From perturbative to non-perturbative AQFT

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Exactly Solvable Models, ESI, 24.07.2024

[1] Based on a joint paper with Romeo Brunetti, Michael Dütsch and Klaus Fredenhagen: AHP 2024 [arXiv:2108.13336].

Outline of the talk

1. Introduction

2. Construction
   - Classical theory
   - Quantum theory

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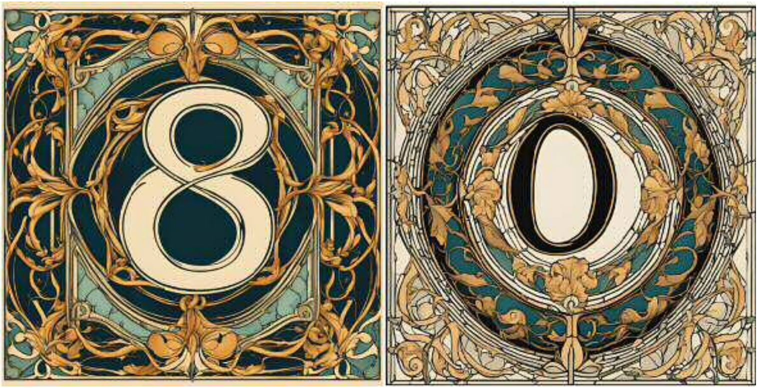
- Three years later, I got invited to ESI in September 2011 to the workshop “Rigorous Quantum Field Theory in the LHC era” to talk about my work and two months later I defended my PhD thesis in Hamburg.

Herzlichen Glückwunsch Harald!



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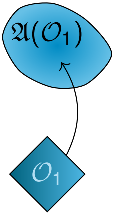

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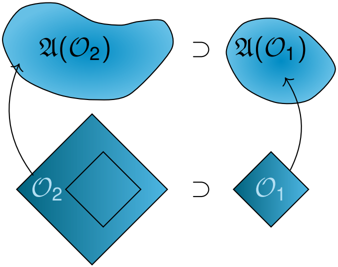
- It started as the axiomatic framework of Haag-Kastler: a model is defined by associating to each region $\mathcal{O}$ of Minkowski spacetime $\mathbb{M}$ an algebra $\mathfrak{A}(\mathcal{O})$ of observables that can be measured in $\mathcal{O}$.
- The physical notion of subsystems is realized by the condition of isotony, i.e.: $\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$. We obtain a net of algebras.



Further properties we want

One can also ask for further, physically motivated properties: causality and time-slice axiom.

- **Causality**: If $\mathcal{O}_1, \mathcal{O}_2 \subset \mathbb{M}$ are spacelike separated (no causal curve joining them), then

$$[\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)] = \{0\},$$

where $[.,.]$ is the commutator in the sense of $\mathfrak{A}(\mathcal{O}_3)$, where $\mathcal{O}_3$ contains both $\mathcal{O}_1$ and $\mathcal{O}_2$.

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  where $[.,.]$ is the commutator in the sense of $\mathfrak{A}(\mathcal{O}_3)$, where $\mathcal{O}_3$ contains both $\mathcal{O}_1$ and $\mathcal{O}_2$.
- **Time-slice axiom:** If $\mathcal{N}$ is a neighborhood of a Cauchy-surface in $\mathcal{O}$, then $\mathfrak{A}(\mathcal{N})$ is isomorphic to $\mathfrak{A}(\mathcal{O})$.

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- Main idea: theory described by an abstract C^*-algebra generated by a collection of unitaries, with a number of relations.
- These unitaries are interpreted as local S-matrices and are labelled by local functionals. This approach is motivated by results from perturbative algebraic quantum field theory (pAQFT).

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- Spacetime (M, g), where g is a globally hyperbolic Lorentzian metric, i.e. has a Cauchy surface (good for specifying initial data for PDEs).

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- Dynamics: we use a modification of the Lagrangian formalism (fully covariant).

