Constructing morphisms for arithmetic subsequences of Fibonacci

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Geuversfest, Nijmegen, 2024

(Picture by Dr. Oetker)

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The Fibonacci sequence: start with 0 and substitute

$$0 \mapsto 01, \qquad 1 \mapsto 0$$

Iterating gives, writing $\mathcal{F}$ for the substitution/morphism,

$$\begin{aligned}
\mathcal{F}^0(0) &= 0 \\
\mathcal{F}^1(0) &= 01 \\
\mathcal{F}^2(0) &= 01\ 0 \\
\mathcal{F}^3(0) &= 010\ 01 \\
&\vdots \\
\mathcal{F}^{k+1}(0) &= \mathcal{F}^k(\mathcal{F}(0)) = \mathcal{F}^k(01) = \mathcal{F}^k(0)\mathcal{F}^k(1) = \mathcal{F}^k(0)\mathcal{F}^{k-1}(0)
\end{aligned}$$

The limit $\sigma := \mathcal{F}^\infty(0)$ is well-defined. Alphabet $\Sigma = \{0, 1\}$

$$\sigma = 010010100100101001010010010100100101001010010010100101001001010\ldots$$

This is a *pure morphic sequence.*

Morphic sequence: pure morphic + coding.

Example: $\Sigma = \{0, 1, 2\}$, morphism $\mathcal{G}$ given by

$$0 \mapsto 01, \qquad 1 \mapsto 12, \qquad 2 \mapsto 0.$$

and coding $\tau : \Sigma \to T$ with $T = \{0, 1\}$ by

$$0 \mapsto 0, \qquad 1 \mapsto 1, \qquad 2 \mapsto 0.$$

Then

$$\begin{array}{rl} \mathcal{G}^\infty(0) = & 0\ 1\ 1\ 2\ 1\ 2\ 0\ 1\ 2\ 0\ 0\ 1\ 1\ 2\ 0\ 0\ 1\ 0\ 1\ 1\ 2\ \ldots \\ \text{coding:} & \downarrow \\ \tau(\mathcal{G}^\infty(0)) = & 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ \ldots \end{array}$$

Same sequence might have different generating morphisms.

- Complexity of a morphism $\mathcal{F}\colon \Sigma \to \Sigma^*$:

$$|\mathcal{F}| = \sum_{a\in\Sigma} |\mathcal{F}(a)|.$$

Example: Fibonacci $0 \mapsto 01, 1 \mapsto 0$ has $|\mathcal{F}| = 3$.

- Complexity of a morphic sequence σ over Σ:

σ might be generated in different ways by

  - a morphism $\mathcal{G} : T \to T^*$ over some alphabet T
  - a coding $\tau : T \to \Sigma$.

Complexity σ: smallest possible complexity of such $\mathcal{G}$.

Example: $\mathcal{F}^\infty(0) = 010010100100101\ldots$ has complexity 3.

Theorem (Dekking, 1994)

Let σ be a morphic sequence. Any arithmetic subsequence of σ is a morphic sequence.

$$\sigma = 0100101001001010010100100101001001010010100100101 0\ldots$$

$$0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad \ldots$$

$$(\sigma[4n+2])_{n\geq 0} = 0101010101010000001010101010101011\ldots$$

Question: Complexity of this arithmetic subsequence? Morphism?

Complexity of $(\sigma[4n+2])_{n\geq 0}$: Split σ in blocks of length 4:

$$\sigma = 0100 \mid 1010 \mid 0100 \mid 1010 \mid 0101 \mid 0010 \mid 0101 \mid 0010 \ \ldots$$

These blocks will be 'letters' in a new alphabet $\mathcal{A}$:

$$a_0 = 0100,\ a_1 = 1010,\ a_2 = 0101,\ a_3 = 0010,\ a_4 = 1001.$$

Then σ looks like

$$\tilde{\sigma} = a_0 a_1 a_0 a_1 a_2 a_3 a_2 a_3 a_2 a_3 a_4 a_3 a_4 a_0 a_4 a_0 a_4 a_0 a_1 \ldots$$

Coding $\tau : \mathcal{A} \to \{0,1\}$: map each letter to its third bit:

$$\tau(a_0) = \tau(a_2) = \tau(a_4) = 0, \qquad \tau(a_1) = \tau(a_3) = 1.$$

Still to find: morphism $\mathcal{G} : \mathcal{A} \to \mathcal{A}^*$ generating $\tilde{\sigma}$.

Naive approach:

$$\mathcal{G}(a_0) \simeq \mathcal{F}(0100) = 0100 \mid 101.$$

Problem: 101 is not a letter in $\mathcal{A}$.

Trick: consider powers of $\mathcal{F}$. We find

$$|\mathcal{F}^6(0100)| = 76, \qquad |\mathcal{F}^6(1010)| = 68, \qquad |\mathcal{F}^6(0101)| = 68,$$
$$|\mathcal{F}^6(0010)| = 76, \qquad |\mathcal{F}^6(1001)| = 68.$$

All multiples of 4. So images can be written as words over $\mathcal{A}$. We can take

$$\mathcal{G} \simeq \mathcal{F}^6.$$

Complexity:

$$\frac{2 \times 76 + 3 \times 68}{4} = 89.$$

Why does this work?

