# A new look at Lie algebras

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# Introduction

There is a well-known isomorphism between special orthogonal Lie algebra $\mathfrak{so}(3)$ and $\mathbb{R}^3$. For the first structure, the Lie bracket is given by the matrix commutator $[X, Y] = XY - YX$ for $X, Y \in \mathfrak{so}(3)$, and for the second by the cross product $\times$ for vectors from $\mathbb{R}^3$. The mapping

$$X = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \longmapsto v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

gives this isomorphism $(\mathfrak{so}(3), [\cdot, \cdot]) \cong \left(\mathbb{R}^3, \times\right)$.
The main goal of my presentation is to show that one can construct a similar isomorphism for any Lie algebra. We will show that Lie algebras have a lot in common with linear maps, and more precisely with linear maps with a fixed eigenvector.

# Lie algebra

A Lie algebra is vector space over a field $\mathbb{R}$ equipped with Lie bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ with is a bilinear, antisymmetric map, which satisfies the Jacobi identity

$$[[x, y], z] + [[z, x], y] + [[y, z], x] = 0$$

for all $x, y, z \in \mathfrak{g}$.

The basic ingredient is a pair $(F, v)$ consisting of a linear mapping $F \in End(V)$ with an eigenvector $v$. This pair allows to build a Lie bracket on a dual space to a linear space $V$.

In our considerations, we will restrict ourselves to the linear space $V$ over a field $\mathbb{R}$. It means that we will analyze in detail only real Lie algebras. However, we want to emphasize that the presented formulas also work for vector spaces over the field of complex numbers.

# Eigenvalue problem

We present some constructions of a Lie bracket on a space $V^*$ having a pair: linear mapping and its eigenvector. A pair $(F, v)$ gives a Lie bracket on a dual space $V^*$:

Theorem

*If $V$ is a vector space, $F : V \longrightarrow V$ is a linear map and $v \in V$ is an eigenvector of the map $F$, then $(V^*, [\cdot, \cdot]_{(F,v)})$, is a Lie algebra, where the Lie bracket is given by*

$$[\psi, \phi]_{(F,v)} = \phi(v)F^*(\psi) - \psi(v)F^*(\phi)$$

*for $\psi, \phi \in V^*$.*

# $V \simeq V^* \simeq \mathbb{R}^N$

We can identify $V$ and $V^*$ with $\mathbb{R}^N$ with the canonical basis $\{e_1, e_2, \ldots, e_N\}$ (i.e. $V \simeq V^* \simeq \mathbb{R}^N$). Then the Lie bracket can be rewritten in the form

$$[u, w]_{(F,v)} = \langle w|v\rangle F^T u - \langle u|v\rangle F^T w \quad \text{for } u, w \in \mathbb{R}^N,$$

where $\langle\cdot|\cdot\rangle$ is the scalar product in $\mathbb{R}^N$.

## Theorem

*Let* $[\cdot,\cdot]_{(F,v)}$ *be given by*

$$[\psi,\phi]_{(F,v)} = \phi(v)F^*(\psi) - \psi(v)F^*(\phi),$$

*then the Lie algebra* $(V^*, [\cdot,\cdot]_{(F,v)})$ *is solvable.*

## Proof.

We say that a linear subspace $\mathfrak{h}$ is an ideal of a Lie algebra $\mathfrak{g}$ when $[\mathfrak{g},\mathfrak{h}] \subseteq \mathfrak{h}$. Of course the set $[\mathfrak{h},\mathfrak{h}]$ is also an ideal. Then we define a sequence of ideals (the derived series $\mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} \supseteq \cdots \supseteq \mathfrak{g}^{(i)} \supseteq \dots$)

$$\mathfrak{g}^{(0)} = \mathfrak{g},\ \mathfrak{g}^{(1)} = [\mathfrak{g},\mathfrak{g}],\ \mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)},\mathfrak{g}^{(1)}], \dots, \mathfrak{g}^{(i)} = [\mathfrak{g}^{(i-1)},\mathfrak{g}^{(i-1)}], \dots$$

A Lie algebra $\mathfrak{g}$ is called solvable if, for some positive integer $i$, $\mathfrak{g}^{(i)} = 0$.