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- Localization of functionals governed by their spacetime support:

$$\begin{aligned}\mathrm{supp}\, F = \{x \in M | \forall \text{ neighbourhoods } U \text{ of } x\ \exists \varphi,\psi \in \mathcal{E}, \\ \mathrm{supp}\,\psi \subset U \text{ such that } F(\varphi+\psi) \neq F(\varphi)\}\ .\end{aligned}$$

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- F is local (notation: $F\in\mathcal{F}_{\mathrm{loc}}$) if it is of the form:

$$F(\varphi)=\int_M \alpha(\varphi(x),\partial\varphi(x),\dots)\, d\mu_g(x)\,,$$

where $d\mu_g\equiv\sqrt{-g}d^4x$.

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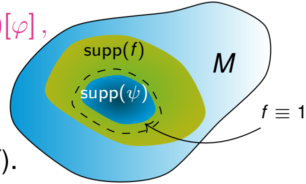
- To simplify notation, I will write L_F simply as F.

Classical Dynamics

- Let L be a generalized Lagrangian and $\varphi \in \mathcal{E}$. Define $\delta L : \mathcal{E}_c \times \mathcal{E} \to \mathbb{R}$ by
$$\delta L(\psi)[\varphi] \doteq L(f)[\varphi + \psi] - L(f)[\varphi]\,,$$
where $\varphi \in \mathcal{E}$, $\psi \in \mathcal{E}_c$ (compactly supported configuration)
and $f \equiv 1$ on $\mathrm{supp}\,\psi$ (the map $\delta L(\psi)[\varphi]$ does not depend on the particular choice of f).



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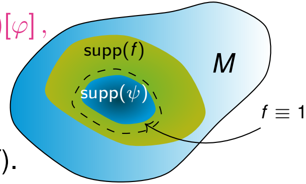

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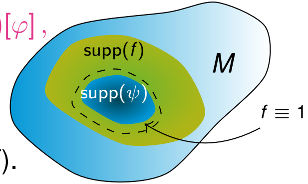

- The above definition can be turned into a difference quotient and we can use it to introduce the Euler-Lagrange derivative of L.
- The Euler-Lagrange derivative of L is a 1-form on $\mathcal{E}$ defined by $\langle dL(\varphi), \psi \rangle \doteq \lim_{t\to 0} \frac{1}{t}\delta L(t\psi)[\varphi] = \int \frac{\delta L(f)}{\delta\varphi(x)}\psi(x)$, with $\psi \in \mathcal{E}_c$ and $f \equiv 1$ on $\mathrm{supp}\,\psi$. The field equation is: $dL(\varphi) = 0$.

Dynamical spacetimes

- Define dynamical spacetimes as pairs (M, L) where M is a globally hyperbolic spacetime and L is a Lagrangian of the form $L = L_0 + V_0$, where L_0 is kinnetic term (so something of the form $g(d\varphi, d\varphi)$) and V_0 depends only of the value of the field φ and the value of its first derivative (no higher derivatives enter.)

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- The corresponding generalized Lagrangians are called interactions w.r.t L and their set is denoted by $\mathrm{Int}(M, L)$.
- Interactions (which can also be understood as perturbations of L) play a key role in this formulation.

Dynamical Algebra

The dynamical algebra $\mathfrak{A}(M, L)$ is a C*-algebra freely generated by unitaries $S_{(M,L)}(F)$, $F \in \mathcal{F}_{\mathrm{loc}}(M, L)$ with $S_{(M,L)}(c) = e^{ic}1$ for constant functionals c, $c \in \mathbb{R}$, modulo the following relations (for simplicity we drop the subscript (M, L)):

A1 **Locality/Causality**: Let $G \in \mathcal{F}_{\mathrm{loc}}(M, L)$ and $F, H \in \mathcal{F}_{\mathrm{loc}}(M, L + G)$. Then

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- **A2** **Dynamical relation (on-shell)** For all $F\in\mathcal{F}_{\mathrm{loc}}(M,L)$ we require
$$S(F)=S(F^{\psi}+\delta L(\psi))\ ,\quad \psi\in\mathcal{D}(M,\mathbb{R}^n)\ ,$$
where
$$F^{\psi}[\phi]\doteq F[\phi+\psi]\ ,\quad \delta L(\psi)\doteq L(f)^{\psi}-L(f)\ ,$$
for any $f\in\mathcal{D}(M)$ satisfying $f\equiv 1$ on $\mathrm{supp}\,\psi$.