Let $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0, F_1 = 1$ (Fibonacci numbers).

n	0	1	2	3	4	5	6	7	8	9
F_n	0	1	1	2	3	5	8	13	21	34

Then

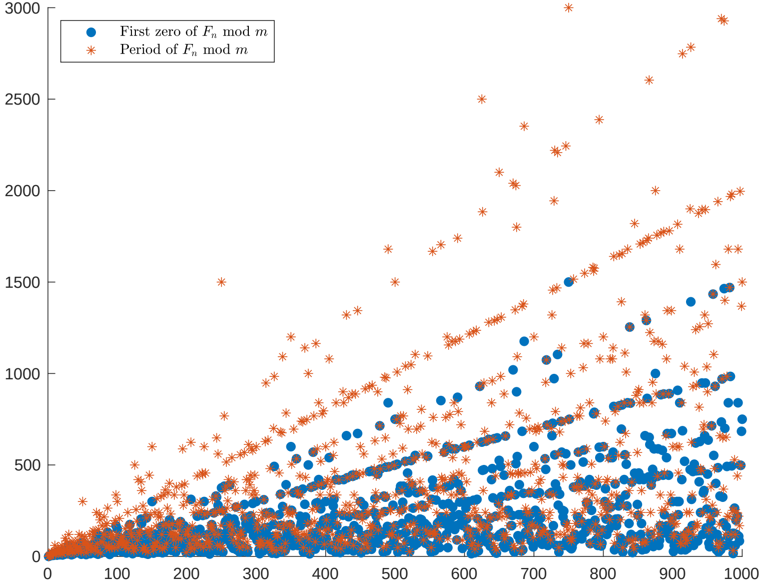
$$|\mathcal{F}^k(0)| = F_{k+2}, \qquad |\mathcal{F}^k(1)| = F_{k+1}$$

In our example

$$\begin{aligned} |\mathcal{F}^6(0100)| &= |\mathcal{F}^6(0)| + |\mathcal{F}^6(1)| + |\mathcal{F}^6(0)| + |\mathcal{F}^6(0)| \\ &= F_8 + F_7 + F_8 + F_8 = 4 \cdot F_7 + 3 \cdot F_6. \end{aligned}$$

It works because $F_6 \equiv 0 \mod 4$.

For $(\sigma[mn+k])_{n \geq 0}$: need F_j with $F_j \equiv 0 \mod m$.



3000
2500
2000
1500
1000
500
0
0
100
200
300
400
500
600
700
800
900
1000
First zero of F_n mod m
Period of F_n mod m

Complexity: $\sum_{a \in \mathcal{A}} |\mathcal{G}(a)|$

For $m = 4$**:** five blocks of length 4 exist, so $|\mathcal{A}| = 5$.

Length of $|\mathcal{G}(a)|$:

$$\begin{aligned} |\mathcal{F}^6(0100)| &= |\mathcal{F}^6(0)| + |\mathcal{F}^6(1)| + |\mathcal{F}^6(0)| + |\mathcal{F}^6(0)| \\ &= F_8 + F_7 + F_8 + F_8 = 4 \cdot F_7 + 3 \cdot F_6, \end{aligned}$$

so $|\mathcal{G}(a)| \leq \frac{4 \cdot F_8}{4} = F_8$ for all a.

$$\implies \sum_{a \in \mathcal{A}} |\mathcal{G}(a)| \leq |\mathcal{A}| \cdot F_8 = 5 \cdot 21 = 105.$$

For general m**:** $|\mathcal{A}| = m + 1$ (factor complexity of σ) and

$$|\mathcal{G}(a)| \leq F_{z(m)+2}$$

with $z(m)$ the index of the first Fibonacci number divisible by m.

Theorem

Let σ be the Fibonacci sequence. The arithmetic subsequence $(\sigma[mn+k])_{n\geq 0}$ has complexity at most

$$(m+1)F_{z(m)+2},$$

where $z(m) = \min\{j : F_j \equiv 0 \mod m\}$.

m	$(m+1)F_{z(m)+2}$
2	15
3	32
4	105
5	78
6	2639
7	440
8	189
9	3770

m	$(m+1)F_{z(m)+2}$
10	17567
11	1728
12	4901
13	476
14	1820895
⋮	⋮
29	29610
30	125634925674311

Remarks:

- The actual complexity of our construction is slightly smaller than $(m+1)F_{z(m)+2}$.
- The true complexity of $(\sigma[mn+k])_{n\geq 0}$ might be much smaller. **Open question!**
- Brute force gives candidate morphisms + one needs a proof. **Example:** $(\sigma[2n])_{n\geq 0}$ has compl. 8, $(\sigma[2n+1])_{n\geq 0}$ has 9. Our construction has 13.

Thank you!