In this case we get $\mathfrak{g}^{(2)} = [[\mathfrak{g},\mathfrak{g}]_{(F,v)}, [\mathfrak{g},\mathfrak{g}]_{(F,v)}]_{(F,v)} = 0$. □

In addition, if we introduce the following sequence of ideals (the lower central series $\mathfrak{g}_{(0)} \supseteq \mathfrak{g}_{(1)} \supseteq \cdots \supseteq \mathfrak{g}_{(i)} \supseteq \dots$)

$$\mathfrak{g}_{(0)} = \mathfrak{g},\ \mathfrak{g}_{(1)} = [\mathfrak{g}_{(0)}, \mathfrak{g}],\ \mathfrak{g}_{(2)} = [\mathfrak{g}_{(1)}, \mathfrak{g}], \dots, \mathfrak{g}_{(i)} = [\mathfrak{g}_{(i-1)}, \mathfrak{g}], \dots,$$

we say that algebra $\mathfrak{g}$ is called nilpotent if the lower central series terminates $\mathfrak{g}_{(i)} = 0$ for some $i \in \mathbb{N}$. Obviously, a nilpotent Lie algebra is also solvable.

**Theorem**

*If $F$ is a nilpotent operator, then $(V^*, [\cdot,\cdot]_{(F,v)})$ is a nilpotent Lie algebra.*

# Lie algebra generalized $ax+b$-group

If we look at this bracket we notice that this is a structure of a Lie bracket for a Lie algebra generalized $ax+b$-group

$$[(w_1,t_1),(w_2,t_2)] = (t_1Dw_2 - t_2Dw_1, 0),$$

where $V = W \ltimes \mathbb{R}$, $W$ is $N-1$-dimensional linear space, $w_1, w_2 \in W$, $t_1, t_2 \in \mathbb{R}$ and $D$ is established endomorphism $End(W)$. Identification is given by association $V \cong V^* \cong \mathbb{R}^N$ and putting $\psi = (w_1,t_1), \phi = (w_2,t_2)$, $v = e_N$, $F = \left(\begin{array}{c|c} -D^\top & 0 \\ \hline 0 & 0 \end{array}\right)$

$$[\psi,\phi]_{(F,v)} = \phi(v)F^*(\psi) - \psi(v)F^*(\phi),$$

I. Beltiță, D. Beltiță, *Quasidiagonality of $C^*$-algebras of solvable Lie groups*, Integr. Equ. Oper. Theory, **90**:5, 2018.

We show that these solvable algebras are the basic bricks of the construction of all other Lie algebras.

# The linear combination of Lie brackets $[\cdot,\cdot]_{(F,v)}$, $[\cdot,\cdot]_{(G,w)}$ gives a Lie bracket

**Theorem**

*Let $V$ be a vector space over $\mathbb{R}$. If $F, G \in End(V)$, $v, w \in V$ are such that:*

- *$v$ is an eigenvector of the map $F$,*
- *$w$ is an eigenvector of the map $G$,*
- *the following condition is true*

$$v \wedge w \wedge [F,G]^* + w \wedge Gv \wedge F^* + v \wedge Fw \wedge G^* = 0.$$

*Then $(V^*, [\cdot,\cdot]^{\lambda}_{(F,v),(G,w)})$, where*

$$[\psi,\phi]^{\lambda}_{(F,v),(G,w)} = [\psi,\phi]_{(F,v)} + \lambda[\psi,\phi]_{(G,w)}$$

*is a Lie algebra for every $\lambda \in \mathbb{R}$.*

# Example – the three-dimensional Lie algebras

Let us take $V = \mathbb{R}^3$ with the standard basis $\{e_1, e_2, e_3\}$. We will show how to easily connect three-dimensional real Lie algebras with the corresponding linear mappings and their eigenvectors. We will restrict ourselves to the eigenvector $v = (0, 0, 1)^\top$. Lie brackets will be defined in the space $V^* = \left(\mathbb{R}^3\right)^\top$ with the dual base $\{e_1^*, e_2^*, e_3^*\}$.


Patera, J., Sharp, R.T., Winternitz, P., Zassenhaus, H.: Invariants of real low dimension Lie algebras. J. Math. Phys. 17. 986 (1976)

If we take

$$F = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$, we obtain the Lie bracket of the form

$$[\psi, \phi]_{(F,v)} = \lambda_1 (\psi_1 \phi_3 - \psi_3 \phi_1) e_1^* + \lambda_2 (\psi_2 \phi_3 - \psi_3 \phi_2) e_2^*,$$

where $\psi = \psi_1 e_1^* + \psi_2 e_2^* + \psi_3 e_3^*$ and $\phi = \phi_1 e_1^* + \phi_2 e_2^* + \phi_3 e_3^*$. The commutator rules are following

$$[e_1^*, e_2^*]_{(F,v)} = 0, \quad [e_1^*, e_3^*]_{(F,v)} = \lambda_1 e_1^*, \quad [e_2^*, e_3^*]_{(F,v)} = \lambda_2 e_2^*.$$