Motivation from pAQFT

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- We add to $\frac{i}{2}\Delta$ a symmetric distribution H with $P_x H(x,y) = 0 = P_y H(x,y)$, so that the resulting $W = \frac{i}{2}\Delta + H$ is of positive type and has a particularly nice singluarity structure (it ends up being the 2-point function of the quantized theory).

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- We introduce the Feynman propagator $\Delta^{\mathrm{F}} = \frac{i}{2}(\Delta^{\mathrm{R}} + \Delta^{\mathrm{A}}) + H$.

Motivation from pAQFT

- Denote $D_{ij} = \left\langle \hbar\Delta^{\mathrm{F}}, \frac{\delta^2}{\delta\varphi_i\delta\varphi_j} \right\rangle$ and for $F_1, \dots, F_n \in \mathcal{F}_{\mathrm{loc}}$ with pariwise disjoint supports define their time-ordered product by

$$\mathcal{T}_n(F_1 \otimes \cdots \otimes F_n)(\varphi) = \exp\Big(\sum_{1\leq i<j\leq n} D_{ij}\Big)(F_1(\varphi_1)\dots F_n(\varphi_n))\Big|_{\varphi_1=\dots=\varphi_n=\varphi}.$$

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- This has to be extended to arguments with arbitrary support and is done by means of Epstein-Glaser renormalization.

Motivation from pAQFT

- Denote $D_{ij} = \left\langle \hbar\Delta^{\mathrm{F}}, \frac{\delta^2}{\delta\varphi_i\delta\varphi_j} \right\rangle$ and for $F_1, \dots, F_n \in \mathcal{F}_{\mathrm{loc}}$ with pariwise disjoint supports define their time-ordered product by

$$\mathcal{T}_n(F_1 \otimes \cdots \otimes F_n)(\varphi) = \exp\Big(\sum_{1\leq i<j\leq n} D_{ij}\Big)(F_1(\varphi_1)\dots F_n(\varphi_n))\Big|_{\varphi_1=\cdots=\varphi_n=\varphi}.$$

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- On the technical level, this procedure amounts to construction of extensions of certain distributions.
- One of the crucial axioms is Causal factorisation property:

$$\mathcal{T}_n(V_1, \dots, V_n) = \mathcal{T}_k(V_1, \dots, V_k) \star \mathcal{T}_{n-k}(V_{k+1}, \dots, V_n) ,$$

if $\mathrm{supp}\, V_{k+1}, \dots, \mathrm{supp}\, V_n$ not later than $\mathrm{supp}\, V_1, \dots, \mathrm{supp}\, V_k$.

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More about time-ordered products

- From $(\mathcal{T}_n)_{n\in\mathbb{N}}$ we can also define a map $\mathcal{T}$ on multilocal functionals $\mathcal{F}$ by: $\mathcal{T}F \doteq \bigoplus_{n\in\mathbb{N}} \mathcal{T}_n \circ m^{-1}$.

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Convergence in sine-Gordon model

Theorem (Bahns, KR 2016)

The formal S-matrix $\mathcal{S}(\lambda V) = e_{\mathcal{T}}^{i\lambda V/\hbar}$ in the sine-Gordon model with $V = \cos(a\Phi(f))$ and $0 < \beta = \hbar a^2/4\pi < 1$, $f \in \mathcal{D}(M)$, converges as a functional on the configuration space in the appropriate topology (related to Hörmander topology on distribution spaces).

