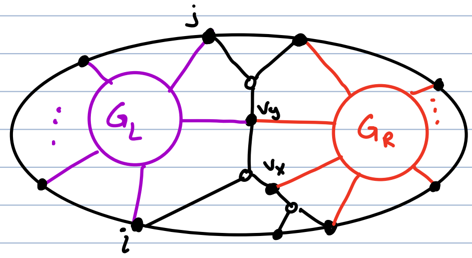
§8. VRCs & the amplituhedron

Zariski open

- To show Z_G a tile, enough to show: ① for $Y \in \mathcal{U} \subseteq Gr^\circ_{m,n}$, $\exists ! \, [\underline{v}, \underline{R}]$ w/ $\partial = Y$ & ② $\tilde{\tilde{Z}}(C_G) \subseteq \mathcal{U}$

  ↳ In this case, the coeffs r_e are rat'l fcns in Plücker coords of Y, so also obtain semi-alg. description of $\tilde{\tilde{Z}}(C_G)$, Z°_G.

- Call G solvable if for Y generic, $\exists ! \, [\underline{v}, \underline{R}]$ w/ $\partial = Y$; i.e. usually $\partial [\underline{v}, \underline{R}]$! determines inside.



Solvability interacts really well w/ recursions

If fix "generic" bdry, coeffs on black edges & v_x, v_y ! determined. If G_L, G_R solvable (& bdries for G_L, G_R "generic"), rest of VRC is also determined, so G is solvable also.
$\rightsquigarrow$ gluing solvable graphs (w/ some care) produces solvable graph. This is why BCFW recursion gives tiles.

- The original BCFW recursion (on rat'l fcns) is essentially

$$F_{G_L \bowtie G_R}\left(\left[\begin{smallmatrix} v_1 & \cdots & v_n \end{smallmatrix}\right]\right) = F_{G_L}\left(\left[\begin{smallmatrix} v_i & \cdots & v_j & v_y \end{smallmatrix}\right]\right) \cdot F_{G_R}\left(\left[\begin{smallmatrix} v_{j+1} & \cdots & v_{i-1} & v_x & v_y \end{smallmatrix}\right]\right) \cdot \frac{1}{\text{all Plückers on 5 indices touched by "core"}}.$$

- On the other hand, we hope that the canonical form of Z_G is

$$\bigwedge_{e \in E'} \frac{dr_e}{r_e} = \left(\bigwedge_{e \in E' \cap G_L} \text{dlog of } r'_e \text{ for } G_L \text{ evaluated on } [v_i \ldots v_j \ v_y] \right) \wedge \left(\bigwedge_{e \in E' \cap G_R} \text{dlog of } r'_e \text{ for } G_R \text{ eval. on } [v_{j+1} \ldots v_{i-1} \ v_x \ v_y] \right) \wedge \left(\text{dlogs of 5 Plückers on "core" indices} \right)$$

edges not gauge-fixed to 1

These look very similar! Encouraging.

New questions:

- Which graphs are solvable?

↳[EPSTW]: combin. characterize solvable trees & how solvable graphs can be glued to produce solvable graphs. ⇝ new recursions for tiles?

↳[Lam]: Schubert calculus answer

- For even m, is G solvable $\Leftrightarrow$ Z_G a tile? (For $m=1$, $\not\Rightarrow$ but $\Leftarrow$ holds).

- Connections to cluster algebras? All coeffs r_e in BCFW are compatible cluster var. for $Gr_{4,n}$. Is this true for all solvable G?