1. For $\lambda_1 = \lambda_2 = 1$, we recognize the Lie structure related to the Lie algebra $\mathfrak{g}_{3,3}$.
2. For $\lambda_1 = -\lambda_2 = 1$, we recognize the Lie structure related to the Lie algebra $\mathfrak{g}_{3,4}$.
3. For $\lambda_1 = 1$, $\lambda_2 = a$, we recognize the Lie structure related to the Lie algebra $\mathfrak{g}_{3,5}^a$.

# Linear mappings and their eigenvectors giving three dimensional Lie algebras

| $F$ | $v$ | Casimir | Name |
|---|---|---|---|
| $F = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ | $x_1$ | $\mathfrak{g}_{3,1}$ |
| $F = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ | $x_1 e^{-\frac{x_2}{x_1}}$ | $\mathfrak{g}_{3,2}$ |
| $F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ | $\frac{x_2}{x_1}$ | $\mathfrak{g}_{3,3}$ |
| $F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ | $x_1 x_2$ | $\mathfrak{g}_{3,4}$ |
| $F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ | $\frac{x_1}{x_2^a}$ | $\mathfrak{g}_{3,5}^a$ |

| | | | |
|---|---|---|---|
| $F=\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$ | $v=\begin{pmatrix}0\\0\\1\end{pmatrix}$ | $x_1^2+x_2^2$ | $\mathfrak{g}_{3,6}$ |
| $F=\begin{pmatrix}a & -1 & 0\\ 1 & a & 0\\ 0 & 0 & 0\end{pmatrix}$ | $v=\begin{pmatrix}0\\0\\1\end{pmatrix}$ | $(x_1^2+x_2^2)e^{2a\operatorname{arctg}\frac{x_1}{x_2}}$ | $\mathfrak{g}_{3,7}^a$ |
| $F=\begin{pmatrix}0 & -2 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$<br>$G=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$ | $v=\begin{pmatrix}0\\0\\1\end{pmatrix}$<br>$w=\begin{pmatrix}0\\1\\0\end{pmatrix}$ | $x_1x_3+x_2^2$ | $\mathfrak{g}_{3,8}$ |
| $F=\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$<br>$G=\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$ | $v=\begin{pmatrix}0\\0\\1\end{pmatrix}$<br>$w=\begin{pmatrix}0\\1\\0\end{pmatrix}$ | $x_1^2+x_2^2+x_3^2$ | $\mathfrak{g}_{3,9}$ |

# General structure

For a Lie algebra $\mathfrak{g}$ with the basis $\{e_1, e_2, \ldots, e_N\}$, given by commutator relations $[e_i, e_j] = \sum_{k=1}^{N} c_{ij}^k e_k$, we can assign $N$-pairs $(F_1, e_N), \ldots, (F_{N-i+1}, e_i), \ldots, (F_N, e_1)$.

$$[e_i, e_N] = \sum_{k=1}^{N} c_{iN}^k e_k$$

$$F_1 = \left(\begin{array}{cccc|c} c_{1N}^1 & c_{1N}^2 & \dots & c_{1N}^{N-1} & 0 \\ c_{2N}^1 & c_{2N}^2 & \dots & c_{2N}^{N-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{N-1\,N}^1 & c_{N-1\,N}^2 & \dots & c_{N-1\,N}^{N-1} & 0 \\ \hline 0 & 0 & \dots & 0 & 0 \end{array}\right),$$

The mapping $F_1$ corresponds as the vector $e_n$. In the above matrix, the structure constants $c_{iN}^N$, $i = 1, \dots, N-1$ do not appear. They will be placed in the next mappings $F_2, \dots, F_N$. To be precise, $c_{iN}^N$ will appear in the mapping $F_{N-i+1}$.