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- Similarly, we have constructed relative S-matrices $\mathcal{S}_{\lambda V}(\Phi(g))$, where $\Phi(g)$ is the smeared field.
- Using these results, and a class of Hadamard states locally normal to the massive vacuum, in [Bahns, Fredenhagen, KR CMP 2021, arXiv:1712.02844] we constructed the local net of von Neumann algebras.

Renormalisation group in pAQFT

- Non-uniqueness in constructing $\mathcal{T}_n$s is described using Stückelberg-Petermann renormalization group. It is essentially a group of maps from formal power series in $\mathcal{F}_{\mathrm{loc}}$ to formal power series in $\mathcal{F}_{\mathrm{loc}}$, satisfying some extra conditions (more later!).

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- The key to understand this is the main theorem of renormalization. It states that given $\mathcal{T}$, another $\widetilde{\mathcal{T}}$ is a valid time-ordered product if and only if there exists an element Z of the renormalization group such that:

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- Intuitively, we can think of Z as adding finite counter terms or changing the values of the coupling constants.

Non-perturbative Renormalization Group

In the C^*-algebraic framework, the renormalization group $\mathcal{R}(M, L)$ for a Lagrangian $L = L_0 + V$, with $V \in \mathrm{Int}(M, L_0)$, is the set of all bijections Z of $\mathcal{F}_{\mathrm{loc}}(M, L)$ which satisfy the following conditions:

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- **R 3** (**Locality**) Let $G \in \mathcal{F}_{\mathrm{loc}}(M, L)$ and $F, H \in \mathcal{F}_{\mathrm{loc}}(M, L + G)$ with $\mathrm{supp}\, F \cap \mathrm{supp}\, H = \varnothing$. Then

$$Z(F + G + H) = Z(F + G) - Z(G) + Z(G + H)\ .$$

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- **R 6** (**Causal Stability**) Z does not change the causal structure.

Symmetry transformations

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- We also introduce an L-dependent action of $\mathbf{G}_c(M)$ on $\mathcal{F}_{\mathrm{loc}}(M, L)$:

$$(\mathbf{g}, F) \mapsto \mathbf{g}_L F \doteq \delta_{\mathbf{g}} L + \mathbf{g}_* F\ .$$

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$$\zeta_{\mathbf{gh}} = \zeta_{\mathbf{h}}(\zeta_{\mathbf{g}})^{\mathbf{h}} \quad \text{where} \quad (\zeta_{\mathbf{g}})^{\mathbf{h}} \doteq \mathbf{h}_L^{-1} \zeta_{\mathbf{g}} \mathbf{h}_L \,, \quad \mathbf{g}, \mathbf{h} \in \mathbf{G}_c(M) \,,$$

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A3 (**Unitary Anomalous Master Ward Identity**) Let (M, L) be some dynamical spacetime. A representation π of $\mathfrak{A}(M, L)$ satisfies the unitary AMWI if there exists some $\zeta \in \mathfrak{Z}(M, L)$ such that:

$$\pi \circ S_{(M,L)} \circ \mathbf{g}_L = \pi \circ S_{(M,L)} \circ \zeta_{\mathbf{g}} \ , \quad \mathbf{g} \in \mathbf{G}_c(M) \ .$$

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In particular: $S_{(M,L,\zeta)}(\delta_{\mathbf{g}} L) = S_{(M,L,\zeta)}\big(\zeta_{\mathbf{g}}(0)\big)$.
- The interpretation of $\zeta_{\mathbf{g}}(0)$ in path integral formulation is as the logarithm of the Jacobian corresponding to the symmetry transformation **g**. The infinitesimal version thereof is the renormalized BV Laplacian Δ [Fredenhagen, KR CMP 2013].

Time-slice axiom

Theorem

Let $N \subset M$ be a causally convex and globally hyperbolic neighbourhood of a Cauchy surface of M with respect to the causal structure induced by the Lagrangian L and let $\zeta \in \mathfrak{Z}(M, L)$ be a cocycle. Then

$$\mathfrak{A}(N, L\restriction_N, \zeta\restriction_N) = \mathfrak{A}(M, L, \zeta)\ .$$

Here, $\zeta\restriction_N : \mathbf{G}_c(N) \to \mathcal{R}(N, L\restriction_N)$ is obtained by the restriction of $\zeta\ :\ \mathbf{G}_c(M) \to \mathcal{R}(M, L)$.

Relative Cauchy evolution

- Morphisms $\alpha_{V,\pm}$ interpolate between the algebra $\mathfrak{A}(M, L')$ and the *perturbed* algebra $\mathfrak{A}(M, L)$, with $L = L' + V$.

$$\alpha_{V,+}(S_{(M,L)}(F)) = S_{(M',L')}(V(f))^{-1} S_{(M',L')}(F + V(f)) \, ,$$

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- Let $N_\pm$ be two neighborhoods of Cauchy surfaces in (M, L'), one in the past and the other in the future of the support of the perturbation V, w.r.t. the causal structure induced by L'. They are embedded into M by $\chi_\pm$ and the induced morphisms are $\alpha_{\chi_\pm}$.

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- We define $\tilde{\alpha}_{V,-} \doteq \alpha_{\chi_-} \circ \overline{\alpha}^{-1}_{V,-}$ and $\tilde{\alpha}_{V,+} \doteq \alpha_{\chi_+} \circ \overline{\alpha}^{-1}_{V,+}$

Relative Cauchy evolution

- Morphisms $\alpha_{V,\pm}$ interpolate between the algebra $\mathfrak{A}(M,L')$ and the *perturbed* algebra $\mathfrak{A}(M,L)$, with $L = L' + V$.

$$\alpha_{V,+}(S_{(M,L)}(F)) = S_{(M',L')}(V(f))^{-1} S_{(M',L')}(F + V(f)) ,$$
$$\alpha_{V,-}(S_{(M,L)}(F)) = S_{(M',L')}(F + V(f)) S_{(M',L')}(V(f))^{-1} .$$

- They induce isomorphisms $\overline{\alpha}_{V,\pm}$ from $\mathfrak{A}(M, L' + V, \zeta^V)$ to $\mathfrak{A}(M, L', \zeta)$, where
$$\zeta^W_{\mathbf{g}}(F) \doteq \zeta_{\mathbf{g}}(F + W(f)) - W(f) \in \mathcal{F}_{\mathrm{loc}}(M, L' + W) .$$
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- We find that the relative Cauchy evolution automorphism β_V of $\mathfrak{A}(M, L', \zeta)$ defined via these maps is given by
$$\beta_V \doteq \tilde{\alpha}_{V,+} \circ \tilde{\alpha}^{-1}_{V,-} = \mathrm{Ad}\big(S_{(M,L',\zeta)}(V(f))^{-1}\big) .$$

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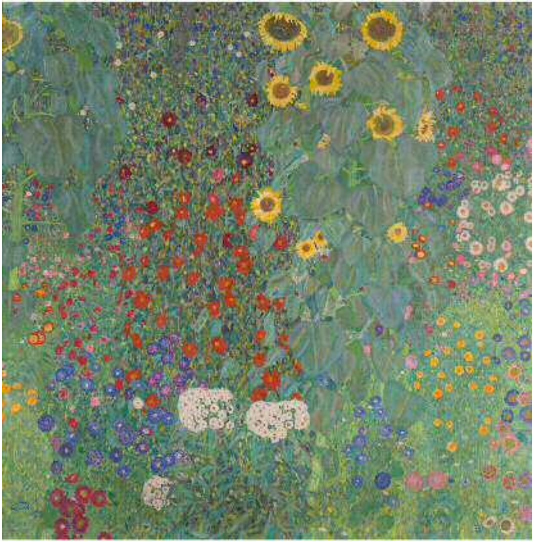
- Hence it is unitarily implemented by S-matrices.



Happy Birthday Harald!