$$F_1 = \left(\begin{array}{cccc|c} c^1_{1N} & c^2_{1N} & \dots & c^{N-1}_{1N} & 0 \\ c^1_{2N} & c^2_{2N} & \dots & c^{N-1}_{2N} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c^1_{N-1\ N} & c^2_{N-1\ N} & \dots & c^{N-1}_{N-1\ N} & 0 \\ \hline 0 & 0 & \dots & 0 & 0 \end{array}\right),$$

$$F_{N-i+1} = \left(\begin{array}{cccc|c|ccc} c^1_{1\ i} & c^2_{1\ i} & \dots & c^{i-1}_{1\ i} & 0 & c^{i+1}_{1\ i} & \dots & c^N_{1\ i} \\ c^1_{2\ i} & c^2_{2\ i} & \dots & c^{i-1}_{2\ i} & 0 & c^{i+1}_{2\ i} & \dots & c^N_{2\ i} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c^1_{i-1\ i} & c^2_{i-1\ i} & \dots & c^{i-1}_{i-1\ i} & 0 & c^{i+1}_{i-1\ i} & \dots & c^N_{i-1\ i} \\ \hline 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \hline 0 & 0 & \dots & 0 & 0 & -c^{i+1}_{i\ i+1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & -c^N_{i\ N} \end{array}\right),$$

$$F_N = \left(\begin{array}{c|cccc} 0 & 0 & \dots & 0 \\ \hline 0 & -c_{1\,2}^{2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & -c_{1\,N}^{N} \end{array}\right).$$

### Theorem

*Every Lie algebra* $(\mathfrak{g}, [\cdot,\cdot])$ *is isomorphic to the corresponding Lie algebra* $(\mathbb{R}^N, [\cdot,\cdot]_{(F_1,v_1),\dots,(F_N,v_N)})$.

The isomorphism $(\mathfrak{g}, [\cdot,\cdot]) \cong (\mathbb{R}^N, [\cdot,\cdot]_{(F_1,v_1),\dots,(F_N,v_N)})$ is not canonical, we can assign the linear mappings and their eigenvectors differently.

# Casimir functions

Using relations between the Lie algebra, the Lie–Poisson structure and the Nambu bracket, we show that the algebra invariants (Casimir functions) are solutions of an equation which has an interesting geometric significance.

**Theorem**

*Casimir functions $c_i$, $i = 1, \ldots, k$, for the Lie algebra $(\mathbb{R}^N, [\psi, \phi]_{(F_1, v_1), \ldots, (F_N, v_N)})$ satisfy the following equation*

$$\nabla c_i(x) \wedge \star \sum_{j=1}^{N} (F_j(x) \wedge v_j) = 0.$$

The Hodge star operator $\star : \bigwedge^2 V \longrightarrow \bigwedge^{N-2} V$

# For a pair $(F, v)$ giving a Lie algebra structure, we always have $N-2$ Casimir functions

Theorem

*Let* $(\mathbb{R}^N, [\cdot,\cdot]_{(F,e_N)})$ *be a Lie algebra, then Casimirs* $c_i, i = 1, 2, \ldots, N-2$*, of the algebra fulfill the following conditions*

$$\langle Fx|\nabla c_i(x)\rangle = 0,$$
$$\langle e_N|\nabla c_i(x)\rangle = 0$$

*for all* $x \in \mathbb{R}^N$*.*

A. Dobrogowska, M. Szajewska, *Eigenvalue problem versus Casimir functions for Lie algebras*, Anal. Math. Phys. **14** (2024), 1-24.

# Casimir functions

**Definition**

A non-zero tensor $t \in \bigwedge^{N} V$ is $s$-partially decomposable if there exist $w_i, i = 1, 2, \ldots, s$, vectors and $N - s$-tensor $u \in \bigwedge^{N-s} V$ such that

$$t = w_1 \wedge w_2 \wedge \ldots \wedge w_s \wedge u.$$

Finally, the following theorem holds

### Theorem

*Let pairs* $(F_j, v_j)$*,* $j = 1, \ldots, N$*, give any Lie algebra* $\mathfrak{g}$*. Functions* $c_i$*,* $i = 1, \ldots, s$*, are functionally independent Casimir functions for* $\mathfrak{g}$ *if and only if* $\star \sum_{j=1}^{N} (F_j x \wedge v_j) \in \bigwedge^{N-2} \mathbb{R}^N$ *is* $s$*-partially decomposable, i.e. if there exist* $w_i \in \mathbb{R}^N, i = 1, 2, \ldots, s$*,* $u \in \bigwedge^{N-s-2} \mathbb{R}^N$ *such that*

$$\star \sum_{j=1}^{N} (F_j x \wedge v_j) = w_1 \wedge w_2 \wedge \ldots \wedge w_s \wedge u.$$

*Furthermore,* $\nabla c_i \sim w_i$*.*

If we have a single pair $(F, e_N)$, then obviously the tensor $Fx \wedge e_N$ is decomposable, so consequently the tensor $\star(Fx \wedge e_N) \in \bigwedge^{N-2} \mathbb{R}^N$ is decomposable. Therefore, algebra with this pair must always have $N-2$ Casimir functions.

If there are $N-2$ smooth Casimir functions $c_1, \ldots, c_{N-2}$, this corresponds to the situation that the Poisson bracket arises from the Nambu bracket by fixing $N-2$ functions as Casimir functions. In this case, the formula has a form

$$\{f,g\}\Omega = u\ df \wedge dg \wedge dc_1 \wedge \ldots \wedge dc_{N-2}, \quad f,g \in C^\infty(\mathbb{R}^N),$$

where $\Omega = dx_1 \wedge \ldots \wedge dx_N$ is the standard volume element on $\mathbb{R}^N$, and $u$ is some function on $\mathbb{R}^N$. The case, where there are less smooth Casimir functions, namely $c_1, \ldots, c_s$, $s < N-2$, then the Poisson bracket has a form

$$\{f,g\}\Omega = df \wedge dg \wedge dc_1 \wedge \ldots \wedge dc_s \wedge u.$$

In details studied in

P.A. Damianou, F. Petalidou, *Poisson Brackets with Prescribed Casimirs*, Canad. J. Math. **64** (5), (2012) 991–1018.

It is connected with $s+2$-linear Nambu bracket in dimension $N$, higher than $s+2$.

Chan C. Chandre, A. Horikoshi, *Classical Nambu brackets in higher dimensions*, J. Math. Phys. **64**, (2023) 052702.

# Nambu bracket

In 1973, Nambu proposed a generalization of the Poisson bracket on $\mathbb{R}^3$ to the Nambu bracket in the form

$$\{f_1, f_2, f_3\}(\mathbf{x}) = \frac{\partial(f_1, f_2, f_3)}{\partial(x_1, x_2, x_3)} = \sum_{i,j,k=1}^{3} \epsilon_{ijk} \frac{\partial f_1}{\partial x_i}(\mathbf{x}) \frac{\partial f_2}{\partial x_j}(\mathbf{x}) \frac{\partial f_3}{\partial x_k}(\mathbf{x}),$$

where $\epsilon$ is Levi-Civita tensor and $f_1, f_2, f_3 \in C^\infty(\mathbb{R}^3)$. In general, a Nambu bracket $\{\cdot, \ldots, \cdot\} : \underbrace{C^\infty(M) \times \cdots \times C^\infty(M)}_{n} \longrightarrow C^\infty(M)$ is a $n$-linear, skew-symmetric map, which satisfies the generalized Jacobi identity (fundamental identity)

$$\{f_1, \ldots, f_{n-1}, \{g_1, \ldots, g_n\}\} = \sum_{i=1}^{n} \{g_1, \ldots, \{f_1, \ldots, f_{n-1}, g_i\}, \ldots, g_n\}$$

and Leibniz rule

$$\{f_1, \ldots, f_{n-1}, fg\} = f\{f_1, \ldots, f_{n-1}, g\} + \{f_1, \ldots, f_{n-1}, f\}g.$$

# Left-symmetric algebras

Let us now remind what is left-symmetric algebra.

**Definition**

Structure $(\mathfrak{g}, \bullet)$ is called left-symmetric algebra, if bilinear multiplication $\bullet \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ has a property that the associator $(\cdot,\cdot,\cdot) \colon \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ given by

$$(x, y, z) = (x \bullet y) \bullet z - x \bullet (y \bullet z) \qquad \text{for all} \qquad x, y, z \in \mathfrak{g}$$

is symmetric in $x$ and $y$, i.e.,

$$(x, y, z) = (y, x, z).$$

Left-symmetric algebras have been introduced by A. Cayley (1896). Then they were forgotten for a long time until Vinberg (1960) and Koszul (1961) introduced them in the context of convex homogeneous cones and affinely flat manifolds.

# Left-symmetric algebras

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is symmetric in $x$ and $y$, i.e.,

$$(x, y, z) = (y, x, z).$$

Every left-symmetric algebra carries a canonical Lie bracket defined as follows

$$[x, y] = x \bullet y - y \bullet x.$$

# Left-symmetric algebras

The Jacobi identity is fulfilled because of the symmetry of the associator mapping in the first two arguments

$$
\begin{aligned}
[x,[y,z]] + [z,[x,y]] + [y,[z,x]] =& (x,y,z) + (y,z,x) + (z,x,y) \\
& - (y,x,z) - (z,y,x) - (x,z,y).
\end{aligned}
$$

# Left-symmetric algebras

A pair $(F, v)$, which gives us the eigenvalue problem, also gives multiplication on dual space $V^*$. The multiplication $\bullet\colon V^* \times V^* \to V^*$ is given by formula

$$\alpha \bullet \beta := -\alpha(v)F^*(\beta).$$

For this multiplication following theorem holds

**Theorem**

*Structure $(V^*, \bullet)$ is left–symmetric algebra.*

$$\begin{aligned}(\alpha, \beta, \gamma) &= (\alpha \bullet \beta) \bullet \gamma - \alpha \bullet (\beta \bullet \gamma) \\ &= \alpha(v)\beta(v)F^*(\lambda\mathbb{I} - F^*)(\gamma) \\ &= (\beta, \alpha, \gamma).\end{aligned}$$

A. Dobrogowska, K. Wojciechowicz, *Lie Algebras, Eigenvalue Problems and Left-Symmetric Algebras*, J. Geometry and Symmetry in Physics **69**, (2024) 59-67.

## Example

| | $F$ | $u \bullet w$ | $\mathfrak{g}$ |
|---|---|---|---|
| 1. | $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} -u_3 w_2 \\ 0 \\ 0 \end{pmatrix}$ | $\mathfrak{g}_{3,1}$ |
| 2. | $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} -u_3(w_1 + w_2) \\ -u_3 w_2 \\ 0 \end{pmatrix}$ | $\mathfrak{g}_{3,2}$ |
| 3. | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} -u_3 w_1 \\ -u_3 w_2 \\ 0 \end{pmatrix}$ | $\mathfrak{g}_{3,3}$ |
| 4. | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} -u_3 w_1 \\ -a u_3 w_2 \\ 0 \end{pmatrix}$ | $\mathfrak{g}_{3,4}^a$ |
| 5. | $\begin{pmatrix} a & -1 & 0 \\ 1 & a & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} -u_3(a w_1 + w_2) \\ -u_3(-w_1 + a w_2) \\ 0 \end{pmatrix}$ | $\mathfrak{g}_{3,5}^a$ |

Table: Three dimensional Lie algebras given by one eigenvalue problem

It is well known that the Lie algebra of an left-symmetric algebra can not be semisimple.

Dietrich Burde, *Left-symmetric structures on simple modular Lie algebras*, Journal of Algebra **169**, (1994) 112-138.

A collection of two eigenvalue problems $(F_1, v_1)$, $(F_2, v_2)$ gives the multiplication on $V^*$

$$\alpha \bullet \beta = -\alpha(v_1)F_1^*(\beta) - \alpha(v_2)F_2^*(\beta).$$

## Example

| | |
|---|---|
| $\left(\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_3\right)$ <br> $\left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_2\right)$ | $u \bullet w = \begin{pmatrix} -u_3 w_2 \\ u_3 w_1 \\ -u_2 w_1 \end{pmatrix}$ |

For the Lie algebra $\mathfrak{so}(3)$ the associator can be written as

$$\begin{aligned}(u, w, z) &= u_2 z_2 \begin{pmatrix} w_1 \\ 0 \\ w_3 \end{pmatrix} + u_3 w_3 \begin{pmatrix} z_1 \\ z_2 \\ 0 \end{pmatrix} - w_1 z_1 \begin{pmatrix} u_2 \\ u_3 \\ 0 \end{pmatrix} \\ &= (u, w, z)_L + (u, w, z)_C + (u, w, z)_R.\end{aligned}$$

## Example

| | |
|---|---|
| $\left(\begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_3\right)$ $\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, e_2\right)$ | $u \bullet w = \begin{pmatrix} -u_2 w_1 \\ 2u_3 w_1 \\ u_2 w_3 \end{pmatrix}$ |

Let us notice that for $\mathfrak{sl}(2,\mathbb{R})$ associator is given by

$$
\begin{aligned}
(u,w,z) &= \begin{pmatrix} -u_2 w_2 z_1 \\ 2(u_2 w_3 + u_3 w_2) z_1 \\ -u_2 w_2 z_3 \end{pmatrix} + u_3 z_3 \begin{pmatrix} 0 \\ 0 \\ 2w_1 \end{pmatrix} + w_1 z_1 \begin{pmatrix} -2u_3 \\ 0 \\ 0 \end{pmatrix} \\
&= (u,w,z)_L + (u,w,z)_C + (u,w,z)_R.
\end{aligned}
$$

# Thank you for your attention